BULLETIN

P17567

OF THE

AMERICAN MATHEMATICAL SOCIETY

CUK-H06289-39-P17567

EDITED BY

NATHAN JACOBSON
DEANE MONTGOMERY
RUDOLPH ERNEST LANGER
ELLIS B. STOUFFER

510.5
0036
39

WITH THE ASSISTANCE OF

R. H. BING
M. M. DAY
E. R. LORCH
HERBERT BUSEMANN
F. G. DRESSEL
I. M. SHEFFER
G. W. WHITEHEAD
A. H. CLIFFORD
MAURICE HEINS
R. M. THRALL

VOLUME 55, NUMBER 7
JULY, 1949

CALCUTTA UNIVERSITY CENTRAL LIBRARY

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK

AMERICAN MATHEMATICAL SOCIETY

General Office: Room 305, Low Memorial Library, 531 West 116th Street, New York 27, N. Y.

This address should be used for all mail which concerns membership, subscriptions, changes of address, etc. Mail for Librarian and Treasurer should be sent to General Office.

Editorial Office, Mathematical Reviews: Brown University, Providence 12, R. I. Dr. R. P. Boas, Executive Editor.

This address should *not* be used for subscriptions and payments, etc. Correspondence about these matters should go to the Society's General Office.

Officers and Editors:

President: Professor J. L. Walsh, Harvard University, Cambridge 38, Mass.
Secretary: Professor J. R. Kline, American Mathematical Society, University of Pennsylvania, Philadelphia 4, Pa.
Associate Secretaries:
  Professor J. W. Green, University of California at Los Angeles, Los Angeles 24, Calif.
  Professor T. R. Hollcroft, Wells College, Aurora, N. Y.
  Professor G. B. Price, University of Kansas, Lawrence, Kan.
  Professor J. W. T. Youngs, Indiana University, Bloomington, Ind.
Treasurer: Dean A. E. Meder, Rutgers University, New Brunswick, N. J.
Librarian: Professor Arnold Dresden, Swarthmore College, Swarthmore, Pa.
Managing Editor of the Bulletin: Professor R. E. Langer, University of Wisconsin, Madison 6, Wis.
Managing Editor of the Transactions: Professor G. T. Whyburn, University of Virginia, Charlottesville, Va.
Editors of Mathematical Reviews:
  Professor William Feller, Cornell University, Ithaca, N. Y.
  Professor M. H. Stone, University of Chicago, Chicago 37, Ill.
  Professor Hassler Whitney, Harvard University, Cambridge 38, Mass.
Chairman, Colloquium Editorial Committee: Professor C. C. MacDuffee, University of Wisconsin, Madison 6, Wis.
Chairman, Mathematical Surveys Editorial Committee: Professor Nelson Dunford, Yale University, New Haven 11, Conn.
Editors of American Journal of Mathematics:
  Professor Richard Brauer, University of Michigan, Ann Arbor, Mich.
  Professor L. M. Graves, University of Chicago, Chicago 37, Ill.
  Professor D. C. Lewis, Johns Hopkins University, Baltimore 18, Md.
  Professor Hassler Whitney, Harvard University, Cambridge 38, Mass.
  Professor Aurel Wintner, Johns Hopkins University, Baltimore 18, Md.

THE BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY is published monthly. Price per annual volume, $11.00. Subscriptions, orders for back numbers, and inquiries in regard to nondelivery of current numbers should be addressed to the American Mathematical Society, 450 Ahnaip Street, Menasha, Wis., or Low Memorial Library, 531 West 116th Street, New York 27, N. Y.

Entered as second class matter April 24, 1926, at the post office at Menasha, Wis., under the act of August 24, 1912. Acceptance for mailing at the special rate of postage provided for in the act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R., authorized May 9, 1935.

Copyright, American Mathematical Society, 1949

PROBABILITY METHODS IN SOME PROBLEMS OF ANALYSIS AND NUMBER THEORY

M. KAC

1. **Introduction.** In 1922 Rademacher [1][1] introduced the functions

$$r_n(t) = \text{sign} \sin 2^n \pi t, \qquad 0 \leqq t \leqq 1, \; n = 1, 2, \cdots, \tag{1.1}$$

and proved that the series

$$\sum_1^\infty c_n r_n(t) \tag{1.2}$$

converges almost everywhere provided

$$\sum_1^\infty c_n^2 < \infty. \tag{1.3}$$

39

In 1925 Kolmogoroff and Khintchine [1] generalized this result and also proved the counterpart to the effect that

$$\sum_1^\infty c_n^2 = \infty \tag{1.4}$$

implies divergence almost everywhere of (1.2). The probabilistic nature of these results (first recognized by Steinhaus [1]) becomes apparent when one notices that the Rademacher functions $r_n(t)$ are *statistically independent*, that is, have the property that

$$\left| \underset{t}{E} \{ r_1(t) < \alpha_1, \cdots, r_n(t) < \alpha_n \} \right| = \prod_1^n \left| \underset{t}{E} \{ r_k(t) < \alpha_k \} \right|,^{2} \tag{1.5}$$

for $n = 2, 3, \cdots$ and all real $\alpha_1, \alpha_2, \cdots$.

Following the natural line of development, Kolmogoroff [1; **2**] was led to his celebrated necessary and sufficient conditions (the "three series theorem") for convergence of series,

$$\sum_1^\infty f_n(t), \tag{1.6}$$

An address delivered before the Annual Meeting of the Society in Columbus, Ohio, on December 28, 1948, by invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings; received by the editors November 1, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] Here, as in the sequel, $E\{\ \}$ denotes the set of t's satisfying the condition inside the braces, and $|A|$ denotes the Lebesgue measure of the set A.

of arbitrary independent functions or, equivalently, series of independent random variables.

Simultaneously with the above development another theory was being evolved. In 1924 Kolmogoroff [3] proved that (1.3) also implies the convergence almost everywhere of the trigonometric gap series,

$$\sum_{1}^{\infty} c_k \sin 2\pi n_k t, \tag{1.7}$$

where $\{n_k\}$ is a sequence of integers satisfying the "Hadamard gap" condition

$$\frac{n_{k+1}}{n_k} > q > 1. \tag{1.8}$$

A few years later Zygmund [1] proved the counterpart of this result to the effect that (1.4) implies divergence almost everywhere of (1.7).

The analogies in behavior of Rademacher series (1.2) and trigonometric gap series (1.7) were so striking that considerable attention was devoted to the subject.

Paley and Zygmund [1; 2; 3] discovered many other important analogies and discussed a variety of related topics while Banach [1] was led to a general study of so-called "lacunary series" which include as special cases both Rademacher and trigonometric gap series.

One important analogy, however, escaped investigation. Since the Rademacher functions are independent, it follows from the central limit theorem of the calculus of probability that

$$\lim_{n\to\infty} \left| E_t \left\{ \sum_{1}^{n} c_k r_k(t) < \omega \left(\sum_{1}^{n} c_k^2 \right)^{1/2} \right\} \right| = (2\pi)^{-1/2} \int_{-\infty}^{\omega} e^{-u^2/2} du \tag{1.9}$$

provided

$$\sum_{1}^{\infty} c_k^2 = \infty \tag{1.10}$$

and

$$\max_{1 \leq k \leq n} |c_k| = o\left\{ \left(\sum_{1}^{n} c_k^2 \right)^{1/2} \right\}. \tag{1.11}$$

It was only natural to inquire whether a similar result holds if the $r_k(t)$ are replaced by $\sin 2\pi n_k t$ or, more generally, by $f(n_k t)$. This problem was the starting point of a series of investigations which will be reviewed in Part I.

Part II will be devoted to a review of a class of results in analytic number theory which again have been strongly influenced by the central limit theorem and other probabilistic results.

The limitations of time prevent us from reviewing other fields (notably the theory of almost periodic functions) in which methods of probability theory were successfully employed.

PART I. GAP SERIES

2. The trigonometric cases. One of the oldest tools of probability theory is the method of moments. The following lemma was first used by Tchebysheff:

If a sequence of distribution functions $\sigma_n(u)$ is such that for $l=0, 1, 2, \cdots$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} u^l d\sigma_n(u) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} u^l e^{-u^2/2}du, \tag{2.1}$$

then

$$\lim_{n\to\infty}\sigma_n(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{\omega} e^{-u^2/2}du \tag{2.2}$$

for every real ω.

By a simple restatement of this result we obtain the following: If $f_n(t)$ is a sequence of measurable functions defined in (0, 1) and if for $l=1, 2, \cdots$

$$\lim_{n\to\infty}\int_0^1 f_n^l(t)dt = (2\pi)^{-1/2}\int_{-\infty}^{\infty} u^l e^{-u^2/2}du, \tag{2.3}$$

then

$$\lim_{n\to\infty}\left| \underset{t}{E}\{f_n(t) < \omega\}\right| = (2\pi)^{-1/2}\int_{-\infty}^{\omega} e^{-u^2/2}du. \tag{2.4}$$

This suggests an attack on the problem of determining the limit of

$$\left|\underset{t}{E}\left\{\frac{\sum_1^n c_k \sin 2\pi n_k t}{\left(\sum_1^n c_k^2\right)^{1/2}} < \omega\right\}\right|, \tag{2.5}$$

as $n\to\infty$.

If instead of the gap condition

(2.6) $$n_{k+1}/n_k > q > 1$$

we have the stronger condition

(2.7) $$n_{k+1}/n_k \to \infty,$$

then, letting

$$f_n(t) = 2^{1/2}\frac{\sum_1^n c_k \sin 2\pi n_k t}{\left(\sum_1^n c_k^2\right)^{1/2}},$$

it is quite easy to verify that (2.3) holds provided $c_k = O(1)$ or, more generally,

(2.8) $$\max_{1\leq k\leq n} |c_k| = o\left\{\left(\sum_1^n c_k^2\right)^{1/2}\right\}$$

and

(2.9) $$\sum_1^\infty c_k^2 = \infty.$$

Thus under conditions (2.7), (2.8), and (2.9) (Kac [1])

(2.10) $$\lim_{n\to\infty}\left| E_t\left\{\sum_1^n c_k \sin 2\pi n_k t < \omega\left(\frac{1}{2}\sum_1^n c_k^2\right)^{1/2}\right\}\right| = (2\pi)^{-1/2}\int_{-\infty}^{\omega} e^{-u^2/2}du.$$

The verification of (2.3) in the case of the gap condition (2.6) is more delicate. It was first done by Erdös in 1940, under the condition $c_k = O(1)$, but the proof was not published. It actually turns out that the proof can be carried out under the less restrictive condition (2.8).[3] In 1947 Fortet and Ferrand [1] announced a proof of (2.10) under the gap condition (2.6) but failed to state any restrictions on the c_k's. This omission led Salem and Zygmund [1] to believe that the Fortet-Ferrand proof must have been incomplete. They then proceeded to give a proof of (2.10) using an adaptation of the method of characteristic functions rather than the method of moments. They also proved that (2.8) is a necessary condition for the validity of (2.10), thus making the analogy with sums $\sum_1^n c_k r_k(t)$ complete.

---

[3] This proof will appear in a paper by Erdös, Ferrand, Fortet and Kac.

In a subsequent paper Salem and Zygmund [2] extended (2.10) to the case where the n_k's are not necessarily integers. This extension had also been announced by Fortet and Ferrand [1]. In this case (unlike the case of integral n_k) there is a definite advantage in using the Salem-Zygmund method because the moment method becomes rather tedious.

Among various extensions of the above results we mention one because we feel that it is of independent interest.

Let

$$F(z) = \sum_0^\infty c_k z^{n_k}, \tag{2.11}$$

where

$$\sum_0^\infty | c_k |^2 = \infty, \qquad \frac{n_{k+1}}{n_k} > q > 1 \tag{2.12}$$

and

$$\max_{1 \leq k \leq n} | c_k | = o\left\{\left(\sum_0^n c_k^2\right)^{1/2}\right\}. \tag{2.13}$$

For $0<r<1$, set

$$M(r) = \left(\frac{1}{2\pi}\int_0^{2\pi} | F(re^{i\theta}) |^2 d\theta\right)^{1/2} \tag{2.14}$$

and consider the set $E_r(\Omega)$ of those θ for which

$$\frac{F(re^{i\theta})}{M(r)} \in \Omega, \tag{2.15}$$

where Ω is a measurable set.

Then

$$\lim_{r\to 1} | E_r(\Omega) | = 2\int\int_\Omega e^{-(u^2+v^2)} du dv \tag{2.16}$$

(Salem and Zygmund [2] and for an earlier weaker version Kac [1]).

3. **The general case.** In view of the results of the preceding section it has been conjectured that for sufficiently restricted periodic (with period 1) $f(t)$ satisfying the normalization condition

$$\int_0^1 f(t) dt = 0 \tag{3.1}$$

and for integers n_k satisfying (2.6) the limit

$$\lim_{m\to\infty} \left| E_t \left\{ \sum_1^m f(n_k t) < \omega m^{1/2} \right\} \right| \tag{3.2}$$

is again given by a properly normalized Gaussian integral. (We first consider the simple case $c_k = 1$, $k = 1, 2, \cdots$.)

It thus came as a surprise when simultaneously and independently of each other, Erdös and Fortet constructed an example showing that the limit (3.2) need not be Gaussian. The example[4] is

$$f(t) = \cos 2\pi t + \cos 4\pi t, \qquad n_k = 2^k - 1, \tag{3.3}$$

in which case the limit (3.2) can be shown to be

$$\pi^{-1/2} \int_0^1 \int_{-\infty}^{\omega/2|\cos \pi x|} e^{-u^2} du dx \tag{3.4}$$

(Kac [2]).

More complicated examples can be constructed for which the limit (3.2) need not even exist.

The interest of the Erdös-Fortet example is enhanced by the fact that for $n_k = 2^k$ one has the following general theorem (Fortet [1], Kac [3]):

If $f(t)$ is periodic with period 1 and either satisfies a Lipschitz condition or is of bounded variation[5] and if furthermore

$$\int_0^1 f(t) dt = 0 \tag{3.5}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \left(\sum_1^n f(2^k t) \right)^2 dt = \sigma^2 \neq 0, \tag{3.6}$$

then

$$\lim_{n\to\infty} \left| E_t \left\{ \sum_1^n f(2^k t) < \omega n^{1/2} \right\} \right| = \frac{(2\pi)^{-1/2}}{\sigma} \int_{-\infty}^{\omega} e^{-u^2/2\sigma^2} du. \tag{3.7}$$

Thus the arithmetic structure of the sequence $\{n_k\}$ is of relevance except in the case $f(t) = \sin 2\pi t$ (or somewhat more generally $f(t) = a \sin 2\pi t + b \cos 2\pi t$). The result (3.7) clearly indicates that, in general, the functions $f(2^k t)$ do not behave as if they were independent. In fact, if they did, one would expect

---

[4] In Salem and Zygmund [2] this example is erroneously credited to Erdös alone.

[5] These conditions can be slightly relaxed. See Kac [3]. It should be clear that here and henceforth 2^k can be replaced by a^k, where a is an integer greater than 1.

$$\sigma^2 = \int_0^1 f^2(t)dt$$

instead of (3.6). As Fortet [1] points out the dependence exhibited by $f(2^n)t$ is very much like that in a Markoff chain. In the treatment of Kac [3] the dependence of $f(2^n t)$ is also brought out but unfortunately the connection with Markoff chains is lost.

It was pointed out by Erdös (see the final remark in Kac [**3**]) that (3.7) can be extended as follows:

If

$$\sum_1^\infty c_k^2 = \infty, \qquad c_k = O(1) \tag{3.8}$$

and if

$$\liminf_{n\to\infty} \frac{\int_0^1 \left(\sum_1^n c_k f(2^k t) \right)^2 dt}{\sum_1^n c_k^2} > 0, \tag{3.9}$$

then

$$\begin{aligned}\lim_{n\to\infty} \left| \underset{t}{E} \left\{ \sum_1^n c_k f(2^k t) < \omega \left(\int_0^1 \left(\sum_1^n c_k f(2^k t) \right)^2 dt \right)^{1/2} \right\} \right| \\ = (2\pi)^{-1/2} \int_{-\infty}^{\omega} e^{-u^2/2} du.\end{aligned} \tag{3.10}$$

It may be worth mentioning that from (3.7) it is not difficult to conclude that

$$\begin{aligned}\lim_{n\to\infty} \left| \underset{t}{E} \left\{ \sum_1^n f((2^k - 1)t) < \omega n^{1/2} \right\} \right| \\ = \frac{1}{\pi} \int_0^1 \int_{-\infty}^{\omega/g(x)} e^{-u^2} du dx,\end{aligned} \tag{3.11}$$

where $g(x)$ is defined as follows:

Assume for the sake of simplicity that $f(t)$ is even, so that

$$f(t) \sim \sum_1^\infty a_n \cos 2\pi n t$$

($a_0=0$ in view of (3.5)) and set, for $s=0, 1, 2, \cdots$,

$$\alpha_s(t) = \sum_{n=0}^{\infty} a_{(2s+1)2^n} \cos 2\pi(2s+1)2^n t,$$

$$\beta_s(t) = \sum_{n=0}^{\infty} a_{(2s+1)2^n} \sin 2\pi(2s+1)2^n t,$$

then

$$g(x) = \left(\sum_{s=0}^{\infty} (\alpha_s^2(x) + \beta_s^2(x)) \right)^{1/2}. \tag{3.12}$$

This, of course, includes (3.4) as a special case. It is somewhat curious to note that if $f(t)$ contains no even harmonics ($a_n=0$, n even) we get

$$g(x) = \text{const.} = \left(2\int_0^1 f^2(t)dt \right)^{1/2} \tag{3.13}$$

and the distribution function on the right of (3.11) is again normal. An extension of (3.11) to sums

$$\sum_1^n c_k f((2^k - 1)t)$$

does not appear to be easy.

In contradistinction to the complex situation which prevails in the case of Hadamard gaps $n_{k+1}/n_k > q > 1$, the case of "big gaps"

$$n_{k+1}/n_k \to \infty \tag{3.14}$$

is relatively simple. In fact we have the following theorem:

If $f(t)$ (periodic with period 1) satisfies a Lipschitz condition or is of bounded variation and if (3.5), (2.8) and (3.14) hold, then

$$\lim_{m\to\infty} \left| E_t \left\{ \sum_1^m c_k f(n_k t) < \omega \left(\sum_1^n c_k^2 \right)^{1/2} \right\} \right| = \frac{(2\pi)^{-1/2}}{\sigma} \int_{-\infty}^{\omega} e^{-u^2/2\sigma^2} du, \tag{3.15}$$

where

$$\sigma^2 = \int_0^1 f^2(t)dt. \tag{3.16}$$

The proof follows from the following two facts:

1. The theorem holds for a finite trigonometric polynomial

$$S_l(t) = \sum_{r=1}^{l} a_r \cos 2\pi r t. \tag{3.17}$$

2. For every $\epsilon>0$ one can choose an $S_l(t)$ (namely, a sufficiently high partial sum of the Fourier series of $f(t)$) in such a way that

$$\int_0^1 \left(\sum_1^m c_k f(n_k t) - \sum_1^m c_k S_l(n_k t) \right)^2 dt < \epsilon \sum_1^m c_k^2, \tag{3.18}$$

for all m.

The first part can be proved by the method of moments and only slight modifications of the procedure used in Kac [1] are required. The proof of the second part parallels proofs of analogous statements in Kac [3] and [4].

It should be mentioned that Salem and Zygmund [2] showed that for

$$f(t) = \sum_{r=j}^{l} a_r \cos 2\pi r t \tag{3.19}$$

the conclusion (3.15) holds with (3.14) replaced by

$$n_{k+1}/n_k \geqq q > l/j. \tag{3.20}$$

In view of the Erdös-Fortet example this condition is best possible. The smoothness conditions imposed on $f(t)$ are needed in the proof of (3.18).[6] The result (3.15) renders obsolete a weaker result in §5 of Kac [3].

4. **Convergence and divergence of gap series.** From (3.15) it follows almost immediately that

$$\sum_1^\infty c_k f(n_k t) \tag{4.1}$$

diverges almost everywhere provided

$$\sum_1^\infty c_k^2 = \infty. \tag{4.2}$$

For the Hadamard gaps the situation again becomes complex. We note first that, in general, divergence of $\sum c_k^2$ does not imply divergence almost everywhere of (4.1). In fact (Kac [4]), setting

---

[6] In proving (3.7) and (3.11) one also needs approximation theorems of type (3.18). Thus the necessity of imposing smoothness conditions on $f(t)$.

$$(4.3)\qquad f(t) = g(t) - g(2t), \qquad n_k = 2^k, \qquad c_k = \frac{1}{k^{1/2}},$$

we see that (4.2) is satisfied while the series (4.1) converges for every t.

However, for the case $n_k = 2^k$, we can appeal to (3.10) and obtain the following:

If (4.2), $c_k = O(1)$ and, in addition,

$$(4.4)\qquad \liminf_{n\to\infty} \frac{\int_0^1 \left(\sum_1^n c_k f(2^k t)\right)^2 dt}{\sum_1^n c_k^2} > 0,$$

then

$$(4.5)\qquad \sum_1^\infty c_k f(2^k t)$$

diverges almost everywhere.

It is seen that in the example (4.3) the condition (4.4) is violated.

It is worth pointing out that the full strength of (3.10) is not really needed. It is sufficient to know that

$$(4.6)\quad \lim_{n\to\infty} \frac{\int_0^1 \left(\sum_1^n c_k f(2^k t)\right)^4 dt}{\left(\int_0^1 \left(\sum_1^n c_k f(2^k t)\right)^2 dt\right)^2} = 3 = (2\pi)^{-1/2} \int_{-\infty}^{\infty} u^4 e^{-u^2/2} du.$$

In fact, setting

$$F_n(t) = \frac{\sum_1^n c_k f(2^k t)}{\left(\int_0^1 \left(\sum_1^n c_k f(2^k t)\right)^2 dt\right)^{1/2}}$$

we have

$$\int_0^1 F_n^2(t) dt = 1, \qquad \lim_{n\to\infty} \int_0^1 F_n^4(t) dt = 3$$

and it is easy to show that $F_n(t)$ cannot approach 0 almost everywhere.[7] Consequently (4.4) cannot converge almost everywhere and hence must diverge almost everywhere. That the series (4.4) must converge or diverge almost everywhere is due to the fact that the gap sequence is 2^k. For general gap sequences this may not be true.

---

[7] In fact, the measure of the set on which $F_n(t)$ converges to 0 cannot exceed 2/3.

It is an interesting open problem to decide whether the series

$$\sum_1^\infty c_k f(n_k t), \qquad \frac{n_{k+1}}{n_k} > q > 1,$$

can converge (diverge) on a set of positive measure without converging (diverging) almost everywhere.

With the exception of the case $n_k = 2^k$, the whole question of divergence of series (4.1) for Hadamard gaps is also open.

In the trigonometric case, divergence of (4.2) implies divergence almost everywhere of

$$\sum_1^\infty c_k \sin 2\pi n_k t$$

even in the case when the n_k's are not integers. This follows from (2.10) which can be regarded as a natural generalization of the result of Zygmund [1] and its extension to the non-harmonic case (Kac [5], Hartman [1]).

As far as convergence is concerned the situation is somewhat more satisfactory. We have, in fact, the following theorem (Kac [4]):

If

$$f(t+1) = f(t), \tag{4.7}$$

$$\int_0^1 f(t)dt = 0, \tag{4.8}$$

and

$$|f(t') - f(t'')| \leqq M|t' - t''|^\alpha, \qquad 0 < \alpha \leqq 1, \tag{4.9}$$

then

$$\sum_1^\infty c_k f(n_k t), \qquad \frac{n_{k+1}}{n_k} > q > 1, \tag{4.10}$$

converges almost everywhere provided

$$\sum_1^\infty c_k^2 < \infty. \tag{4.11}$$

In contrast with previous results we are unable here to relax condition (4.9). However, at the expense of further restrictions on the sequence

$\{n_k\}$, condition (4.9) can be considerably weakened. More specifically, we have the following:

If $f(t)$ satisfies (4.7), (4.8) and if for some α, $0<\alpha\leqq 1$,

$$\int_0^1 (f(t+h)-f(t))^2 dt \leqq Mh^{2\alpha} \tag{4.12}$$

and

$$\sum_1^\infty 2^{-\alpha(m_{k+1}-m_k)} < \infty, \tag{4.13}$$

then (4.11) implies convergence almost everywhere of the series

$$\sum_1^\infty c_k f(2^{m_k}t) \tag{4.14}$$

(m_k integers). (For a weaker result see Izumi and Kawata [1] and for a still weaker one Kac [6]). The interest of this theorem, in spite of its specialized character, lies in the fact that it can be deduced from Kolmogoroff's theorem concerning convergence of series of independent functions.

The proof is based on constructing a sequence of *independent* functions $g_k(t)$ such that

$$\int_0^1 g_k(t)dt = 0, \qquad k = 1, 2, \cdots, \tag{4.15}$$

$$\sum_1^\infty c_k^2 \int_0^1 g_k^2(t)dt < \infty \tag{4.16}$$

and

$$\sum_1^\infty \int_0^1 \left| f(2^{m_k}t) - g_k(t) \right| dt < \infty. \tag{4.17}$$

That the existence of $g_k(t)$ implies our theorem is obvious since by Kolmogoroff's theorem (Kolmogoroff [1; 2]), (4.15) and (4.16) imply that

$$\sum_1^\infty c_k g_k(t)$$

converges almost everywhere and (4.17) implies (since $c_k=O(1)$) that

$$\sum_1^\infty c_k g_k(t) \quad \text{and} \quad \sum_1^\infty c_k f(2^{m_k}t)$$

are equiconvergent almost everywhere.

To construct the $g_k(t)$ divide the interval (0, 1) into 2^r equal subintervals and set

$$f_r(t) = 2^r \int_{k2^{-r}}^{(k+1)2^{-r}} f(x)dx, \tag{4.18}$$
$$k2^{-r} \leqq t < (k+1)2^{-r}; \quad k = 0, 1, \cdots, 2^r - 1.$$

It is evident that

$$\begin{aligned} &\int_0^1 (f(t) - f_r(t))^2 dt \\ &\qquad = \sum_{k=0}^{2^r-1} 2^r \frac{1}{2} \int_{k2^{-r}}^{(k+1)2^{-r}} \int_{k2^{-r}}^{(k+1)2^{-r}} (f(t) - f(s))^2 dtds. \end{aligned} \tag{4.19}$$

By an elementary change of variables we have

$$\begin{aligned} \frac{1}{2} \int_{k2^{-r}}^{(k+1)2^{-r}} \int_{k2^{-r}}^{(k+1)2^{-r}} (f(t) - f(s))^2 dtds &= \int_0^{2^{-r}} \int_{k2^{-r}}^{(k+1)2^{-r}-h} (f(t+h) - f(t))^2 dtdh \\ &\leqq \int_0^{2^{-r}} \int_{k2^{-r}}^{(k+1)2^{-r}} (f(t+h) - f(t))^2 dtdh \end{aligned}$$

and consequently

$$\begin{aligned} \int_0^1 (f(t) - f_r(t))^2 dt &\leqq 2^r \int_0^{2^{-r}} \int_0^1 (f(t+h) - f(t))^2 dtdh \\ &\leqq \frac{M}{2\alpha + 1} 2^{-2r\alpha}. \end{aligned}$$

Finally,

$$\begin{aligned} \int_0^1 \left| f(2^{m_k}t) - f_r(2^{m_k}t) \right| dt &= \int_0^1 \left| f(t) - f_r(t) \right| dt \\ &\leqq \left(\frac{M}{2\alpha + 1}\right)^{1/2} 2^{-r\alpha}. \end{aligned}$$

Setting

$$g_k(t) = f_{m_{k+1}-m_k}(2^{m_k}t) \tag{4.20}$$

it is easy to verify that the $g_k(t)$ are independent and satisfy (4.15), (4.16) and (4.17)

5. Further results concerning convergence of gap series. Let $f(t)$ satisfy the conditions (4.7) and (4.8), and in addition assume that

$$\int_0^1 (f(t) - S_n(t))^2 dt = O(\lg n)^{-\alpha}, \tag{5.1}$$

where $\alpha > 0$ and $S_n(t)$ is the nth partial sum of the Fourier series of $f(t)$. It can then be shown (Kac, Salem and Zygmund [1]) that:

(a) If $\alpha > 2$ the series

$$\sum_1^\infty c_k f(n_k t), \qquad \frac{n_{k+1}}{n_k} > q > 1, \tag{5.2}$$

converges almost everywhere, provided

$$\sum_1^\infty c_k^2 \log^2 k < \infty. \tag{5.3}$$

(b) If $\alpha \leqq 2$, the series

$$\sum_1^\infty \frac{f(n_k t)}{k^{1-\delta}}, \qquad \frac{n_{k+1}}{n_k} > q > 1, \tag{5.4}$$

converges almost everywhere, provided

$$\delta < \alpha/4. \tag{5.5}$$

The n_k's need not be integers.

It follows immediately that under condition (5.1)

$$\lim_{m\to\infty} \frac{1}{m} \sum_1^m f(n_k t) = 0 \tag{5.6}$$

almost everywhere.

Erdös [1] pointed out that (5.6) holds under the weaker condition

$$\int_0^1 (f(t) - S_n'(t))^2 dt = O(\lg \lg n)^{-1-\epsilon}, \qquad \epsilon > 0. \tag{5.7}$$

What is, however, more interesting is that he was able to construct an $f(t) \in L^2$ and a gap sequence $\{n_k\}$ of integers for which

$$\limsup_{m\to\infty} \frac{1}{m} \sum_1^m f(n_k t) = \infty. \tag{5.8}$$

From (5.8) it follows a fortiori that

$$\sum_{1}^{\infty} \frac{f(n_k t)}{k} \tag{5.9}$$

diverges almost everywhere.

In view of this example it is apparent that in convergence theorems of gap series some restrictions on the function $f(t)$ (like (4.9), (4.12), or (5.1)) are necessary.

Let us finally mention that, for $n_k = 2^k$, (5.6) follows from the ergodic theorem (F. Riesz [1], Raikov [1]) for every $f \in L$. We have here another example of the extreme sensitivity of gap theorems to the arithmetic structure of the gap sequence.

It is an open question whether for $n_k = q^k$ and $q > 1$ not an integer, (5.6) holds for all $f \in L$, or at least for all $f \in L^2$.

PART II. DISTRIBUTIONS OF ADDITIVE NUMBER-THEORETIC FUNCTIONS

6. Preliminary remarks. A function $f(n)$, $n = 1, 2, \cdots$, is called additive if

$$f(mn) = f(m) + f(n) \tag{6.1}$$

whenever

$$(m, n) = 1. \tag{6.2}$$

An additive function is thus completely determined if one knows

$$f(p^\alpha), \qquad p \text{ a prime}, \alpha = 1, 2, \cdots. \tag{6.3}$$

The function $f(n)$ is called *strongly additive* if

$$f(p^\alpha) = f(p), \qquad \alpha = 2, 3, \cdots. \tag{6.4}$$

The simplest examples of strongly additive functions are

$$\nu(n) = \text{number of prime divisors of } n \tag{6.5}$$

(each prime divisor counted once) and

$$\log \phi(n)/n, \tag{6.6}$$

where $\phi(n)$ is Euler's function.

In the first example $f(p) = 1$ and in the second $f(p) = \log(1 - 1/p)$. Let, for p a prime,

$$\rho_p(n) = \begin{cases} 1, & p \mid n, \\ 0, & p \nmid n. \end{cases} \tag{6.7}$$

With this notation the strongly additive function $f(n)$ can be written in the form

$$(6.8)\qquad f(n) = \sum_p f(p)\rho_p(n).$$

If $p_1, p_2, \cdots, p_r$ are distinct primes it follows trivially that

$$(6.9)\qquad \mathcal{D}\{\rho_{p_1}(n) = \epsilon_1, \cdots, \rho_{p_r}(n) = \epsilon_r\} = \prod_{k=1}^{r} \mathcal{D}\{\rho_{p_k}(n) = \epsilon_k\},$$

where $\mathcal{D}\{\ \}$ denotes the density of integers which satisfy the condition inside the braces and each ϵ can be either 0 or 1. Formula (6.9) expresses the fact that the functions $\rho_p(n)$ are *statistically independent* and one might expect that the theory of addition of independent random variables can be applied to distribution problems of additive arithmetic functions.

Since the density is only a finitely additive measure, probability theorems will be directly applicable only if a finite number of ρ_p's is involved. Thus one can apply probability theorems to "truncated" functions

$$f_k(n) = \sum_{p<k} f(p)\rho_p(n).$$

The passage from theorems concerning $f_k(n)$ to theorems concerning $f(n)$ must by necessity employ number theoretic arguments and it is these arguments that usually constitute the deeper parts of the proof. However, without the probabilistic connections it would have been extremely difficult even to guess some of the theorems.

7. The Hardy-Ramanujan theorem and its generalizations. In 1917 Hardy and Ramanujan [1] proved that almost every integer m has approximately lg lg m prime divisors.

The precise formulation is as follows: If $\psi(m)\to\infty$ (arbitrarily slowly) the density of integers for which

$$(7.1)\qquad \lg\lg m - \psi(m)(\lg\lg m)^{1/2} < \nu(m) < \lg\lg m + \psi(m)(\lg\lg m)^{1/2}$$

is 1.

Due to the slowness with which lg lg m increases we have the equivalent formulation: If q_n is the number of integers m, $1\leqq m\leqq n$, for which

$$(7.2)\qquad \lg\lg n - \psi(n)(\lg\lg n)^{1/2} < \nu(m) < \lg\lg n + \psi(n)(\lg\lg n)^{1/2},$$

then

$$(7.3)\qquad q_n/n \to 1.$$

The original proof of this theorem was quite lengthy and was based on estimates of the number $\pi_k(n)$ of integers not greater than n which have exactly k prime divisors.

In 1936 Turán [1] gave an extremely simple and ingenious proof of (7.3) based on the easily verifiable relationship

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} (\nu(m) - \lg \lg n)^2}{n \lg \lg n} = 1. \tag{7.4}$$

To get (7.3) one need only notice that

$$\frac{\sum_{m=1}^{n} (\nu(m) - \lg \lg n)^2}{n \lg \lg n} \geqq \frac{(n - q_n)\psi^2(n)}{n}.$$

One recognizes here immediately the familiar device of estimating probabilities of deviations from the second moment (Tchebysheff's inequality). A slight restatement of (7.4) is highly suggestive.

Let $k_n(\omega)$ be the number of integers m, $1 \leqq m \leqq n$, for which

$$\frac{\nu(m) - \lg \lg n}{(\lg \lg n)^{1/2}} < \omega. \tag{7.5}$$

Set

$$\sigma_n(\omega) = k_n(\omega)/n, \tag{7.6}$$

note that $\sigma_n(\omega)$ is a distribution function and

$$\frac{\sum_{1}^{n} (\nu(m) - \lg \lg n)^2}{n \lg \lg n} = \int_{-\infty}^{\infty} \omega^2 d\sigma_n(\omega). \tag{7.7}$$

Formula (7.4) can now be written as

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \omega^2 d\sigma_n(\omega) = 1 \tag{7.8}$$

and it also follows simply that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \omega d\sigma_n(\omega) = 0. \tag{7.9}$$

Since

$$\nu(m) = \sum_{p} \rho_p(m) \tag{7.10}$$

is a sum of independent functions one might guess that, in some sense, the distribution of $\nu(m)$ will be normal. Formulas (7.8) and (7.9) suggest then that we might have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\omega^k d\sigma_n(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\omega^k e^{-\omega^2/2}d\omega \tag{7.11}$$

for $k=3, 4, \cdots$, in which case it would follow that

$$\lim_{n\to\infty}\sigma_n(\omega) = \lim_{n\to\infty}\frac{k_n(\omega)}{n} = (2\pi)^{-1/2}\int_{-\infty}^{\omega}e^{-u^2/2}du. \tag{7.12}$$

Unfortunately it does not seem easy to carry out the proof along these lines because of the difficulties involved in calculating the higher moments

$$\frac{\sum_{m=1}^{n}(\nu(m) - \lg\lg n)^k}{n(\lg\lg n)^{k/2}}. \tag{7.13}$$

We mention this approach because it is the most straightforward and, in our opinion, well worth pursuing.

A special case of (7.12), namely,

$$\lim_{n\to\infty}\sigma_n(0) = 1/2, \tag{7.14}$$

was proved by Erdös [2] in 1937. This proof used a simplified version (due to Landau) of Brun's method. By a slight modification of Erdös' argument one can obtain (7.12) for every ω (LeVeque [1]). In fact, the only reason why Erdös did not prove (7.12) in its full generality seems to be that he was unaware of the formula

$$\lim_{x\to\infty}e^{-x}\sum_{k<x+\omega x^{1/2}}\frac{x^k}{k!} = (2\pi)^{-1/2}\int_{-\infty}^{\omega}e^{-u^2/2}du, \tag{7.15}$$

except for the special case $\omega=0$.

It might be of interest to give a heuristic argument which indicates how (7.15) is related to the problem of the distribution of $\nu(m)$.

Denoting by $\pi_k(n)$ the number of integers not greater than n having exactly k prime divisors we have

$$k_n(\omega) = \sum_{k<\lg\lg n+\omega(\lg\lg n)^{1/2}}\pi_k(n). \tag{7.16}$$

The classical result of Landau to the effect that

$$\pi_k(n) \sim \frac{n}{\lg n} \frac{(\lg\lg n)^{k-1}}{(k-1)!} \tag{7.17}$$

makes it plausible that

$$\frac{k_n(\omega)}{n} \sim \frac{1}{\lg n} \sum_{k<\lg\lg n+\omega(\lg\lg n)^{1/2}} \frac{(\lg\lg n)^{k-1}}{(k-1)!}. \tag{7.18}$$

Now (7.12) follows from (7.15) by putting $x=\lg\lg n$.

The method of Erdös [2] was applicable only to $\nu(m)$ but in 1939 Erdös and Kac [1'] proved a general theorem which also established a pattern of proof of rather wide applicability. The theorem in question is as follows:

Let

$$f(m) = \sum_p f(p)\rho_p(m) \tag{7.19}$$

be a strongly additive function and set

$$A_n = \sum_{p<n} \frac{f(p)}{p}, \tag{7.20}$$

$$B_n = \sum_{p<n} \frac{f^2(p)}{p}. \tag{7.21}$$

Assume that

$$B_n \to \infty, \qquad f(p) = O(1), \tag{7.22}$$

and denote by $k_n(\omega)$ the number of integers m, $1 \leqq m \leqq n$, for which

$$f(m) < A_n + \omega B_n^{1/2}; \tag{7.23}$$

then

$$\lim_{n\to\infty} \frac{k_n(\omega)}{n} = (2\pi)^{-1/2} \int_{-\infty}^{\omega} e^{-u^2/2} du. \tag{7.24}$$

Considering first the truncated function

$$f_k(m) = \sum_{p<k} f(p)\rho_p(m)$$

one shows by a direct application of the central limit theorem that the density of integers m for which

$$f_k(m) < A_k + \omega B_k^{1/2} \tag{7.25}$$

approaches, as $k\to\infty$, the normal integral

$$(2\pi)^{-1/2}\int_{-\infty}^{\omega} e^{-u^2/2}du.$$

This can be restated as follows:

If $l(n, k; \omega)$ is the number of integers m, $1\leqq m\leqq n$, for which (7.25) holds, then

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{l(n, k; \omega)}{n} = (2\pi)^{-1/2}\int_{-\infty}^{\omega} e^{-u^2/2}du. \tag{7.26}$$

The crucial point is now to convert (7.26) into

$$\lim_{n\to\infty}\frac{l(n, \alpha_n, \omega)}{n} = (2\pi)^{-1/2}\int_{-\infty}^{\omega} e^{-u^2/2}du, \tag{7.27}$$

where $\alpha_n\to\infty$.

This can be done if

$$\alpha_n = n^{\psi(n)}, \qquad \psi(n)\to 0,\ \alpha_n\to\infty, \tag{7.28}$$

but in the proof one needs a precise estimate of the number of integers not greater than N not divisible by primes less than K_N (in the limit as $N\to\infty$ faster than each fixed power of K_N). Such an estimate can be obtained by Brun's method.[8] In contrast to Erdös [2] one needs here the full strength of this method. Finally, choosing $\psi(n)$ in such a way that

$$1/\psi(n) = o(B_n) \tag{7.29}$$

and using the assumption $f(p)=O(1)$, one shows that the errors introduced by replacing $f(m)$, A_n, B_n by $f_{\alpha_n}(m)$, A_{α_n}, B_{α_n} are negligible and thus derives (7.24) from (7.27).

Let us also mention that from (7.12) one obtains by a very simple argument the following result:

If $d(m)$ denotes the number of divisors of m and $r_n(\omega)$ the number of integers m, $1\leqq m\leqq n$, for which

$$d(m) < 2^{\lg\lg n+\omega(\lg\lg n)^{1/2}}, \tag{7.30}$$

then (Kac [7])

---

[8] A book devoted to Brun's method and its applications is being prepared by W. J. Harrington and J. B. Rosser. We also take this opportunity to correct a few minor omissions and misprints in Erdős and Kac [1]: (a) in the statement of Lemma 3 it should be added that the 0-estimate is uniform in i; (b) in formula (iii) on p. 741 $+$ should be replaced by $-$; (c) in the last formula of §4 (p. 742) $O(1)$ should be replaced by $o(1)$.

$$\lim_{n\to\infty} \frac{r_n(\omega)}{n} = (2\pi)^{-1/2} \int_{-\infty}^{\omega} e^{-u^2/2} du.$$

8. **Further extensions.** An important extension of (7.24) was stated without proof by Erdös [3]. His theorem is as follows:

Denoting by $k_n(\omega_1, \omega_2)$ the number of integers m, $1 \leqq m \leqq n$, for which simultaneously

$$f(m) < A_n + \omega_1 B_n^{1/2}, \qquad f(m+1) < A_n + \omega_2 B_n^{1/2} \tag{8.1}$$

we have under the assumptions (7.22)

$$\lim_{n\to\infty} \frac{k_n(\omega_1, \omega_2)}{n} = \frac{1}{2\pi} \int_{-\infty}^{\omega_1} \int_{-\infty}^{\omega_2} e^{-(u^2+v^2)/2} du dv. \tag{8.2}$$

The intrinsic interest of this result (which can be extended to $f(m)$, $f(m+1), \cdots, f(m+r)$, for each fixed r), and the fact that the method of proof is little known to number theorists made it seem worth while to supply a detailed proof. This was done by LeVeque [1] who also pointed out the following corollaries:

If $t_n(\omega)$ and $s_n(\omega)$ denote, respectively, the number of integers m, $1 \leqq m \leqq n$, for which

$$\nu(m) < \nu(m+1) + \omega(2 \lg \lg n)^{1/2}, \tag{8.3}$$

$$d(m) < d(m+1) 2^{\omega(2 \lg \lg n)^{1/2}}, \tag{8.4}$$

then

$$\lim_{n\to\infty} \frac{t_n(\omega)}{n} = \lim_{n\to\infty} \frac{s_n(\omega)}{n} = (2\pi)^{-1/2} \int_{-\infty}^{\omega} e^{-u^2/2} du. \tag{8.5}$$

In the particular case $\omega = 0$, (8.5) was proved by Erdös [4] in 1936.

The proof of (8.2) utilizes the fact that $f_k(m)$ and $f_k(m+1)$ are statistically independent (for every fixed k) and again makes use of Brun's method. The application of Brun's method is in this case considerably more tedious inasmuch as one needs the so-called two residue case.

Several other applications of the central limit theorem can be found in Erdös [3].

9. **Kolmogoroff's "three series" theorem in the theory of additive functions.** An important and interesting problem which received considerable attention in the literature was to decide under what conditions a strongly additive function

P17567

$$f(m) = \sum_p f(p)\rho_p(m) \tag{9.1}$$

possesses a distribution function. We say that the distribution function $\sigma(\omega)$ is the distribution function of $f(m)$, if the density of integers for which $f(m) < \omega$ exists and is equal to $\sigma(\omega)$ at each continuity point of σ. It may also be noted that the case of general additive functions can be reduced to the case of strongly additive ones (Erdös [5]).

Generalizing a whole series of results (Schoenberg [1; 2], Davenport [1], Erdös [5; 6]) Erdös [7] proved in 1938 that the convergence of the series

$$\sum_p \frac{f^*(p)}{p} \quad \text{and} \quad \sum_p \frac{f^{*2}(p)}{p}, \tag{9.2}$$

where

$$f^*(p) = \begin{cases} f(p), & |f(p)| \leqq 1, \\ 1, & |f(p) > 1, \end{cases} \tag{9.3}$$

is a sufficient condition for the existence of the distribution function of $f(m)$. The result is best understood by bringing out its probabilistic nature (Erdös and Wintner [1]). If we consider the truncated function

$$f_k(m) = \sum_{p<k} f(p)\rho_p(m), \tag{9.4}$$

we see that it possesses a distribution function $\sigma_k(\omega)$ identical with the distribution function of

$$S_k = \sum_{p<k} f(p)X_p, \tag{9.5}$$

where the X_p's are independent random variables such that

$$\text{Prob. } \{X_p = 0\} = 1 - 1/p, \qquad \text{Prob. } \{X_p = 1\} = 1/p. \tag{9.6}$$

The Fourier-Stieltjes transform of $\sigma_k(\omega)$ is thus

$$L_k(\xi) = \int_{-\infty}^{\infty} e^{i\xi\omega} d\sigma_k(\omega) = \prod_{p<k}\left(1 - \frac{1}{p} + \frac{1}{p} e^{i\xi f(p)}\right). \tag{9.7}$$

By Kolmogoroff's three series theorem (Kolmogoroff [1; 2]) convergence of the series (9.2) implies convergence with probability 1 (and hence convergence in probability) of the sequence (9.5) and consequently the existence of a distribution function $\sigma(\omega)$ such that

$$L(\xi) = \int_{-\infty}^{\infty} e^{i\xi\omega} d\sigma(\omega) = \prod_p \left(1 - \frac{1}{p} + \frac{1}{p} e^{i\xi f(p)}\right). \tag{9.8}$$

The distribution function $\sigma(\omega)$ is then the distribution function of the infinite series

$$\sum_p f(p) X_p. \tag{9.9}$$

In the case of number theoretic functions, convergence, as $k \to \infty$, of $f_k(m)$ to $f(m)$ for every m is trivial, but because density is only a finitely additive measure, one cannot draw the conclusion that $f(m)$ has a distribution function. Even if we knew that the distribution function of $f(m)$ exists it still would not follow that it must be the limit of $\sigma_k(\omega)$. It is, however, not difficult to show (Erdös [7, Lemma 1]) that if the series (9.2) converge, $f_k(m)$ approaches $f(m)$ *in measure*, that is, the *upper* density of integers for which $|f(m) - f_k(m)| \geqq \epsilon$ approaches 0, as $k \to \infty$, for every $\epsilon > 0$. This, together with the fact that $\sigma_k(\omega) \to \sigma(\omega)$, implies that $\sigma(\omega)$ is indeed the distribution function of $f(m)$. Since, by Kolmogoroff's theorem, convergence of the series (9.2) is also a necessary condition for convergence of $\sigma_k(\omega)$ to $\sigma(\omega)$, it was natural to conjecture that it is also a necessary condition for the existence of the distribution function of $f(m)$. This indeed is the case (Erdös and Wintner [1]) although the proof is somewhat less elementary (it depends, for instance, on the result of Erdös and Kac [1]).

Finally, let us mention that from a general result of P. Lévy [1] it follows that $\sigma(\omega)$ is continuous if and only if the series

$$\sum_{f(p) \neq 0} \frac{f(p)}{p} \tag{9.10}$$

diverges. This was also proved by Erdös [7] by elementary number theoretic considerations.

REFERENCES

S. BANACH

1. *Ueber einige Eigenschaften der lakunären trigonometrischen Reihen*, Studia Mathematica vol. 2 (1930) pp. 207–220.

H. DAVENPORT

1. *Ueber Numeri Abundantes*, Preuss. Akad. Wiss. Sitzungsber. (1930) pp. 830–837.

P. ERDÖS

1. *On the strong law of large numbers*, Bull. Amer. Math. Soc. Abstract 54-5-191. The paper will appear in Trans. Amer. Math. Soc.
2. *Note on the number of prime divisors of integers*, J. London Math. Soc. vol. 12 (1937) pp. 308–314.

3. *On the distribution function of additive functions*, Ann. of Math. vol. 47 (1946) pp. 1–20.
4. *On a problem of Chowla and some related problems*, Proc. Cambridge Philos. Soc. vol. 32 (1936) pp. 530–540.
5. *On the density of some sequences of numbers* I, J. London Math. Soc. vol. 10 (1935) pp. 120–125.
6. *On the density of some sequences of numbers* II, ibid. vol. 12 (1937) pp. 7–11.
7. *On the density of some sequences of numbers* III, ibid. vol. 13 (1938) pp. 119–127.

P. Erdős and M. Kac

1. *The Gaussian law of errors in the theory of additive number theoretic functions*, Amer. J. Math. vol. 62 (1940) pp. 738–742.

P. Erdős and A. Wintner

1. *Additive arithmetical functions and statistical independence*, Amer. J. Math. vol. 61 (1939) pp. 713–721.

W. Feller

1. *The fundamental limit theorems in probability*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 800–832.

J. Ferrand and R. Fortet

1. *Sur des suites arithmétiques equiréparties*, C. R. Acad. Sci. Paris vol. 224 (1947) pp. 514–515.

R. Fortet

1. *Sur une suite également repartie*, Studia Mathematica 9 (1940) pp. 54–69.

G. H. Hardy and S. Ramanujan

1. *The normal number of prime factors of a number* n, Quart. J. Math. Oxford Ser. vol. 48 (1917) pp. 76–92.

P. Hartman

1. *The divergence of non-harmonic gap series*, Duke Math. J. vol. 9 (1942) pp. 404–405.

S. Izumi and T. Kawata

1. *On certain series of functions*, Tôhoku Math. J. vol. 46 (1939) pp. 91–105.

M. Kac

1. *Note on power series with big gaps*, Amer. J. Math. vol. 61 (1939) pp. 473–476.
2. *Distribution properties of certain gap sequences*, Bull. Amer. Math. Soc. Abstract 53-7-290. Details will be given in the paper by Erdős, Ferrand, Fortet and Kac (see footnote 2). The following misprint should be corrected: $\omega/|\cos \pi k|$ should be replaced by $\omega/2|\cos \pi x|$.
3. *On the distribution of values of sums of the type* $\sum f(2^k t)$, Ann. of Math. vol. 47 (1946) pp. 33–49.
4. *Convergence of certain gap series*, ibid. 44 (1943) pp. 411–415.
5. *Convergence and divergence of non harmonic gap series*, Duke Math. J. vol. 8 (1941) pp. 541–545.
6. *Sur les fonctions indépendantes* V, Studia Mathematica vol. 7 (1938) pp. 96–100.
7. *Note on the distribution of values of the arithmetic function* $d(m)$, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 815–817.

M. Kac, R. Salem and A. Zygmund

1. *A gap theorem*, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 235–243.

S. Kaczmarz and H. Steinhaus

1. *Le système orthogonal de M. Rademacher*, Studia Mathematica vol. 2 (1930) pp. 231–247.

A. Khintchine and A. Kolmogoroff

1. *Ueber Konvergenz von Reihen deren Glieder durch den Zufall bestimmt werden*, Rec. Math. (Mat. Sbornik) vol. 32 (1925) pp. 668–677.

A. Kolmogoroff

1. *Ueber die Summen durch den Zufall bestimmter unabhängiger Grössen*, Math. Ann. vol. 99 (1928) pp. 309–319.
2. *Bemerkungen zu meiner Arbeit "Ueber die summen Zufälliger Grössen,"* ibid. vol. 102 (1929) pp. 484–489.
3. *Une contribution à l'étude de la convergence des séries de Fourier*, Fund. Math. vol. 5 (1924) pp. 96–97.

P. Lévy

1. *Sur les séries dont les termes sont des variables éventuelles indépendantes*, Studia Mathematica vol. 3 (1931) pp. 119–155.

W. J. LeVeque

1. *On the distribution of values of number-theoretic functions*, Thesis, Cornell University.

R. E. A. C. Paley and A. Zygmund

1. *On some series of functions* (1), Proc. Cambridge Philos. Soc. vol. 26 (1930) pp. 337–357.
2. *On some series of functions* (2), ibid. pp. 458–474.
3. *On some series of functions* (3), ibid. vol. 28 (1932) pp. 190–205.

H. Rademacher

1. *Einige Sätze ueber Reihen von allgemeinenen Orthogonalfunktionen*, Math. Ann. vol. 87 (1922) pp. 112–138.

D. Raikov

1. *On some arithmetical properties of summable functions*, Rec. Math. (Mat. Sbornik) N.S. vol. 1 (1936) pp. 377–383.

F. Riesz

1. *Sur la théorie ergodique*, Comment. Math. Helv. vol. 17 (1945) pp. 221–239.

R. Salem and A. Zygmund

1. *On lacunary trigonometric series*, Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) pp. 333–338.
2. *On lacunary trigonometric series*, ibid. vol. 34 (1948) pp. 54–62.

I. J. Schoenberg

1. *Ueber die asymptotische Verteilung reeller Zahlen* mod 1, Math. Zeit. vol. 28 (1928) pp. 171–200.
2. *On asymptotic distribution of arithmetical functions*, Trans. Amer. Math. Soc. vol. 39 (1936) pp. 315–330.

H. Steinhaus

1. *Les probabilités dénombrables et leur rapport à la théorie de la mesure*, Fund. Math. vol. 4 (1922) pp. 286–310.

P. Turán

1. *Ueber einige Verallgemeinerungen eines Satzes von Hardy und Ramanujan*, J. London Math. Soc. vol. 11 (1936) pp. 125–133.

A. Zygmund

1. *On the convergence of lacunary trigonometric series*, Fund. Math. vol. 16 (1930) pp. 90–107.

Cornell University

THE APRIL MEETING IN DURHAM

The four hundred forty-fifth meeting of the American Mathematical Society was held at Duke University, Durham, North Carolina, on Friday and Saturday, April 1–2, 1949. The total attendance was about 130 including the following 86 members of the Society:

Louise Adams, D. F. Barrow, W. S. Beckwith, Felix Bernstein, R. G. Blake, T. A. Botts, C. P. Brady, A. T. Brauer, E. T. Browne, G. S. Bruton, R. S. Burington, E. A. Cameron, Leonard Carlitz, M. D. Clement, A. C. Cohen, J. B. Coleman, J. C. Currie, D. C. Dearborn, F. G. Dressel, L. A. Dye, Myrtle Edwards, W. W. Elliott, F. A. Ficken, Tomlinson Fort, A. S. Galbraith, J. R. Garrett, I. C. Gentry, J. J. Gergen, Wallace Givens, P. E. Guenther, O. G. Harrold, I. R. Hershner, Temple Hollcroft, T. R. Hollcroft, Harold Hotelling, G. B. Huff, M. P. Jarnagin, V. L. Klee, J. R. Kline, F. W. Kokomoor, S. H. Lachenbruch, G. B. Lang, J. W. Lasley, C. G. Latimer, M. A. Lee, Marguerite Lehr, Jack Levine, R. J. Levit, P. E. Lewis, E. J. McShane, E. L. Mackie, M. H. Martin, E. P. Miles, B. E. Mitchell, E. E. Moots, C. G. Mumford, J. D. Novak, T. E. Oberbeck, H. V. Park, W. V. Parker, Mary Pettus, W. T. Puckett, C. N. Reynolds, T. D. Reynolds, R. G. D. Richardson, H. E. Robbins, J. H. Roberts, L. V. Robinson, V. N. Robinson, Clarence Ross, Walter Rudin, H. M. Schwartz, R. E. Smith, W. S. Snyder, H. K. Sohl, R. W. Stokes, B. B. Townsend, H. S. Thurston, L. A. M. Van Dyke, R. Z. Vause, L. I. Wade, J. A. Ward, G. T. Whyburn, W. M. Whyburn, R. L. Wilson, J. W. Young.

Professor J. H. Roberts of Duke University delivered an hour address at 8:30 P.M. on Friday entitled *Transformations which change dimension*. Professor G. T. Whyburn presided at this session.

Following the invited address a brief informal business meeting was held, Professor W. M. Whyburn presiding. It was agreed that the group would recommend to the Council the acceptance of the invitation of the Oak Ridge National Laboratory to hold the 1950 southeastern meeting in Oak Ridge, Tennessee.

Two sections for contributed papers were held at 9:30 A.M. on Saturday: one in Analysis, Statistics and Applied Mathematics, in which Professor Tomlinson Fort presided, and one in Algebra, Geometry and Topology, in which Professor C. G. Latimer presided.

A dinner for mathematicians and guests was held at 7:00 P.M. on Friday, Professor J. J. Gergen, toastmaster. President A. H. Edens, of Duke University, gave an address of welcome and Professor J. R. Kline spoke regarding the current activities of the Society and the International Congress. A resolution of thanks to Duke University for its hospitality was presented by Professor L. A. Dye and adopted by those present at the dinner.

Abstracts of the papers presented follow below. Abstracts whose numbers are followed by the letter "*t*" were presented by title. Paper number 312 was read by Professor Townsend, number 321 by Professor Robbins. Mr. Gadsden was introduced by Professor T. R. Hollcroft, Mr. Peters by Professor Tomlinson Fort, Mr. Blumen and Dr. Nicholson by Professor H. E. Robbins, and Mr. Keesee by Professor A. D. Wallace.

ALGEBRA AND THEORY OF NUMBERS

282. A. T. Brauer: *Limits for the roots of algebraic equations.*

The results of the papers *Limits for the characteristic roots of a matrix* I, II, III, Duke Math. J. vol. 13 (1946) pp. 387–395, vol. 14 (1947) pp. 21–26, vol. 15 (1948) pp. 871–877, are used to prove well known theorems on the location of the roots of algebraic equations in a unified way while the original proofs are very different. Moreover, most of the results obtained in this paper give better bounds for the roots than the original theorems. For instance, it is proved that all the zeros of $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$ lie in the closed unit circle or in the oval of Cassini $|z(z+a_1) \leq |a_2|+|a_3|+\cdots+|a_n|$. (Received February 17, 1949.)

283. Leonard Carlitz: *Hurwitz series: Eisenstein criterion.*

By an H-series is meant one of the form $\sum_0^\infty c_m x^m/m!$, where the c_m are rational integers. (For some previous results see Duke Math. J. vol. 8 (1941) pp. 689–700.) The H-series evidently form a domain of integrity; factorization is not in general unique. The principal result of the present paper is an analogue of the Eisenstein criterion for series with rational c_m, that is, a necessary condition that such a series satisfy an equation $\sum_0^r A_i w^i=0$, where the A_i denote H-series. (Received February 17, 1949.)

284*t*. Leonard Carlitz: *Some properties of Hurwitz series.*

Let $f(x)=\sum_1^\infty c_m x^m/m!$, $c_1=1$, c_m integral, and assume that the inverse function is of the form $\lambda(x)=\sum_1^\infty e_m x^m/m$, e_m integral (see Duke Math. J. vol. 8 (1941) pp. 689–700). Among the results of this paper are: (1) $c_{m+p-1}\equiv c_m \pmod{p}$; a like result holds for powers of $f(x)$; (2) $x/f(x)$ has integral coefficients if and only if $x/\lambda(x)$ has integral coefficients. A number of results on the coefficients of the reciprocal of an arbitrary Hurwitz series are also derived. (Received February 23, 1949.)

285*t*. Graham Higman, B. H. Neumann, and Hanna Neumann: *Embedding theorems for groups.* Preliminary report.

The following theorem is proved: Two subgroups A and B of a group G are conjugate in a suitable group H containing G as a subgroup if and (trivially) only if they are isomorphic; and then every isomorphic mapping of A onto B can be obtained by a transformation in a suitable H. Using this theorem one can construct infinite groups in which all elements except the unit element are conjugate to each other. Another consequence is the theorem that every countable group G is a subgroup of a group H with two generators. If G can be defined by n defining relations, then one can define the two-generator group H also by n relations. (Received February 17, 1949.)

286. R. J. Levit: *A criterion for the irreducibility of polynomials in the rational field.*

Several tests for the irreducibility of a polynomial in the field of rationals are known which depend upon the values assumed by the polynomial on a finite set of integers. G. Pólya proved (Jber. Deutschen Math. Verein. vol. 28 (1919) pp. 31–40) the following: Let $P(x)$ be an integral polynomial of degree n. If a set of n integers $x_1, x_2, \cdots, x_n$ can be found such that $0<|P(x_i)|<M$, $i=1, 2, \cdots, n$, where M is a certain number depending only on n, then $P(x)$ is irreducible in the field of rationals. Later writers have sharpened this test by obtaining higher values for the bound M, the highest previously known being due to A. T. Brauer and G. Ehrlich (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 844–856). In the present paper a still larger bound is obtained, namely, $M=a(a+1)(a+2)\cdots(a+b)/2^b$, where $a=2^{-1}[n/2]$ and $b=[(n-1)/2]$. The proof depends upon the properties of a family of polynomials which are a generalization of the Tchebycheff polynomials. (Received February 17, 1949.)

287. W. V. Parker: *On the matrix equation $AX=XB$.*

A method is given for writing the matrix X such that $AX=XB$ in terms of paramêters when A is in rational canonical form. Let S be the set of all matrices commutative with A and let k be the number of invariant factors of $\lambda I-A$. Associated with every matrix X of S is a matrix $M(\lambda)$ whose order is k and whose elements are polynomials in λ. If $F(\mu, \lambda)=|\mu I_k-M(\lambda)|$ then $F(X, A)=0$. This reduces to the Hamilton-Cayley theorem for $k=n$ and to $X=M(A)$ for $k=1$. (Received February 11, 1949.)

288t. K. G. Ramanathan: *On certain identical relations between modular functions.*

If $f_\nu(\tau)=T[(\eta(p^2\tau)/\eta(\tau))^\nu]$ where $\eta(\tau)$ is Dedekind's modular form and T is Hecke's linear operator (Math. Ann. vol. 117, p. 1) it is shown that between any $p+1$ consecutive $f_\nu(\tau)$ there is a linear relation, with coefficients which are independent of ν. In the cases $p=5, 7, 13$ this linear relation with $\nu=1, \cdots, p+1$ is exactly the irrational modular equation connecting $\eta(p^2\tau)/\eta(p\tau)$ and $\eta(p\tau)/\eta(\tau)$. When $p=11$ it is shown that the linear relation could be used in an attack on Ramanujan's problem of the congruences of the partition function. (Received February 16, 1949.)

289t. Walter Rudin: *A theorem on Hurwitz series.*

By an H-series of type k ($k\geqq 0$) is meant a formal power series $f(x)=\sum a_m x^m/\{m,k\}$ where the coefficients a_m are rational integers, and $\{m, k\}=m!(m+1)!\cdots(m+k)!/1!\cdots k!$, $\{m, 0\}=m!$. For a fixed k, the H-series of type k form a domain of integrity, denoted by H_k. The following theorem is proved. Let P be an infinite set of distinct rational primes. Let $f(x)\in H_{k+1}$, and suppose that $a_p\not\equiv 0 \pmod{p}$ for $p\in P$. Then $f(x)$ is transcendental relative to H_k. (Received March 24, 1949.)

290. L. I. Wade: *Multiply periodic functions over $GF[p, x]$.*

Using the valuation of $GF(p, x)$ determined by the degree in x, let $\mathfrak{M}$ denote the algebraically closed, complete field containing this topological field. The interest is in certain entire functions, that is, power series in t over $\mathfrak{M}$ that converge for all elements of M. Such a function $f(t)$ is called periodic over $GF[p, x]$ with period ω if $f(t+A\omega)=f(t)$ for $A\in GF[p, x]$. A necessary and sufficient condition that an entire function exist with the periods $\omega_1, \cdots, \omega_n$, linearly independent over $GF(p, x)$, is that $\omega_1, \cdots, \omega_n$ be linearly independent over the completion of $GF(p, x)$. The function $f(t)$ with the zeros $A_1\omega_1+\cdots+A_n\omega_n$, $A_i\in GF[p, x]$, and no other zeros, is a linear function (of the form $\sum \alpha_i t^{p^i}$); and, furthermore, has the multiplication theorem $f(Mt) = \sum_0^n \gamma_i f^{p^i}(t)$, $\gamma_i \in \mathfrak{M}$. Conversely, this relation determines an n-ply periodic function of this type. Similar theorems are proved about entire functions with infinitely many periods (in particular, their existence). Also, multiple periodicity over $GF[p; x_1, \cdots, x_n]$ is treated. (Received February 18, 1949.)

291. J. A. Ward: *Solution of generalized Cauchy-Riemann differential equations by linear algebras.*

The author proves the following theorem: If $A_i = (a_{ijr})$ $(i = 1, 2, \cdots, n)$ is a set of linearly independent n by n matrices over a field F such that $a_{i1r} = \delta_{ir}$ and $A_i A_j = A_j A_i$ for all i and j, then $A_i A_j = \sum_{k=1}^n a_{ijk} A_k$. This theorem is used to establish necessary and sufficient conditions that the system $\sum_{i,j=1}^n d_{ijk} \partial y_i / \partial x_j = 0$, $k = 1, 2, \cdots, (n^2 - n)$, where the d_{ijk} are constants in F, shall be the generalized Cauchy Riemann differential equations of a linear, associative, commutative algebra A of order n with an identity element over F. (See J. A. Ward, *A theory of analytic functions in linear associative algebras*, Duke Math. J. vol. 7 (1940) pp. 233–248.) If the conditions are satisfied the author is able to determine the algebra A and give the solutions of the differential equations as a power series in the algebra. (Received February 14, 1949.)

ANALYSIS

292. Felix Bernstein: *A new integral relation of the theta zero function from which its natural boundary can be obtained.*

The characteristic integral equation of the Volterra type of the theta zero function derived by the author in 1926 in order to solve a problem in heat conduction enables one to establish all the known relations of the theta zero function to problems of the conduction of heat. An integral relation which is a generalization of this integral equation leads to a simple proof of the existence of a natural boundary for the theta zero function. This is a new approach to the gap theorems and to the study of the behavior of an analytic function in the neighborhood of its natural boundary. (Received April 2, 1949.)

293. C. P. Gadsden: *On the zeros of the solutions of systems of homogeneous linear differential equations.*

The system considered is (1) $x' = A(t)x$, where $x = [x_i(t)]$ is a column matrix and $A(t) = [a_{ij}(t)]$ is an $n \times n$ matrix of class C^m on R: $-\infty < t < \infty$, for some $m = 0, 1, \cdots, \infty$. For a solution $x \neq 0$ of (1) define the set $X_i = x_i^{-1}[0]$ of zeros of $x_i(t)$. Then (i) X_i is closed and (ii) $\bigcap X_i = 0$. The conditions (i) and (ii) are also sufficient, in the following sense. Let $[X_i]$ be n subsets of R satisfying (i) and (ii). Then there exist matrices $A(t)$ and x of class C^∞ satisfying (1) and such that $x_i^{-1}[0] = X_i$. The proof depends on the fact that, for any closed set $Z \subset R$, a function $f(t)$ of class C^∞ on R can be defined so that $f^{-1}[0] = Z$. Similar results are given for $A(t)$ analytic. For the case $n = 2$, some separation theorems are given, assuming that one of the conditions $\pm A_{ij}(t) \geqq 0$, $i \neq j$, holds. (Received February 16, 1949.)

294*t*. V. L. Klee: *A note on $\wp$-functionals.*

Let R denote the real number field and $R^+ = R \cup \{\infty\}$. A $\wp-((\wp^+-))$ functional on a real linear system L is a positively homogeneous subadditive function on L to

$R((R^+))$. The following results are established: (1) If p is a $\wp^+$-functional and $-q$ is a $\wp$-functional such that $q \leqq p$, then there is a linear functional f on L such that $q \leqq f \leqq p$; (2) If p_1 and p_2 are $\wp$-functionals and f is a linear functional such that $f \leqq \sup (p_1, p_2)$, then there are linear functionals $f_i \leqq p_i$ and $a \in [0, 1]$ such that $f = af_1 + (1-a)f_2$. By means of these a new proof is given for some results of Krein and Smulian (Ann. of Math. vol. 41 (1940) pp. 561–564, 567) on regularly convex sets. (Received February 23, 1949.)

295*t*. V. L. Klee: *A second note on $\wp$-functionals.*

(For the definition of $\wp$- and $\wp^+$-functionals see the preceding abstract.) A new proof of the Hahn-Banach extension theorem (concerning $\wp$-functionals on an arbitrary linear system) is given. On finite-dimensional systems there is obtained a simple characterization of those $\wp^+$-functionals for which the Hahn-Banach theorem remains valid, and it is shown that each $\wp^+$-functional majorizes at least one linear functional. (Received February 23, 1949.)

296*t*. V. L. Klee: *On dense convex sets.*

Let E be an infinite-dimensional normed linear space. It is shown that for subspaces S of E the property of having a dense complementary subspace depends only on the deficiency of S. In particular, E must have a dense proper subspace. From this it is shown that if D is the affine dimension and N the cardinal number of E, and either $2 \leqq M \leqq D$ or $M = N$, then E can be expressed as the union of M pairwise disjoint dense convex sets. This strengthens a result of J. W. Tukey (Portugaliae Mathematica vol. 3 (1942) p. 101). (Received February 23, 1949.)

297*t*. V. L. Klee: *On a conjecture of Bourgin.*

A subset X of a Hausdorff linear space H is called "regularly H convex" if to each $y \in H - X$ corresponds a continuous linear functional f on H such that $f(y) > \sup_{x \in X} f(x)$. D. G. Bourgin (Amer. J. Math. vol. 65 (1943) pp. 637–659) has shown that $(\alpha) H$ is locally convex implies (β) every closed convex subset of H is regularly H convex. The present paper the author describes a class of Hausdorff linear spaces for which (β) is true but (α) is not, thus substantiating a conjecture of Bourgin. (Received February 23, 1949.)

298. V. L. Klee: *On convex sets in linear spaces.*

A $(T\alpha)$-space E is at once a real linear system and a topological space, the two structures being so related that for each $y \in E$ and real r, the function $y + rx \mid x \in E$ is continuous. It is shown that many of the known properties of convex sets in normed linear spaces are still present in $(T\alpha)$-spaces, and of those which fail in some $(T\alpha)$-spaces several important ones are valid at least in every locally convex Hausdorff linear space. A fundamental result is the following: If A and B are disjoint convex subsets of the real linear system L and one of them has a radially interior point, there is a linear functional $f \neq 0$ on L and a constant c such that $f \geqq c$ on A and $f \leqq c$ on B. Among the topics treated are radial interiority, existence of continuous linear functionals, and support, separation, and non-support properties of convex sets. The principal theorems extend to less restricted spaces results of D. G. Bourgin, J. P. LaSalle, S. Mazur, J. W. Tukey, and the author. (Received February 23, 1949.)

299*t*. N. Metropolis and S. M. Ulam: *On motions of systems of mass points randomly distributed on the infinite line.*

Assume that $x = .\alpha_1\alpha_2 \cdots \alpha_n \cdots$ is a normal number (in the sense of Borel) in the binary system corresponding to a system of mass points, each of mass 1 located on the infinite line; or the coordinate i if $\alpha_{2i} = 1$, or $-i$ if $\alpha_{2i+1} = 1$. This distribution corresponds to time $t = 0$. The points attract each other with forces inversely proportional to the squares of their respective distances. A study is made of asymptotic properties of the motion of such systems. In particular, one proves the existence of condensations and super-condensations of various orders. (Received February 18, 1949.)

300*t*. Paolo Nesbeda: *On the minimum number of self-intersections of a locally simple closed curve.* Preliminary report.

Let g be a L-S-closed curve intersecting itself in a finite number of points, and let $[g]$ denote the 0-homotopy class of g. A curve of g which is admissibly deformable into a curve g' without altering the self-intersections and so that on a sufficiently small circle with center at an arbitrary self-intersection Q of g' the four subarcs of g' with end point Q are straight is defined as a curve of model intersections. If the model self-intersections are defined to be of type A if the incoming straight arcs are adjacent, it is shown that the intersections of g if model and minimum in number are necessarily of type A. It is shown further that the minimum number of self-intersections is $N = |q| - |p - q| - 1$ where p and q denote the angular order and the order of g with respect to 0, respectively. The proof is combinatorial in character. Some of the results and methods used are taken from the work of M. Morse, *Topological methods in the theory of functions of a complex variable.* (Received February 18, 1949.)

301. G. O. Peters: *Bernoulli polynomials and numbers of the second kind of higher order and of negative order.* Preliminary report.

Nörlund, in his *Vorlesungen über Differenzenrechnung*, defines Bernoulli polynomials and numbers of higher order and of negative order. Jordan, in his *Calculus of finite differences*, defines Bernoulli polynomials and numbers of the second kind, but does not treat the higher orders or negative orders of these polynomials and numbers. The author denotes Bernoulli polynomials and numbers of the second kind, of order n and degree v, by $b_v^n(x)$ and b_v^n, respectively, defined by: $b_v^0(x) = x^{(v)} = x(x-1)(x-2) \cdots (x-v+1)$; $b_v^n(x) = \int_0^x v b_{v-1}^{n-1}(x)dx + b_v^n$; $b_v^n(x) = \sum_{i=0}^{v} {}^{v}C_{i} b_i^n x^{(v-i)} = (x+b^n)^{(v)}$; $v = 0, 1, 2, \cdots$; $n = 0, \pm 1, \pm 2, \cdots$. Properties and applications of these polynomials are found; one of which allows integration (or differentiation) of certain polynomials, which would be tedious by ordinary means, to be performed by summation (or differencing), which becomes simple for these cases. Formulas, involving Sterling's numbers of the second kind, are found for evaluating the Bernoulli numbers of the second kind of higher order. The symmetry of the polynomials is found to be given by $b_v^n(x) = (-1)^v b_v^n(v-n-1-x)$. (Received February 18, 1949.)

302. L. V. Robinson: *Intermediate integrals of nonlinear second order partial differential equations by operational methods.*

By means of a differential operator, it is shown how an intermediate integral of a nonlinear second order partial differential equation can be easily obtained. (Received April 1, 1949.)

303. Walter Rudin: *Uniqueness theory for Laplace series.*

Let P denote a point on S, the surface of the unit sphere. Let $Y_n(P)$ be a spherical surface harmonic of degree n. Put $\Delta F(P)=\lim_{h\to 0}(m(F;P,h)-F(P))/\sin^2(h/2)$, where $m(F;P,h)$ denotes the mean of F on the circle of intersection of S and the sphere of radius 2 sin $(h/2)$ about P. Define $\Delta^*F(P)$, $\Delta_*F(P)$ similarly, with lim sup, lim inf in place of lim. The series $\Sigma=\sum_0^\infty Y_n(P)\in K$ if $-\sum_1^\infty Y_n(P)/n(n+1)$ is the Laplace series of a function $F(P)$ continuous on S. Let Z be a closed set of capacity zero on S. The following two theorems are the main result of the paper. I. Let $\Sigma\in K$. Suppose (a) $\Delta^*F(P)>-\infty$, $\Delta_*F(P)<+\infty$ on $S-Z$; (b) there exists $y\in L$ on S such that $\Delta^*F(P)\geqq y(P)$ on S. Then $\Delta F(P)$ exists p.p. on S, and Σ is the Laplace series of $Y_0+\Delta F(P)$. II. Let $\Sigma\in K$. Suppose (a) the upper and lower Poisson sums (f^* and f_*) of Σ are finite on $S-Z$; (b) there exists $y\in L$ on S such that $f_*(P)\geqq y(P)$ on S. Then Σ is Poisson summable p.p. on S, and is the Laplace series of its Poisson sum. (Received February 9, 1949.)

304. H. M. Schwartz: *Contributions to existence theory of ordinary differential equations in the real domain.* II.

In part I (Bull. Amer. Math. Soc. Abstract 55-5-521) were discussed conditions that are sufficient for the existence, unique or otherwise, of a solution of a system of differential equations $y_i'(x)=f_i(x, y_1, \cdots, y_n)$ $(i=1, 2, \cdots, n)$ with preassigned value at a given point of the interval of definition I. In the present paper that discussion is extended in two directions. The requirement of finiteness of the interval I is removed, and the conditions of existence of solutions of the required type are shown to be the best in a certain specified sense. The extension of the results to the infinite interval is based on the following lemma (for related results in the finite interval cf. Carathéodory, *Reelle Funktionen*, I, 1939, pp. 172–180): a sequence of functions uniformly bounded and equicontinuous in an interval I, finite or infinite, contains a subsequence uniformly convergent in I. (Received February 17, 1949.)

305. W. S. Snyder: *Regular points of families of sets.* Preliminary report.

Consider families F of closed subsets of a bounded open set G in Euclidean N-space. A point x is an α-regular point for F if there is a subfamily of disjoint sets $F_1\subset F$ such that the union $x+\sum_{X\in F_1}X$ has parameter of regularity (Saks, *Theory of the integral*, 1937, p. 106) greater than α. Let $R(F, \alpha)$ denote the set of points which are α-regular for F, and $R(F)=\sum_{n=2}^\infty R(F, 1/n)$ is termed the set of regular points for F. The theorem is proved: If the sequence of families $\{F_n\}_1^\infty$ satisfies the conditions (1) each F_n is a finite family of disjoint, closed subsets of G, (2) the norm of F_n converges to 0 as $n\to\infty$, (3) $\left|\sum_{x\in F_n} x\right|\geqq\beta>0$, then $\left|\sum_{j=1}^\infty\{\limsup_n R(F_n, 1/j)\}\right|\geqq\beta$ and $0<\gamma<1$ implies that $\left|R(\sum_{n=1}^\infty F_n, \gamma)\right|\geqq\beta$. In the proof, the author uses standard methods of Lebesgue theory. The interest of the theorem is due to its being effectively a converse of the Vitali covering theorem. It can be used to prove that certain derivatives (R. C. Young, *Functions of* Σ, and so on, Math. Ann. vol. 29 (1928)) are essentially independent of the parameter of regularity involved in their definitions. (Received February 23, 1949.)

306. W. M. Whyburn: *Existence and oscillation theorems for second order, nonlinear, differential systems.*

The differential system $y'=K(x, y, z; \lambda)z$, $z'=G(x, y, z; \lambda)y$, together with two linear boundary conditions at two points, is considered. Under mild hypotheses on the coefficients of the equations and boundary conditions, the existence of sets of characteristic numbers for the system is established. Oscillation theorems are given for the characteristic solutions and other properties of these solutions are demonstrated. In the present paper, results previously obtained by the author (Trans. Amer. Math. Soc. vol. 30 (1928) pp. 630–640, 848–854) are extended. The methods used are similar to those of the earlier paper. (Received February 14, 1949.)

307*t*. Bertram Yood: *Banach algebras of continuous functions.*

Let X be a bicompact Hausdorff space and $C=C(X, B)$ be the Banach algebra of all continuous functions $f(x)$ with domain X and values in a commutative Banach algebra B with a unit. The norm $\|f\|$ is given by l.u.b. $\|f(x)\|$, $x\in X$. If the maximal ideals of C are all of the form, the set of all f in C such that $f(x_0)\in M$, where $x_0\in X$ and M is a maximal ideal of B; then the space of maximal ideals of C is homeomorphic to the Cartesian product $X\times\mathfrak{M}$ where $\mathfrak{M}$ is the space of maximal ideals of B. The hypothesis of this theorem is verified if B is a *-algebra in the sense of Arens (Duke Math. J. vol. 14 (1947) pp. 269–282). Other aspects of the ideal theory of C are also considered. (Received February 18, 1949.)

APPLIED MATHEMATICS

308*t*. R. A. Clark, Eric Reissner, and G. B. Thomas: *On the theory of generalised plane stress.*

The authors are concerned with a problem of three-dimensional elasticity for bodies bounded by cylindrical surfaces and by two parallel planes at right angles to the cylindrical surfaces. It is assumed that the plane portions of the surface are free of stress and that the stresses applied to the cylindrical surfaces do not vary along the generators of these surfaces. The object of the study is the determination of three-dimensional corrections for the two-dimensional theories of plane stress and plane strain. For this purpose the direct methods of the calculus of variations are used to obtain a system of approximate two-dimensional partial differential equations for the three-dimensional problem. The general equations obtained are used for the solution of a specific problem. (Received February 23, 1949.)

309. P. E. Guenther: *Numerical solution of panel heat flow problems.*

The two-dimensional problem describing the steady flow of heat in a panel from sources embedded in a panel to the panel surfaces is complicated by nonlinear boundary conditions imposed by radiative and convective transfer conditions at the panel surfaces. Temperature distributions within the panel for a number of cases involving typical operating conditions are determined by relaxation methods. The panel surface-temperature distributions so obtained and the rates of heat flow across the panel surfaces computed therefrom are compared with observed values of the same quantities determined from operation of a full-scale model panel. The effect of the refinement of net size upon improving the agreement between computed and observed results at the expense of complicating and extending the calculations is investigated. (Received February 23, 1949.)

310. M. H. Martin: *Plane rotational flows.*

Taking the stream function ψ and pressure p for independent variables, the determination of the steady, rotational flow of a nonviscous fluid subject to no external forces reduces to the integration of the quasi-linear partial differential equation $(\partial/\partial\psi)((q_{pp}-q\theta_p^2)/\theta_\psi)+q\theta_{pp}+2q_p\theta_p=0$, where θ denotes the angle of inclination of the velocity vector and $q = q(\psi, p)$, the speed of flow is determined by Bernoulli's theorem when the equation of state for the fluid is given, and the variation of entropy and energy from stream line to stream line are prescribed. Flows for which the stream lines are isobars or for which $\theta = \theta(p)$ are excluded from the analysis and are treated separately. In the latter case a generalization of Prandtl-Meyer flows to rotational flows is obtained. The case $\theta = \theta(p)$ leads to a treatment of those flows in which the stream lines are all straight lines. (Received February 24, 1949.)

311*t*. Eric Reissner: *Note on the problem of twisting of a circular ring sector.*

It is shown that the partial differential equation for this problem (which is discussed on pp. 355–361 in *Theory of elasticity* by S. Timoshenko) can be integrated in closed form when the cross section of the ring sector is a hollow section with thin walls. The results obtained correspond in nature to those of R. Bredt for St. Venant torsion of cylindrical rods. (Received February 14, 1949.)

GEOMETRY

312. H. V. Craig and B. B. Townsend: *On certain extended components of connection.*

The derivatives with respect to the curve parameter of tensors whose first intrinsic derivatives vanish are shown to be linear in the tensors and have the extended components of connection, $L_d^{\alpha *}$ and $L^{*}_{\alpha a}$, for coefficients. These relations are applied to the problem of obtaining higher order intrinsic derivatives of absolute tensors of all orders as complete or reduced range contractions of certain extensors with the extended components of connection. These results are used to investigate the reduced range and full range contractions of the extended components of connection, L_d^{α}, $L_{\alpha a}$, $g_d^{\alpha *}$, and $g^{*}_{\alpha a}$. The particular contraction, $g_d^{\alpha *}L^{*}_{\alpha a}$, for $M = 2$, yields a tensor which may be used as a characterization of the flatness of a space. (Received February 23, 1949.)

313. F. A. Ficken: *Elementary reduction of a matrix to Jordan normal form.*

If λ is an eigenvalue of M, and $L = M - \lambda I$, one can distinguish, among the vectors annulled by L, those which are and those which are not in the column space C_L of L. If $LZ_1 = 0$, with Z_1 in C_L, find Z_2 such that $LZ_2 = Z_1$, and continue until $LZ_{k+1} = Z_k$ with Z_{k+1} not in C_L. Do this for each vector of a basis for the intersection of C_L with the nullspace of L. All vectors so produced, along with those eigenvectors not in C_L, span the root space corresponding to λ. In this way obtain a basis for the root space for each eigenvalue λ. In the resulting basis for the whole space, M will be in Jordan normal form. It is assumed that the eigenvalues are known; but successive Z_k are obtained, for a given λ, merely by introducing a succession of right members for the same system of linear equations; neither the adjoint nor powers of M are needed. Justification of the process is based on the Hamilton-Cayley theorem and an analysis of several spaces associated with M; elementary divisors are not used. (Received February 17, 1949.)

314. G. B. Huff: *Cremona's equations and the properness inequalities.*

A solution $x=\{x_0; x_1, x_2, \cdots, x_n\}$ of Cremona's equations, (1) $x_0^2-x_1^2-x_2^2-\cdots-x_n^2=d+p-1$, $3x_0-x_1-x_2-\cdots-x_n=d-p+1$, is said to be *proper* if there exists a linear system of plane curves of genus p, dimension d, order x_0, and having multiplicites $x_1, x_2, \cdots, x_n$ at a set of n general points. A proper solution x of (1) also satisfies the *properness inequalities*, (2) $y_0x_0-y_1x_1-y_2x_2-\cdots-y_nx_n\geqq 0$, where y ranges over the finite number of characteristics of Bertini L-curves of $y_0<x_0$. By means of lemmas concerning minimal order solutions of (1) and (2), the following theorems are proved. I. Any solution of (1) satisfying the inequalities (2) is proper for $p=0$ and $d\geqq 0$, and there are proper solutions for each $d\geqq 0$. II. Any solution of (1) satisfying the inequalities (2) for $p=1$, $d\geqq 1$ is proper; and there exists a proper solution for $1\leqq d\leqq 9$. For $d>9$, there are no proper solutions. (Received February 16, 1949.)

315. Jack Levine: *Field of parallel vectors in conformally flat spaces.*

If $C_n(p, q)$ denotes a conformally flat space admitting p non-null and q null fields of parallel vectors, it is shown that the only possibilities are $C_n(1,0)$ and $C_n(0, 1)$. The $C_n(1, 0)$ is a symmetric Cartan space of class one, and both types are spaces K_n considered by Ruse in which $R_{hi;k,m}=t_{,m}R_{hi;k}$, J. London Math. Soc. vol. 21 (1946) pp. 243–247. Invariant characterizations of a $C_n(1, 0)$ and $C_n(0, 1)$ are also obtained. Canonical forms of the type $u^{-2}\sum e_i(dx^i)^2$ are derived for the metrics of the spaces. The theory of product-spaces as developed by Ficken, Ann. of Math. (2) vol. 40 (1939) pp. 892–913, is used to obtain some of the results. (Received February 4, 1949.)

316. B. E. Mitchell: *A real representation of the imaginary line.*

The Laguerre representation of the imaginary point by an ordered pair of real points readily lends itself to construction in the complex plane. Such construction, however, cannot proceed far nor long, however, without a corresponding dual representation of the imaginary line. Furthermore, since the complex line and the complex point are dual elements in the complex plane, when they are united in position their representations should be organically united as Study has suggested. The ideal would be that if the point and the line are united in position, then their respective representations should also be united in position. This ideal is achieved, however, only when the line has a real trace and the point is real and lies on the trace. There is, nevertheless, a strong organic union in the general case. The real representation of the imaginary line is revealed in the discovery of a certain Gudermannian equivalent of the imaginary angle of the complex line and its conjugate. (Received April 2, 1949.)

317t. J. M. Thomas: *Nomographic disjunction.*

The constants are from a commutative field. The variable domain consists of the triples (x, y, z) obtained as each of the variables independently assumes the values in a corresponding given set. Each function assumes for each triple in the variable domain a single value from the field of constants. A disjoint is a determinant of order three with the following properties: the elements on the ith row are functions of only the ith one of the variables x, y, z; at least two of the functions in each row are linearly independent (with respect to the field of constants); and the algebraic complements of the three functions on each row are linearly independent. A function which equals a disjoint is a nomographic function. A function F being given, the author develops an algebraic process (disjunction) for finding a disjoint D which must equal F if F has a disjoint. The problem of finding whether F is nomographic is thereby reduced to examining whether the single equation $F=D$ is an identity in (x, y, z). (Received February 23, 1949.)

STATISTICS AND PROBABILITY

318. Isadore Blumen: *The effect of serial correlation on the distributions of certain statistics.*

In obtaining the distributions of many commonly used statistics the assumption is made that the observations are independent. However, in time series problems, in agricultural experiments, and in other applications, it is frequently more reasonable to assume that the observations have a correlation depending on the "distance" between them. The exact probability distributions of certain statistics are obtained, as well as useful approximations, when it is assumed that the observations are sets of normally distributed variables with circular serial and lag correlations. (Received February 18, 1949.)

319. A. C. Cohen: *Estimating the mean and variance of truncated normal distribution.*

In certain types of scientific experiments it sometimes happens that the technique of measurement has a known limit beyond which the variate cannot be measured but the number of observations falling beyond this limit in any given sample can be counted. In this paper the method of "maximum likelihood" has been employed to obtain estimates of the mean and variance of normally distributed populations when the sample data are thus restricted. Variances of these estimates for large values of n are also obtained. (Received February 17, 1949.)

320. G. E. Nicholson: *On the estimation of the parameters of a normal p-variate distribution from an incomplete sample.*

In statistical investigations involving multicharacter observations on individuals it sometimes happens that certain observations on some of the characters are missing. The problem of utilizing these incomplete measurements in the estimation of the parameters of the population is treated when it is assumed that the underlying population is multivariate normal. The estimates are obtained in terms of familiar statistical quantities and certain conclusions pertaining to statistical prediction are obtained. (Received February 18, 1949.)

321. H. E. Robbins and E. J. G. Pitman: *Application to the method of mixtures to quadratic forms in normal variates.*

If $F_n(x)$ are distribution functions and if c_n are constants such that $c_n \geq 0$ and $\sum c_n = 1$, then $F(x) = \sum c_n F_n(x)$ is a distribution function, called a mixture. Some useful properties of mixtures are derived and the results are applied to the distribution of a positive quadratic form in normal variates and of the ratio of two such forms. The solutions are given respectively as mixtures of gamma and of beta distributions, with an upper bound to the error involved in replacing the series by finite sums. (Received February 18, 1949.)

TOPOLOGY

322. I. C. Gentry: *On extending a homeomorphism between subsets of 2-dimensional orientable manifolds.* Preliminary report.

Let M_i $(i=1, 2)$ denote two 2-dimensional orientable manifolds of genus n. Let $K_i \subset M_i$ be a Peano continuum. Let $\phi(K_1) = K_2$ be a homeomorphism defined over

K_1 onto K_2. It is proved that in order that this homeomorphism shall be extendable to a homeomorphism ϕ', $\phi'(M_1)=M_2$, it is necessary and sufficient that the following conditions hold: (1) $\epsilon\{\phi(\gamma_1)\}=\epsilon(\gamma_1)$ for every index curve γ_1 of K_1 where $\epsilon(\gamma_1)$ is the signature of γ_1. (2) If L_1 is a linear graph in K_1 composed of a finite number of simple closed curves $\leqq$ genus of M_1 and M_1-L_1 is the sum of components σ_j^1, $j=1, 2, \cdots, m$, then $M_2-\phi(L_1)$ is the sum of components σ_j^2, $j=1, 2, \cdots, m$, such that (a) if $x\in K_1\cdot\sigma_j^1$, then $\phi(x)\in\sigma_j^2$ and (b) for each j, $\overline{\sigma_j^1}$ is homeomorphic to $\overline{\sigma_j^2}$. This is a generalization of similar theorems by Adkisson and MacLane for the sphere and Knobelauch for the torus. (Received February 18, 1949.)

323. O. G. Harrold: *Euclidean domains with uniformly abelian local fundamental groups*. II.

In the first paper with this title the author showed that if the complementary subset to a topological k-cell C in the n-dimensional spherical space S^n has uniformly abelian local fundamental groups, then the group of S^n-C is trivial. In this note the same hypothesis is placed on S^n-C and the question is considered of what conditions may be placed on a subset $T\subset C$ in order that S^n-T have a trivial group. If dimension $C=n$, the problem reduces essentially to one of the first paper. If dimension $C<n-1$, it is sufficient that T be closed. If dimension $C=n-1$, it is sufficient that T be an absolute retract. The main obstacle to be overcome when dim $C=n-1$ is that of establishing a sense of two sides for an arbitrary closed, connected subset in the interior of C relative to S^n-C. (Received February 16, 1949.)

324t. Samuel Kaplan: *Extensions of the Pontrjagin duality*. II. *Direct and inverse sequences*.

Let $\{H_\lambda\}$ be a direct system of topological groups, that is, the index set is directed (for every pair λ, μ, there is a ν such that $\lambda<\nu$, $\mu<\nu$) and for each pair $\lambda<\mu$ there is a continuous homomorphism $p_{\lambda\mu}:H_\lambda\rightarrow H_\mu$ such that if $\lambda<\mu<\nu$, then $p_{\lambda\nu}=p_{\mu\nu}p_{\lambda\mu}$. The *direct limit* $\mathfrak{H}$ of the system $\{H_\lambda\}$ is defined as follows: Let $H=P^*H_\lambda$ topologized as in Kaplan, *Extensions of the Pontrjagin duality*. I, Duke Math. J. vol. 15 (1948); let H_0 denote the subgroup of H generated by all elements of the form $h_\lambda-p_{\lambda\mu}h_\lambda$, $h_\lambda\in H_\lambda$, λ running through the entire index set $\{\lambda\}$; then $\mathfrak{H}=H/\overline{H}_0$. It is proved that if $\{G_n\}$ $(n=1, 2, \cdots)$ is an inverse sequence of locally compact groups and $\{H_n\}$ the "dual" direct sequence of the corresponding character groups, then the inverse limit of the G_n's and the direct limit of the H_n's are character groups of each other. (Received February 9, 1949.)

325. J. W. Keesee: *Finitely-valued cohomology groups*.

Following the notation of Spanier (Ann. of Math. (1948) pp. 405–427) define $F^p(X, A)$ to be the subgroup of $\phi^p(X, A)$ consisting of those p-functions that assume only a finite number of elements of the coefficient group as values. $H_f^p(X, A)$ is then the cohomology group of X mod A based on elements of $F^p(X, A)$. A homomorphism τ on $H_f^p(X, A)$ into $H^p(X, A)$ is exhibited, and it is proved that if X is compact Hausdorff and A is closed, then τ is an isomorphism onto. (Received February 16, 1949.)

T. R. HOLLCROFT,
Associate Secretary

THE APRIL MEETING IN PHILADELPHIA

The four hundred forty-sixth meeting of the American Mathematical Society was held at The University of Pennsylvania, Philadelphia, Pennsylvania, on Friday and Saturday, April 29–30. The attendance was about three hundred, including the following two hundred forty-three members of the Society.

C. R. Adams, R. B. Adams, E. J. Akutowicz, A. A. Albert, C. B. Allendoerfer, Warren Ambrose, R. D. Anderson, S. I. Askovitz, Iacopo Barsotti, F. D. Bateman, P. T. Bateman, F. W. Beal, F. P. Beer, E. G. Begle, Stefan Bergman, Lipman Bers, R. J. Bickel, Garrett Birkhoff, A. L. Blakers, W. A. Blankinship, J. H. Blau, R. P. Bóas, Enrico Bompiani, W. M. Boothby, D. G. Bourgin, R. C. Briant, R. C. Campbell, P. A. Caris, K. Chandrasekharan, S. S. Chern, P. L. Chessin, Randolph Church, Edmund Churchill, J. A. Clarkson, I. S. Cohen, L. W. Cohen, R. H. Cole, Damian Connelly, A. H. Copeland, Natalie Coplan, G. F. Cramer, C. W. Crouse, A. B. Cunningham, H. B. Curry, J. H. Curtiss, D. A. Darling, M. M. Day, C. S. Dias, E. M. DiDomenica, M. P. Dolciani, J. L. Doob, C. H. Dowker, Y. N. Dowker, R. H. Downing, L. A. Dragonette, Aryeh Dvoretzky, J. J. Eachus, Samuel Eilenberg, Bernard Epstein, Paul Erdős, G. C. Evans, O. J. Ferrell, William Feller, N. J. Fine, C. D. Firestone, E. E. Floyd, R. M. Foster, R. H. Fox, Gerald Freilich, Orrin Frink, David Gale, G. N. Garrison, J. B. Giever, W. M. Gilbert, B. P. Gill, O. E. Glenn, H. E. Goheen, H. H. Goldstine, H. H. Goode, M. J. Gottlieb, W. H. Gottschalk, Arthur Grad, Emil Grosswald, C. C. Grove, V. H. Haefeli, M. R. Hafner, Harish-Chandra, K. E. Hazard, G. A. Hedlund, A. E. Heins, M. H. Heins, Alex Heller, Erik Hemmingsen, Coleman Herpel, A. D. Hestenes, T. H. Hildebrandt, Joseph Hilsenrath, T. R. Hollcroft, E. M. Hull, Witold Hurewicz, L. C. Hutchinson, K. S. Jacoby, M. L. Juncosa, Shizuo Kakutani, Hyman Kamel, Irving Kaplansky, Gilbert Kaskey, W. H. Keen, M. E. Kellar, D. E. Kibbey, Fred Kiokemeister, J. R. Kline, T. L. Koehler, D. D. Kosambi, H. W. Kuhn, V. V. Latshaw, Solomon Lefschetz, Joseph Lehner, Marguerite Lehr, R. A. Leibler, R. B. Leipnik, George Lewis, M. A. Lipschutz, S. R. Lipsey, Charles Loewner, E. R. Lorch, Lee Lorch, H. M. Lufkin, D. L. McDonough, E. A. McDougle, J. A. McKee, G. W. Mackey, Murray Mannos, M. H. Martin, W. T. Martin, W. S. Massey, A. E. Meder, L. I. Mishoe, E. E. Moise, Deane Montgomery, T. W. Moore, K. A. Morgan, C. R. Morris, Marston Morse, G. D. Mostow, T. S. Motzkin, W. R. Murray, A. C. Nelson, Paolo Nesbeda, Morris Newman, C. V. Newsom, R. Z. Norman, C. O. Oakley, Paul Olum, F. W. Owens, H. B. Owens, J. C. Oxtoby, J. S. Oxtoby, O. O. Pardee, A. J. Penico, F. W. Perkins, I. D. Peters, R. P. Peterson, H. P. Pettit, Joel Pitcairn, Everett Pitcher, Pasquale Porcelli, W. W. Proctor, M. H. Protter, W. T. Puckett, Gustave Rabson, Hans Rademacher, John Raleigh, K. G. Ramanathan, H. E. Rauch, A. G. Rawling, G. E. Raynor, C. J. Rees, K. W. Reidemeister, Eric Reissner, Russell Remage, M. M. Resnikoff, C. N. Reynolds, J. E. Robinson, S. L. Robinson, P. C. Rosenbloom, J. B. Rosser, C. W. Saalfrank, Arthur Sard, S. W. Saunders, A. B. Schacknow, A. T. Schafer, R. D. Schafer, Albert Schild, Pincus Schub, Abraham Schwartz, G. E. Schweigert, D. B. Shaffer, I. M. Sheffer, Seymour Sherman, S. P. Shugert, Edward Silverman, M. L. Slater, M. H. Slud, Andrew Sobczyk, C. F. Stephens, M. H. Stone, W. J. Strange, A. H. Taub, W. R. Thickstun, J. M. Thomas, M. L. Tomber, C. B. Tompkins, D. L. Thomsen, C. A. Truesdell, A. W. Tucker, W. J. Turanski, J. R. Van Andel, J. L. Vanderslice, H. E.

Vansant, A. H. Van Tuyl, Oswald Veblen, Sylvan Wallach, J. L. Walsh, J. B. Walton, W. R. Wasow, G. C. Webber, M. A. Weber, J. V. Wehausen, Alexander Weinstein, D. W. Western, P. M. Whitman, A. M. Whitney, Hassler Whitney, Albert Wilansky, W. G. Wolfgang, Y. K. Wong, Marcella Yarosheski, G. S. Young, J. W. Young, L. C. Young, Oscar Zariski, P. W. Zettler-Seidel.

On Friday afternoon Professor G. W. Mackey of Harvard University gave an address on *Functions on locally compact groups*. Professor M. H. Stone presided.

President J. L. Walsh presided at the general session Saturday morning at which an address was given by Professor K. W. Reidemeister of the University of Marburg and Institute for Advanced Study on *The complexes of a group*. Following the address, President Walsh presented embossed sets of resolutions of appreciation from the Board of Trustees and the Council of the American Mathematical Society to Professor Bennington P. Gill, retiring Treasurer of the Society. The resolutions adopted by the Council appear in the report of the 1948 Annual Meeting. Professor Gill thanked the Board of Trustees and the Council and spoke briefly on the expanding activities of the Society during his eleven years' service.

President Walsh introduced Professor D. G. Bourgin who presented resolutions of thanks to the President, the Mathematics Department, and the Museum officials of the University of Pennsylvania for their excellent arrangements and cordial hospitality.

On Friday afternoon there were two sections, one for papers in Statistics and Applied Mathematics in which Professor Eric Reissner presided, and one for papers in Analysis, Geometry, and Topology in which Professor Samuel Eilenberg presided. On Saturday afternoon there were two sections, one for papers in Analysis in which Professor C. R. Adams presided, and one for papers in Algebra and Topology in which Professor A. H. Copeland presided.

President and Mrs. Harold E. Stassen of the University of Pennsylvania received the mathematicians and their guests at a tea in the Middle American Hall of the University Museum at 4:30 P.M. on Friday.

The Council met at 2:00 P.M. on April 30.

The Secretary announced the election of the following ninety-five persons to ordinary membership in the Society:

Mr. Allan Henry Albert, University of Michigan;
Miss Linda Allegri, Hunter College High School;
Mr. Raymond William Allen, St. Louis University;
Mr. Richard Lewis Arnowitt, Cambridge, Mass.;
Professor Antonio Astorga, Reserve Officers Preparatory Center, Rio de Janeiro, Brazil;

Mr. Robert Baer, Roosevelt College, Chicago, Illinois;
Mr. David Campbell Barton, Roberts Junior College, North Chili, N. Y.;
Mr. Mohammedali Abdul Ameer Bassam, University of Texas;
Mr. Vernon Nelson Behrns, University of Buffalo;
Mr. Fred George Berghoefer, Chicago, Ill.;
Helen Virena Betz (Mrs. G. P.), Northwestern University;
Miss Lollie Belle Bienvenu, Louisiana State University and Agricultural and Mechanical College;
Professor John Henry Binney, Agricultural and Mechanical College of Texas;
Mr. Jacob Borsuk, Evans Signal Laboratory, Belmar, N. J.;
Mr. Joseph Edward Bosshart, Cleveland, Ohio;
Dr. John R. Bowman, Mellon Institute;
Mr. Frank Herbert Brownell III, Princeton University;
Mr. George Hench Butcher, Howard University;
Mr. William Albert Catenaro, University of Oklahoma;
Mr. Philip Cooperman, New York University;
Miss Grace Virginia Crenshaw, Danville, Va.;
Mr. Fred T. Daly, Ithaca, N. Y.;
Dr. Walter Lorraine Deemer, Lt. Col., U. S. Air Forces, University of North Carolina;
Mr. George Henry DeMers, Urbana, Ill.;
Mr. Bernard Emile Dethier, Nazareth College;
Professor Louis E. Diamond, Oklahoma City University;
Mr. John Allen Doggett, Howard University;
Mr. Robert Leonard Evans, University of Minnesota;
Mr. Frank David Faulkner, University of Michigan;
Professor James Kern Feibleman, Tulane University of Louisiana;
Mr. Marvin Julian Forray, Polytechnic Institute of Brooklyn;
Mr. Jerome Bernard Freier, New York University;
Mr. William Barkley Fritz, Ballistic Research Laboratory, Aberdeen Proving Ground, Md.;
Mr. Robert Arthur Fuchs, University of California;
Dr. Boris Garfinkel, Ballistic Research Laboratory, Aberdeen Proving Ground, Md.;
Mr. Dal Charles Gerneth, Sherman, Texas;
Mr. Gene Heiden Gleissner, College of Pharmacy, Columbia University;
Professor Frederick George Graff, Oberlin College;
Mr. Oliver Alfred Gross, Rand Corporation, Santa Monica, Calif.;
Miss Rose Ann Grundman, University of Arizona;
Mr. Benjamin Franklin Handy, Jr., National Bureau of Standards, Washington, D. C.;
Mr. Frank Sylvester Hawthorne, Hofstra College;
Dr. Alberta H. Henry, Brooklyn College;
Professor Mildred Elizabeth Hull, Wilkes College;
Professor Lamek Hulthen, Royal Institute of Technology, Stockholm, Sweden;
Mr. William Elliott Jenner, University of British Columbia;
Commander Walter Hughes Keen, U. S. Navy, Princeton University;
Mr. Ralph Craig Kennedy, National Broacasting Company, New York, N. Y.;
Miss Evelyn Kendrick Kinney, University of Illinois;
Mr. Robert Elmer Krundklin, American Viscose Corporation, Philadelphia, Pa.;
Mr. Ke-chan Kuo, University of Illinois;
Mr. Paul Gustav Kvick, Columbia University;
Dr. Robert Steven Ledley, Columbia University;

Mr. James A. Lemon, Fenn College;
Miss Dinah G. Levine, Polytechnic Institute of Brooklyn;
Mr. Leo Meyer Levine, Naval Shipyard, Brooklyn, N. Y.;
Dr. Mark Lotkin, Ballistic Research Laboratory, Aberdeen Proving Ground, Md.;
Miss Hilda Marie Lucy, Howard University;
Mr. Robert Kinsey McConnell, Jr., New York University;
Dr. Clifford Joseph Maloney, Research and Development Department, Camp Detrick, Frederick, Md.;
Mr. John Maurice Marr, University of Missouri;
Mr. Norman M. Martin, University of Illinois;
Dr. John George Meiler, Coos Bay Lumber Company, Coos Bay, Ore.;
Professor John Graham Millar, University of Alberta, Calgary, Alta., Canada;
Mr. Paul Dixon Minton, University of North Carolina;
Miss Mabel D. Montgomery, University of Buffalo;
Mr. August Carl Nelson, Jr., University of Delaware;
Mr. Cesar Dacorso Netto, Rio de Janeiro, Brazil;
Professor Helen Olney, Hiram College, Hiram, Ohio;
Professor Richard Robert Otter, University of Notre Dame;
Mr. Bertram Samuel Page, Brotherton Library, University of Leeds;
Mr. Stephen Tyler Paine, Institute for Numerical Analysis, National Bureau of Standards, University of California at Los Angeles;
Dr. Otway O'Meara Pardee, Syracuse University;
Mr. Jerome Kenneth Percus, Columbia University;
Professor Iland Dee Peters, West Virginia University;
Mr. Aihud Pevsner, New York, N. Y.;
Mr. George Walter Reitwiesner, Ballistic Research Laboratory, Aberdeen Proving Ground, Md.;
Professor Antonio Romero-Juarez, University of Mexico;
Mr. Nicholas John Rose, Stevens Institute of Technology;
Mr. Paul Talbot Rotter, Mutual Benefit Life Insurance Company, Newark, N. J.;
Mr. Theodore Rubin, Fairchild Engine and Airplane Corp., Oak Ridge, Tenn.;
Mr. Edward Lee Schlain, Moore School of Electrical Engineering, University of Pennsylvania;
Mr. David Jerome Smith, University of Notre Dame;
Mr. Harry Smith, Jr., University of Delaware;
Mr. James Edward Snover, Sampson College, Sampson, N. Y.;
Mr. Anson Donald Solem, Naval Ordnance Laboratory, White Oak, Silver Spring, Md.;
Mr. Nathaniel Hawthorne Stevens, Franklin School, New York, N. Y.;
Mr. Joseph A. Sullivan, Indiana University;
Professor Henry William Syer, Boston University;
Mr. Gerald Luther Thompson, University of Michigan;
Mr. William Thomas Tutte, University of Toronto;
Mr. Robert Ullman, Polytechnic Institute of Brooklyn;
Mr. James Hatton Wahab, University of North Carolina;
Mr. William Guinn Wolfgang, Altoona Center, Pennsylvania State College;
Dr. Albert Wolinsky, New York University.

It was reported that the following twenty-two persons had been elected to membership on nomination of institutional members as indicated:

Brown University: Mr. Edward Forrest Moore;
Harvard University: Messrs. Seymour Hayden, Henry Helson, George Graham Murray, Henry Gardner Rogers, William J. Turanski, Neal Thomas Watson, Harold Weintraub, John Wermer, and Vidar Michael Wolontis;
Haverford College: Mr. Cloyd Marvin;
Institute for Advanced Study: Dr. Verena Huber Haefeli, Professor Kazimierz Kuratowski, Dr. Kurt Mahler, and Professor Andrzej Mostowski;
University of Minnesota: Mr. Philip George Kirmser;
Pennsylvania State College: Messrs. Winton Laubach and Richard Wesley Wonn;
Purdue University: Messrs. James Dawson McKnight, Jr., Bill VanTrump Ritchie, Eugene Usdin;
University of Rochester: Mr. Arthur Edward Danese.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: London Mathematical Society: Professor Joseph Langley Burchnall, Durham University; Dr. Mary Lucy Cartwright, Girton College, Cambridge University; Professor John Kronsbein, Evansville College, Evansville, Ind.; Dr. Samuel Verblunsky, Queen's University, Belfast, Northern Ireland; Dr. Thomas James Willmore, Durham Colleges, University of Durham; Société Mathématique de France: Mr. Raymond George Ayoub, McGill University; Professor Georges Bourion, University of Algiers; Dr. Ion Carstoiu, National Center of Scientific Research, Paris; Professor Pierre Jacques Lelong, University of Lille; Swiss Mathematical Society: Dr. Harry Ernest Rauch, Rutgers University; Wiskundig Genootschap te Amsterdam: Professor Hans Freudenthal, University of Utrecht.

The following actions taken by mail vote of the Council were reported: election of Professors Samuel Eilenberg, T. H. Hildebrandt, Saunders MacLane and W. T. Martin as members of the Executive Committee of the Council, the first two to serve for a term of one year beginning January 1, 1949, and the second two for a term of two years; acceptance of statement of policy in connection with papers submitted to the Transactions; acceptance of a manuscript by Professors A. C. Schaeffer and D. C. Spencer, entitled *Coefficient regions of schlicht functions*, for publication in the Colloquium Series; authorization of the appointment of a committee to organize an experiment in the reproduction of research by methods other than letter press; acceptance of a manuscript by Professor J. L. Walsh, entitled *The location of critical points*, for publication in the Colloquium Series.

The following appointments of representatives of the Society were reported: Professor J. L. Walsh at Mid-Century Convocation and inauguration of James Rhyne Killian, Jr., as President of Massachusetts Institute of Technology on March 31–April 2, 1949; Professor C. O. Oakley at Fifty-third Annual Meeting of the American Academy of Political and Social Science on April 8–9, 1949; Professor Gillie A. Larew at the inauguration of Walter Stephenson Newman as President of Virginia Polytechnic Institute on April 18, 1949; Professor A. E. Landry at the inauguration of Rev. Hunter Guthrie as President of Georgetown University on April 30–May 1, 1949; Professor S. S. Cairns at the inauguration of Delyte Wesley Morris as President of Southern Illinois University on May 5, 1949; Professor D. C. Lewis, Jr., at the inauguration of Otto Frederick Kraushaar as President of Goucher College on May 6–7, 1949; Professor Lincoln LaPaz at the inauguration of Thomas Lafayette Popejoy as President of the University of New Mexico on June 4, 1949.

The following additional appointments by the President were reported: Professor R. W. Brink as a member for the period 1949–1951 and Professor C. B. Allendoerfer as Chairman for 1949 of the Committee on Places of Meetings (committee now consists of Professors C. B. Allendoerfer, Chairman, R. W. Brink, and R. L. Wilder); Professors William Feller (Chairman), G. A. Hedlund, and M. H. Heins as a Committee to Select Gibbs Lecturers for 1950 and 1951; Professors T. H. Hildebrandt (Chairman), S. S. Cairns, L. M. Graves, J. W. Green, and W. T. Martin as a Committee to Nominate Officers and Members of the Council for 1950; Professors A. J. Kempner (Chairman), J. R. Britton, H. M. Gehman, J. W. Green, C. A. Hutchinson, B. W. Jones, Claribel Kendall, and O. H. Rechard as a Committee on Arrangements for the 1949 Summer Meeting; Professor E. R. Lorch (Chairman), Miss Grace Bolton, Professors T. F. Cope, H. M. Gehman, B. P. Gill, T. R. Hollcroft, S. B. Littauer, and Walter Prenowitz as a Committee on Arrangements for the 1949 Annual Meeting; Professors B. P. Gill (Chairman), Samuel Eilenberg and P. A. Smith as a Committee to Study Special Methods of Reproducing Mathematical Research; Professors A. W. Tucker (Chairman), R. P. Boas, Samuel Eilenberg, J. R. Kline, R. E. Langer, and C. J. Rees as an Emergency Publication Committee; Professors T. F. Cope and B. P. Gill as auditors of the Society's accounts for 1949.

The following items were reported for the information of the Council: appointment of Professor L. M. Graves as Chairman of the Editorial Committee for the International Congress (to replace Professor Salomon Bochner, resigned); appointment of Dean A. E. Meder as an additional member of the Financial Committee of the Congress; selection of Professors Nelson Dunford and M. H. Stone as Chairmen of the Mathematical Surveys and Mathematical Reviews Editorial Committees, respectively, for 1949; establishment of a reciprocity agreement between the Society and the Norwegian Mathematical Society; receipt of a letter of thanks from London Mathematical Society for a gift of journals and books; acceptance from Professor Norbert Wiener of invitation to deliver the Gibbs Lecture at the 1949 Annual Meeting; approval by the Committee on Applied Mathematics of sponsorship by the Society of the following: Conference on Applied Mathematics of the American Institute of Electrical Engineers in New York City on January 31, 1949; Heat Transfer and Fluid Mechanics Institute at the University of California in June, 1949; Conference on Fluid Dynamics at the University of Illinois in the spring of 1950.

On recommendation of the Organizing Committee of the International Congress, the Council authorized extending invitations for hour addresses to twenty-seven mathematicians instead of to twenty-four, the maximum number recommended by the Eisenhart Report adopted by the Council in 1937. It was also reported that Congress fees had been set at $15.00 for full membership and $7.50 for associate membership.

The Council voted to recommend to the Trustees that the subscription prices for Mathematical Reviews be increased to the following: list price, $20.00 per volume; price to members of sponsoring organizations, $10.00; price to reviewers, $5.00.

It was reported that Professors Børge Jessen and F. B. Jones had been invited to deliver hour addresses at the 1949 Summer Meeting.

Recommendations were presented from the Emergency Publication Committee which had also been approved by the Committee on the Role of the Society in Mathematical Publication. The Council adopted the following recommendations, designed to relieve the Society's publication crisis:

(1) That the present green issues of the Bulletin be published as a separate new journal, to be known as the Proceedings of the American Mathematical Society. It is hoped that it may be possible to print the Bulletin (present green issues) by photo-offset process, with considerable saving to the Society. Until the question of individual membership dues is re-studied, members will receive both the Bulletin and the Proceedings.

(2) That a new medium of publication be created, as soon as possible, for longer papers and groups of cognate papers, this medium to be known as the Memoirs of the American Mathematical Society. The Memoirs are to be under the editorial control of the Transactions Editorial Committee, with the same standards as the Transactions. The Memoirs are to be a non-periodical but serial publication, in

this respect like Mathematical Surveys and the Colloquium Publications, and are to be issued and sold in separate volumes. The Memoirs are to be reproduced by photo-offset from copy completely prepared by the authors, following instructions formulated in a manual, but not restricted in style. Because of the financial commitments involved, the question of the Memoirs was referred to the Board of Trustees for final action. If approved by the Board, it is hoped that the Memoirs may be started late in 1949 or early in 1950.

The following items were referred to a special committee to be appointed by the President: the recommendation that the per page rate on which institutional membership dues are based be increased from $2.75 to $5.00; the suggestion that the terms of individual membership be reconsidered, in light of the changes in the Society's publication program suggested above; the question of the relation between the Society and institutional members which publish mathematical journals.

It was reported that the Policy Committee for Mathematics had been studying questions concerned with the possible establishment of a mathematical foundation for the purpose of securing financial support from industry and other interested groups, the recommendations to Selective Service in connection with students in all fields of academic endeavor, and an International Mathematical Union.

The Council, on recommendation of the Policy Committee, approved the by-laws and statutes of the International Union of Theoretical and Applied Mechanics and voted to continue the Society's representation on the United States Committee on Theoretical and Applied Mechanics, the group through which the United States has adhered to the Union, and to recommend to the Trustees that a contribution be made to this Committee.

The Council voted to schedule meetings as follows: November 25–26, 1949, at Illinois Institute of Technology; southeastern meeting of 1950 at Oak Ridge Laboratories; spring eastern meeting of 1950 in Washington, D. C.

The Council voted not to make available to authors reprints of articles appearing in the Proceedings of Applied Mathematics Symposia but voted that one copy of the whole volume should be given to each author.

Abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 336 was read by Mr. Kuhn, paper number 339 by Mr. Porcelli, paper number 342 by Professor Rosenbloom, paper number 365 by Miss Whitney, and paper number 390 by Dr. Blakers. Mr.

Szekeres was introduced by Professor T. R. Hollcroft, Mr. Moore by Professor Herbert Federer, and Mr. Shanks by Dr. A. H. Van Tuyl.

ALGEBRA AND THEORY OF NUMBERS

326t. A. A. Albert: *A theory of trace-admissible algebras.*

Power associative algebras have been defined by many sets of identities (such as those defining Jordan algebras, alternative algebras, right alternative algebras, and standard algebras) and have been studied successfully only by the use of a trace function. The author now takes a major step forward in the study of such algebras by the derivation of a complete structure theory of all trace-admissible power associative algebras whose radical is defined to be the maximal nilideal. Then semisimple algebras are direct sums of simple algebras, the attached algebra $A^{(+)}$ is always a semisimple Jordan algebra, and all (semisimple) simple algebras are either Jordan algebras, flexible algebras of degree two, or simple quasi-associative algebras (that is, algebras obtained from simple associative algebras by redefining the product to be $x \cdot y = \lambda xy + (1-\lambda)yx$ for λ in the base field). This reduces the study of power associative algebras defined by sets of identities to the study of the radical and the proof of trace-admissibility. It shows that algebras defined by new sets of identities will either require study by a new tool or will yield no new simple algebras. (Received April 7, 1949.)

327t. P. T. Bateman and Sarvadaman Chowla: *Averages of character sums.*

The authors consider the arithmetic means of the sums $S(n) = \sum_{m=1}^{n} \chi(m)$, where χ is a primitive residue character modulo k. Among the results obtained are a stronger form of Paley's Ω result for $S(n)$ and, for the case $\chi(-1) = 1$, a simple proof of Davenport's theorem that $|L(s, \chi)| < Ck^{(1-\sigma)/2}$, where s is a fixed complex number such that $0 < \sigma = R(s) < 1$ and C depends only on s. (Received March 18, 1949.)

328t. P. T. Bateman and Sarvadaman Chowla: *Remarks on the size of $L(1, \chi)$.*

Among other things it is proved that $\left| \sum_{n=1}^{\infty} (n/p)/n \right| < C \log p/(\log \log p)$ for infinitely many primes p, (n/p) being the Legendre symbol. (Much better results in this direction are of course known for real primitive characters to unrestricted moduli.) In the other direction it is proved that if χ is a primitive residue character modulo k, then $L(1, \chi) < C\phi(k) \log k/k$, where $L(s, \chi)$ is Dirichlet's function, ϕ is Euler's function, and C is a constant; this is valuable for those k which have many distinct prime factors. (Received March 18, 1949.)

329t. P. T. Bateman and Sarvadaman Chowla: *Proof of a statement of Paley.*

The statement proved is that if χ is a primitive residue character modulo k, b is a fixed positive number, and $n < k^{1/2}(\log k)^b$, then $\left| \sum_{m=1}^{n} \chi(m) \right| < Ck^{1/2} \log \log k$, where C is a constant depending upon b. (Received March 18, 1949.)

330t. Richard Bellman: *Ramanujan functions and expansions in algebraic number fields.* Preliminary report.

It is shown that Ramanujan functions, $C_q(n)$ (cf. Hardy, *Ramanujan's work*,

§11), can be defined over any algebraic number field, and that expansions analogous to those found by Ramanujan for $\sigma_{-s(n)}, d(n)$ and other number-theoretic functions hold for the arithmetic functions $s_{-s}(a) = \sum N(b)^{-s}$, $D(a) = \sum 1$, $(D(a) = S_0(a))$, where the summation is over the ideals b that divide the ideal a. These expansions can then be used to obtain mean values (cf. Bellman, Proc. Nat. Acad. Sci. U.S.A. (1948)). (Received March 21, 1949.)

331*t*. R. C. Buck: *Canonical mappings and algebraic structures.* I.

Let S and K be sets, $S \subset K$, such that no element of S is a subset of K. The ***hierarchy*** of S is defined to be the minimal set hS such that: (1) $S \subset hS \subset K$, (2) if $A \subset hS$ and $P(A) \subset K$, then $P(A) \subset hS$. Denote by $A:B$ the set of all functions mapping A into B. Then to each $f \in A:B$ there corresponds a unique function $\bar{f} \in hA:hB$, the maximal canonical extension of f. An element α of hS is said to be canonical if $\bar{f}(\alpha) = \alpha$ for every f which is a one-to-one mapping of S onto S. It is then shown that a large body of familiar structures, some of which are termed "natural" in the Eilenberg-MacLane theory, are, in the present sense, canonical elements, mappings, or equivalences. Illustration: Let G and F be sets, $G^* = G:F$, $G^{**} = G^*:F$. Then any multiplication $\bigcirc$ on G induces canonically a corresponding operation $\otimes$ on G^{**} which agrees with $\bigcirc$ under the canonical map of G into G^{**}. In the familiar case, $\otimes$ is convolution. (Received March 18, 1949.)

332*t*. R. C. Buck: *Canonical mappings and algebraic structures.* II.

An algebraic structure on the set S is a subset $\mathfrak{S}$ of hS, the hierarchy of S. Two algebraic systems $(A, \mathcal{A})$ and $(B, \mathcal{B})$ are isomorphic if there is a function f which maps A one-to-one onto B such that $\bar{f}$ maps $\mathcal{A}$ onto $\mathcal{B}$. Let $R = E:S$, the cartesian product of S over E. Problem: given a structure on S, is it possible to describe a general procedure for canonically inducing a "corresponding" structure on R, and which sorts of structure are admissible? Partial answers are given for this decision problem in the case of "elementary" algebras and topological algebras, formulated in terms of an analysis of the structure by canonical elements in general relation algebras. Example: if $(S, \mathfrak{S})$ is an elementary algebra, then any statement about the structure of S which may be expressed as a conjunction of statements $p \rightarrow q$, where p and q are conjunctions of atomic statements, is admissible. (Received March 18, 1949.)

333*t*. Leonard Carlitz: *Congruences for the coefficients of the Jacobi elliptic functions.*

Let k^2 be integral (mod p), where p is a fixed odd prime, and put $\mathrm{sn}^h (x, k^2) = \sum_m a_m^{(h)} x^m/m!$, so that the $a_m^{(h)}$ are integral (mod p). It is proved that $\sum_{i=0}^{r} (-1)^i C_{r,i}\, a_p^{r-i}\, a_{m+i(p-1)}^{(h)} \equiv 0$ (mod p^r) for $m \geqq r \geqq 1$. Using this result and the multiplication theorem for sn x, it is proved that $\sum_{i=0}^{r} (-1)^i C_{r,i}\, a_p^{r-i} \tau_{m+i(p-1)} \equiv 0$ (mod p^r), where $m > r \geqq 1$, $p-1 \nmid m$, $\tau_m = \beta_m/m$ and $x/\mathrm{sn}\, x = \sum \beta_m/x^m/m!$. (Received April 8, 1949.)

334. Mary P. Dolciani: *On Waring's problem with non-integral exponents.*

Let c be a real, non-integral number greater than one. B. Segal (Ann. of Math. vol. 36 (1935) pp. 507–520) has discussed the number of representations of any sufficiently large positive integer in the form $\sum_{i=1}^{s} [x_i]$ where each x_i is an integer. The result which he obtained corresponds fairly well to the inequality $G(k) \leqq (k-2)2^{k-1}+5$ by Hardy and Littlewood in Waring's problem with integral kth powers. The author improves Segal's result by employing methods and inequalities developed largely by Vinogradow. (Received March 18, 1949.)

335*t*. Herbert Federer and Bjarni Jónsson: *Some properties of free groups.*

In this paper the methods introduced into the theory of free groups by Jacob Nielsen are applied and extended. It is shown that if Nielsen reductions are applied *at random* to a finite sequence of elements of a free group, then the process will terminate after finitely many steps and yield a free generating set for the subgroup generated by the given elements. It is proved that if G is freely generated by the set X and if a subgroup H of G is well ordered in any manner consistent with X-length, then H is freely generated by the set of all those elements which are not expressible in terms of preceding elements. The homomorphisms of free groups into free groups are completely described by the theorem that if f is a homomorphism of a free group G onto a free group H, then G is expressible as a free product of two subgroups such that f maps one factor isomorphically onto H and f maps the other factor onto the identity element of H. The answer to the question whether a given finite subset of a free group generates a free factor of this group is shown to be finitely computable. (Received February 15, 1949.)

336. David Gale, H. W. Kuhn, A. W. Tucker: *Symmetrizing a game.*

Let G be a zero-sum two-person game with m pure strategies for "white" and n for black." Form a symmetric game S with $m+n+1$ pure strategies for each player to correspond, respectively, to the m white and the n black strategies of G and the privilege of "calling off." For a white strategy against a black, or vice versa, the pay-off in S is just that in G; against a white or a black strategy a call-off wins or loses one unit, respectively; otherwise the pay-off in S is zero. *Theorem*: A necessary and sufficient condition that the game G have a *positive* value v for white and have optimal mixed strategies $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_n)$ for white and black, respectively, is that the symmetric game S have an optimal mixed strategy proportional to $(x_1, \cdots, x_m, y_1, \cdots, y_n, v)$. (Received March 18, 1949.)

337*t*. W. J. LeVeque: *On number-theoretic functions of two variables.*

The author investigates the possibility of extending the definition of multiplicativity for functions of one variable to functions of two variables, under the restriction that the function corresponding to $\sum_{d|a} f(d)$ should also then be multiplicative in the same sense. It is shown that of the four definitions proposed, the only one satisfying this requirement is such that every function f of this sort is the product of two multiplicative functions of one variable, that is, that $f(a, b) = g(a)h(b)$. It is also shown that if $f(a, b)$ is a function of the single variable $d = (a, b)$, say $f(a, b) = \psi(d)$, and if $\sum_{t=1}^{\infty} \psi(t)/t^2$ converges absolutely to the number A, then $\lim (1/n^2) \sum_{a,b=1}^{n} \psi((a, b)) = 6A/\pi^2$. (Received March 15, 1949.)

338*t*. W. J. LeVeque: *On the number of representations of an integer as a sum of consecutive integers.*

Let $\gamma(m)$ be the number of representations of m in the form $(r+1)+(r+2)+ \cdots$

$+s$, where $s > r \geqq 0$. It is shown that $(1/n) \sum_1^n \gamma(m) = (1/2) \log n + (2\gamma + 2 \log 2 - 1)/4 + O(n^{-1/2})$, and that if $s_n(\omega)$ is the number of positive integers $m \leqq n$ for which $\gamma(m) \leqq 2^{\log \log n + \omega(\log \log n)^{1/2} - 1}$, then $\lim_{n \to \infty} s_n(\omega)/n = 1/(2\pi)^{1/2} \int_{-\infty}^{\omega} e^{-x^2/2} dx$, for every real ω. The proofs depend on the relation $\gamma(m) = \tau(\bar{m})$, where $\tau(n)$ is the number of divisors of n, and $\bar{m}$ is the largest odd divisor of m. Generalizations are indicated for the number of representations of m as a sum of integers in any of the arithmetic progressions with fixed difference $d > 1$. (Received March 15, 1949.)

339. Pasquale Porcelli and Gordon Pall: *A theorem on quadratic residues.*

By using a fundamental property of Farey Series the following theorem is proved: "If p is an odd prime and D is the quadratic residue of p, and if the integers h and k satisfy $h - 1 \leqq p$ and $k = [p/h]$, then at least one of $1^2, 2^2, \cdots, k^2$ will be congruent to at least one of $D, 4D, \cdots, (h-1)^2D$. The case $h = 2$ is well known. (Received March 16, 1949.)

340. R. D. Schafer: *Inner derivations of non-associative algebras.*

A definition of inner derivation for non-associative algebras is given, and is applied to alternative and Jordan algebras. (Received April 21, 1949.)

341*t*. G. Szekeres: *Determination of a certain family of finite metabelian groups.*

The object of the author is to determine all the finite abstract groups G which have an abelian invariant subgroup A with cyclic quotient group $F = G/A$. It is assumed that for any $p \mid o(A)$, $p^2 \nmid o(F)$. The algebraic equivalent of the problem is to set up all the finite abelian operator p-groups whose ring of operators is homomorphic to $R = I[x]/(x^n - 1, p^k)$ where $I[x]$ is the ring of polynomials in x with integral coefficients, p is a prime and $(n, p^2) < p^2$. R is not a principal ideal ring, hence an indecomposable R-group is not necessarily "cyclic." The different types of indecomposable R-groups are completely determined and characterized by a set of numerical invariants. The solution for the corresponding abstract group-theoretical problem is then obtained in a fairly straightforward manner. Among the numerous applications perhaps the most significant is the determination of all the p-groups which have abelian subgroups of index p. An asymptotic formula is obtained for the number of such groups of order p^n if n is fixed and p is large, also an estimate for the order of magnitude of this number if $p = 2$ and n is large. (Received February 11, 1949.)

ANALYSIS

342. J. Aczel and P. C. Rosenbloom: *On quasi-linear operations.*

Let S be a linear vector space over the field of real numbers, E be a convex subset of S, and let U be a real-valued function on E such that (1) $U((f_1+g)/2) = U((f_2+g)/2)$ whenever $f_1, f_2, g \in E$ and $U(f_1) = U(f_2)$, (2) $U(tf + (1-t)g)$ is continuous in $0 < t \leqq 1$ for every $f, g \in E$. Then there is a linear functional L on S and a strictly monotonic continuous function ϕ such that $U(f) = \phi(L(f))$ for all $f \in E$. This implies results of de Finetti and others on mean values. Conditions are obtained under which (1) and the monotonicity of U in a suitable sense imply (2). A representation theorem is also obtained in the case of an operation U whose range is an arbitrary linear topological space. (Received February 15, 1949.)

343. Lipman Bers: *Boundary value problems for minimal surfaces with a singularity at infinity.*

The method announced previously for proving the existence of a gas flow obeying the linearized pressure-volume relation past a given convex obstacle (see the paper *On existence theorem in two-dimensional gas dynamics* to appear in the Proceedings of the First Symposium on Applied Mathematics) is extended to various other boundary value problems for minimal surfaces with a prescribed singularity at infinity. A typical problem is: to determine a minimal surface which winds itself around a given cylinder, cuts the cylinder under a right angle, possesses at infinity a tangent plane orthogonal to the directrix of the cylinder. Existence theorems (but no uniqueness theorems) are proved for this and other problems involving the integration of the minimal surface equation in a domain exterior to a curve P. The condition that P be convex used in the previous paper is replaced by a weaker (though probably still unnecessarily restrictive) condition. The existence proof consists in reducing the boundary value problem to a mapping problem which is shown to be equivalent to a nonlinear integral equation. Using the Léray-Schauder method and the theory of conjugate functions, it is shown that the integral equation has a solution. (Received March 18, 1949.)

344*t*. Salomon Bochner and K. Chandrasekharan: *On Fourier series of L_2 functions.*

If $f(x)=f(x_1, \cdots, x_k)$ is a periodic function in each variable, belonging to the Lebesgue class L_2, and if (*) $\int_u^{2u} \{S^\delta(R)\}^2 dR = o(u)$ as $u \to \infty$, for some $\delta > (k-1)/2$, where $S^\delta(R)$ is the δth Riesz mean of the Fourier series of f, when summed spherically, then $\int_0^t \{f_p(u)\}^2 du = o(t)$ for $p=\delta+(3-k)/2$, $t \to 0$, where $f_p(u)$ is the pth spherical mean of f; and the statement holds for $\delta=(k-1)/2$ provided that (*) holds uniformly in an interval $(k-1)/2 \leqq \delta < \delta_0$. This is reciprocal to a previous theorem of the authors (Ann. of Math. vol. 49 (1948) pp. 966–978), and seems to be new even in the case of one variable. (Received March 16, 1949.)

345. W. M. Boothby: *Topology of the level curves of harmonic functions with critical points.*

It was proved by W. Kaplan, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 514–522, that (1) for every curve family F filling the xy-plane which is regular (that is, locally homeomorphic to parallel lines) at every point of the plane, there exists a simply-connected domain D of the xy-plane and a homeomorphism carrying the plane onto D such that F is transformed onto the family of level curves of a function $f(x, y)$ harmonic in D; and (2) the family F may be decomposed into non-overlapping, simply-connected subfamilies in each of which F is homeomorphic to the parallel lines of a half-plane. (1) gives a topological characterization of the level curves of a harmonic function with nonvanishing derivatives in a simply-connected domain; and (2) gives a decomposition of the Riemann surface of the inverse of a function which is analytic with nonvanishing derivative in a simply-connected domain into fundamental domains. In the present paper the author extends (1) and (2) to the case of curve families with a finite or infinite number of isolated singularities of the saddle or branch point type, and incidentally proves that a family of this type is homeomorphic to the solution family of a system of differential equations. (Received February 23, 1949.)

346*t*. D. G. Bourgin: *Approximately multiplicative transformations on continuous function rings.*

Several definitions are given for approximately multiplicative transformations. For instance, the departure from strict multiplicativity of the transform of a product may be dominated by the infimum of one function multiplied by the norm of the other when, say, the underlying spaces are compact. Under weak auxiliary conditions the homeomorphism of the spaces can be asserted. (Received March 28, 1949.)

347*t*. O. J. Farrell: *On approximation by nonvanishing functions.* I.

Let $w=f(z)$ be analytic and bounded and have k zeros in a limited simply connected region G. Let S denote the closed set of functions $F(z)$ which are analytic in G and vanish there either identically or not at all. For every $F(z)$ there is a $\mu=$upper bound $[|F(z)-f(z)|$, z in $G]$ and there is at least one $F(z)$ for which μ equals the greatest lower bound M of all μ. Let R denote the Riemann configuration over the w-plane onto which G is mapped by $w=f(z)$. Let $w=\delta$ be a point nearest the origin $w=0$ from among the boundary points of R which do not correspond also to interior points of G. Let D denote the largest of the radii of k-valence (see Seidel and Walsh, Trans. Amer. Math. Soc. vol. 52 (1942) p. 162) at the points $w_0^{(1)}, w_0^{(2)}, \cdots, w_0^{(m)}$ of R which lie over the origin $w=0$. It is found that $D \leqq M \leqq |\delta|$ and that every function $F_1(z)$ for which upper bound $[|F_1(z)-f(z)|$, z in $G] \leqq D' < D_h(w_0^{(h)})$ has at least as many zeros in G as the number of sheets belonging to the region $R_h(w_0^{(h)})$. In this initial paper better appraisals on M are obtained in certain restricted cases. Approximation by nonvanishing rational functions is considered. (Received March 17, 1949.)

348. O. J. Farrell: *On approximation to a k-valent function by functions of lesser valence.* I.

Let $w=f(z)$ be analytic and bounded and k-valent, $1<k$, in a limited simply connected region G. The class of functions which are analytic and either h-valent at most or identically constant in G form a closed set T_h. For every function $g(z)$ of this set there is a $\lambda_h=$upper bound $[|g(z)-f(z)|$, z in $G]$ and in the set T_h there is at least one function whose λ_h equals the greatest lower bound L_h of all λ_h. With each point $w=w_0$ covered by the Riemann configuration R onto which G is mapped by $w=f(z)$ there can be associated a greatest radius $C_j(w_0)$ of precise j-coverage. Let C_j denote the greatest of all $C_j(w_0)$ and C_j^* the greatest of all C_i for which $j \leqq i$. Denote by K the radius of the smallest circular biscuit cutter which will contain R. For $h<k$ it is found that $C_{h+1} \leqq L_h \leqq K$. Some results are also obtained on the order of multivalence of approximating functions. Included is an initial study of approximation by rational functions of specified valence. (Received March 17, 1949.)

349. Gerald Freilich: *On the measure of Cartesian product sets.*

Let F_n^k represent the k-dimensional integral-geometric (Favard) measure over Euclidean n-space, E_n (see H. Federer, *The* (ϕ, k) *rectifiable subsets of n-space*, Trans. Amer. Math. Soc. vol. 62 (1947) p. 122). If the following five conditions hold: (1) A and B are perpendicular subspaces of E_n of dimensions a and b respectively, (2) $k \leqq a$, (3) S is an F_n^k-measurable subset of A with $F_n^k(S) < \infty$, (4) T is an F_n^b-measurable subset of B with $F_n^b(T) < \infty$, (5) $S+T=\{(s+t) | s \in S \text{ and } t \in T\}$, then it is shown that $F_n^{k+b}(S+T)=F_n^k(S) \cdot F_n^b(T)$. Besicovitch's example, which shows that the correspond ing statement for Hausdorff measure is false, is examined in detail. In addition, for $X \subset E_n$, $k \leqq m \leqq n$, a new formula is obtained which, in terms of the multiplicity integral with respect to F_m^k of the perpendicular projections of X into m-planes, expresses $F_n^k(X)$ as an integral over the set of all these m-planes. (Received March 1, 1949.)

350*t*. W. H. Ingram: *The 2-point 1-dimensional generalized boundary value problem.*

The problem $dy - dH(x, \mu) = 0$, $Ly(a) + Ry(b) = 0$, H an $n \times n$ matric function in $\mathfrak{B}'(x) \subset \mathfrak{B}(x)$ uniformly and analytic in μ, $dy = y(x+) - y(x) + y'(x+)dx$, $dx > 0$, $y'(x+) \equiv \lim\ [y(x+\delta) - y(x+)]/\delta$, $\delta > 0$, requires the integration of weighted differentials fdg in which f and g have simultaneous discontinuities. The integral $\int_a^b fdg$ is the Smith-More limit of the interval function $\sum f^U(x_i+)[g(x_{i+1}) - g(x_i)]$ $f^U(x+) \equiv \mathrm{glb}_{\delta > 0}\ \mathrm{lub}_{\xi \text{ in } S,\ x < \xi \leqq x+\delta} f(\xi)$, $g(x)$ is in $\mathfrak{B}(x)$ the class of functions of bounded variation, and f is any bounded function in $\mathfrak{D}_{(2)}$, the class consisting of all functions on $\mathfrak{S}$ to $\mathfrak{R}$ with the property that to each $\epsilon > 0$ corresponds a set σ_ϵ of real points on $[ab]$ such that the oscillation, defined on $\mathfrak{S}$ to $\mathfrak{R}$, at every other real point is less than ϵ. If g is in $\mathfrak{B}'$, the continuous part of g has a finite derivative everywhere and the discontinuities of g form a point-set of the first species. The set $\mathfrak{S}$ contains the real numbers and all *satellite* numbers $x(+)$, $x(-)$; if $x_1(+)$ denotes the point, asserted to exist, common to all the open intervals $(x_1 < x < x_1 + \delta)$, $\delta > 0$, the corresponding number is $x_1(+)$; similarly for $x_1(-)$. The Archimedean property does not extend to $\mathfrak{S}$ and $\mathfrak{S}$ is not a field, but $(\mathfrak{S}, >)$ forms a linearly ordered set having axcribable, requisite metric properties and the Dedekind property is asserted for $\mathfrak{S}$. (Received April 8, 1949.)

351*t*. Hermann Kober: *On a monotone singular function and on the approximation of analytic functions by nearly analytic functions in the complex domain.*

First certain monotone functions $\omega(t)$ are treated which are continuous and are constant in $\langle 0, T \rangle$ on the intervals of an open set of measure T $(0 < T < \infty)$. A subclass of them is well known. Here they are dealt with from two new points of view. (A) The inverse functions, strictly increasing jump-functions, admit of plain representations. (B) $\omega(t)$ is defined by two very simple functional equations; it is subadditive, and it satisfies some more equations and inequalities. Then by means of $\omega(t)$, "nearly analytic" functions are constructed. They satisfy a Lipschitz condition, and this property and others are discussed. If $f(z)$ and $g(z)$ $(z = x + iy)$ are any two functions which, for $|z| < 1$, are analytic and have bounded derivatives, then, given $\epsilon > 0$, there exists a nearly analytic $H(z)$ such that $|f(z) - H(z)| < \epsilon$ $(|z| \leqq 1)$ and that $H'(z) = g'(z)$ for almost all z. (Received February 17, 1949.)

352*t*. Norman Levinson. *An ordinary differential equation with an interval of stability, a separation point, and an interval of unstable oscillation.*

The second order differential equation $\epsilon u'' - 4u'/[3 + (u')^4] + u = 0$, where $u' = du/dx$, is considered for small $\epsilon > 0$. A solution with initial values at $x = 0$ given by u_0 and u_0' will, if $|u_0| < 1$ and $|u_0'|$ is not large, stay very close to a monotone solution $z(x)$ of the degenerate equation $-4z'/(3 + (z')^4] + z = 0$ obtained by putting $\epsilon = 0$. However as $z(x) \to 1$ (or -1), which will occur for some finite value of $z = z_1$, $z(x)$ cannot be continued further in the range of real values. As x gets near to x_1 the solution $u(x)$ nears 1 and then turns rapidly downward to approximately -1. The solution $u(x)$ then continues to oscillate rapidly between approximately $+1$ and -1 with a "period" of approximately $2\pi\epsilon^{1/2}$. In the range $0 < x < x_1$, $u(x)$ is highly stable. Beyond x_1, $u(x)$ is highly oscillatory and is unstable in "phase." Separation between the stable and oscillatory intervals occur at a value of x which tends to x_1 as $\epsilon \to 0$. The method can be used for a broad class of equations of the form $\epsilon u''+f(x, u, u', \epsilon)=0$. (Received March 4, 1949.)

353t. Norman Levinson: *Boundary value problem for* $\epsilon\Delta u+A(x, y)u_x+B(x, y)u_y+C(x, y)u=D(x, y)$ *for small* ϵ.

Under quite general conditions it is shown that the solution $u(x, y, \epsilon)$ of the first boundary value problem for small $\epsilon>0$ exists in a simply or multiply connected domain R and that at interior points in most of R, lim $u(x, y, \epsilon)$ as $\epsilon\rightarrow+0$ exists and is a solution of the first order equation with $\epsilon=0$, $AU_x+BU_y+CU=D$ which assumes the given boundary value on certain arcs of the boundary. Let S be the boundary of R. More specifically (*) $u(x, y, \epsilon)=U(x, y)+z(x, y, \epsilon)+O(\epsilon^{1/2})$ for small ϵ uniformly in each of a finite number of regions Q_i into which $R+S$ can be divided where z is of the form $h(x, y) \exp(-g(x, y)/\epsilon)$ near certain arcs of the boundary and is zero on the rest of the boundary. Moreover $g=0$ on these arcs of the boundary and $g>0$ in R. The term z is the "boundary layer" term. More precise results are possible where the error is $O(\epsilon^{n+1/2})$ for any fixed n and more general cases of the equation can be handled. In the exact formulation the solutions of the equation $dx/A=dy/B$ play a fundamental role. (Received March 4, 1949.)

354t. Norman Levinson: *The inverse Sturm-Liouville problem.*

It is assumed that the boundary conditions and characteristic values of $y''+(\lambda-P(x))y=0$, $0\leqq x\leqq\pi$, are known and the problem is to find $P(x)$. $P(x)$ is assumed integrable over $(0, \pi)$. Let the boundary conditions be $y(0)\cos\alpha+y'(0)\sin\alpha=0$, $y(\pi)\cos\beta+y'(\pi)\sin\beta=0$. Then if (*) $P(\pi-x)=P(x)$ and if $\alpha+\beta=\pi$, $P(x)$ is uniquely determined by the spectrum. It is also shown without assumption (*) that if the spectrum is known for the above boundary condition and for another boundary condition with β replaced by γ (where of course $\sin(\beta-\gamma)\neq0$) then $P(x)$ is uniquely determined. These results are similar to earlier results of Borg. The method used is a simplified form of a proof that under certain conditions for the differential equation with $0<x<\infty$, the asymptotic phase determines $P(x)$ and makes use of a pseudo Green's function involving λ where λ is regarded as a complex variable. (Received March 4, 1949.)

355t. Norman Levinson: *Uniqueness of potential for given asymptotic phase with inverse square term present in wave equation.*

The solution $y(x, \lambda)$ of $y''+(\lambda^2-V(x)-l(l+1)/x^2)y=0$ with boundary condition $y(x, \lambda)/x^{l+1}\rightarrow1$ as $x\rightarrow+0$ exists if $\int_0^\infty x|V(x)|dx<\infty$. Moreover for any $\lambda\rightarrow0$, $\lambda^{l+1}y(x, \lambda)-A(\lambda)\sin(\lambda x-l\pi/2-\phi(\lambda))\rightarrow0$ as $x\rightarrow\infty$. The function $\phi(\lambda)$ is the asymptotic phase. If $V(x)+l(l+1)/x^2\geqq0$ then $\phi(\lambda)$ determines $V(x)$ uniquely. Also there exists a function of λ, $F(\lambda)$, analytic in the upper half λ plane and free of zeros such that on the real axis, $F(\lambda)=A(\lambda)\exp(i\phi(\lambda))$. Thus $\phi(\lambda)$ and $\log A(\lambda)$ are conjugate functions on the real axis. The proof is similar to that for the case $l=0$ in a paper to appear shortly in Det. Kgl. Danske Videnskabernes Selskab. (Received March 4, 1949.)

356. E. R. Lorch: *Differentiable inequalities.* Preliminary report.

The type of inequality treated is $\sum_1^n x_iy_i\leqq\psi(x)\phi(y)$. Here x represents the vector $(x_1, \cdots, x_n)$ and ϕ and ψ are functions which characterize the inequality. Contrary to first impressions, existing means do not seem to provide "correct" methods for establishing these inequalities. The method used applies to convexity and concavity. The principal features are the use of differentiability, homogeneity, the definition of convexity (concavity) by means of positive definite quadratic forms, and the introduction of a characteristic transformation between subsets of two spaces. Consider the case of convexity. Let $G(x)=G(x_1, \cdots, x_n)$ be a function positively homogeneous of degree $r>1$ and differentiable sufficiently often; let $G_i(x)$ and $G_{ij}(x)$ denote $\partial G/\partial x_i$ and $\partial^2 G/\partial x_i \partial x_j$. Let Ω be a ray-space in the x space. Let $\sum G_{ij}(x)z_i z_j$ be a positive definite form for all $x \in \Omega$. Then $x_i^* = G_i(x)$ defines a 1–1 transformation of Ω to a ray-space Ω^* in the x^* space. The inverse transformation is $x_i = F_i(x^*)$. The functional equation for $\phi(x)>0$: $\max_y(\sum x_i^* y_i)/\phi(y) = (\sum x_i^* x_i)/\phi(x)$ has a unique solution $\phi(x) = [\sum x_i^* x_i]^{1/r}$. Setting $\psi(x^*) = [\sum x_i^* x_i]^{(r-1)/r}$ we have $\sum x_i^* y_i \leqq \psi(x^*)\phi(y)$ for all $y \in \Omega$, $x^* \in \Omega^*$. The situation between the x and x^* space is completely symmetric. The theory of approximation of non-differentiable by differentiable inequalities will be treated in another communication. (Received March 21, 1949.)

357. G. W. Mackey: *On the unitary representations of subgroups of locally compact groups.* Preliminary report.

Let G be a separable locally compact group and let G_0 be a closed subgroup of G. Let $h \rightarrow V_h$ be a suitably continuous representation of the ring of all complex Borel functions on the space G/G_0 of right cosets of G_0 into the ring of all bounded linear operators on a Hilbert space H. Let $\sigma \rightarrow U_\sigma$ be a continuous unitary representation of G in H. V, U is called a system for G, G_0 if $U_\sigma V_h = V_{h_\sigma} U_\sigma$ for all h and σ where $h_\sigma(x) = h(x\sigma)$. Every system has associated with it in a natural way a continuous unitary representation of G_0 in a Hilbert space H'. Two systems are unitary equivalent if and only if the corresponding representations of G_0 are unitary equivalent. A system is irreducible if and only if the corresponding representation of G_0 is irreducible. Under a (probably superfluous) hypothesis regarding the possibility of selecting an element from each coset of G_0 in a suitably regular manner it can be shown that every continuous unitary representation of G_0 is associated with some system. If G_0 is the identity these results are equivalent to a theorem announced in an earlier abstract (Bull. Amer. Math. Soc. Abstract 55-3-153). (Received March 21, 1949.)

358t. E. F. Moore: *The two-dimensional measures of Carathéodory and Hausdorff.* Preliminary report.

Let C and H be the two-dimensional measures of Carathéodory and Hausdorff over Euclidean 3-space E_3. It is shown that $C(A) \leqq H(A) \leqq (\pi(1+3^{1/2})/2)C(A)$ for all subsets A of E_3, and that $0 < C(A) < H(A) < \infty$ in case A is the set of all those points of E_3 each of whose three coordinates is between 0 and 1 and has an octal expansion in which only the digits 0, 1, 2, 3 occur. Generalizations to higher dimensions are also considered. (Received February 21, 1949.)

359t. J. C. Oxtoby: *A minimal set that is not metrically transitive.*

In the compact metric space of all mappings $x(n)$ of the integers into the set $\{0, 1\}$, acted on by the homeomorphism $Tx = x(n+1)$, it is shown that there exists a (regularly) almost periodic point x with the property that the sequence $x(n)$, $n>0$, does not have a density. The orbit closure X of any such point provides an example of a minimal set in which two distinct invariant Borel measures with $\mu(X) = 1$ can be introduced. For the method used in defining these measures, see Oxtoby and Ulam (Ann. of Math. vol. 40 (1939) pp. 560–566). The sum of these two measures is a measure with respect to which T is not metrically transitive. (Received March 23, 1949.)

360*t*. A. M. Peiser: *Uniform approximations to a class of Bessel functions.*

The functions $K_n(x, y) = (y/x)^{n/2} I_n(2(xy)^{1/2}) \exp(-x-y)$, $n = 0, 1, 2, \cdots$, arise in the solution of the differential equation $\partial^2 t/\partial x \partial y + \partial t/\partial x + \partial t/\partial y = 0$. When boundary values $t(x, 0)$ and $r(0, y)$ are specified, $t(x, y)$ can be expressed in terms of integrals of the form $\int_0^\infty f_n(x-u) K_n(u, y) du$, $n = 0, 1$, where f_n is known. Boundary conditions of the form $t(x, 0) = a + bt(S-x, P)$ lead to the integral equation $F(x) + bt(x, 0)e^{-P} + b\int_0^x t(u, 0) K_1(x-u, P) du = t(S-x, 0)$, where F is known. In either case, it is convenient to have approximations to K_n which hold *uniformly* in x. In this paper, it is shown that $K_n(x, y)$ can be approximated uniformly in $0 \leqq x < \infty$, for sufficiently large values of y, by means of the normal probability function $\Phi(x) = \exp(-x^2/2)/(2\pi)^{1/2}$. It is shown, for example, that for $y \geqq 10$, $|K_1(x, y) - \Phi((x-y)/(2y)^{1/2})/(2y)^{1/2}| < 0.167y^{-1} + 0.800y^{-3/2} + 0.694y^{-1/2}e^{-y}$. A similar estimate is obtained for $K_0(z, y)$. Better approximations to K_n are obtained by asymptotic series involving the successive derivatives of $\Phi(x)$. (Received February 23, 1949.)

361. F. W. Perkins: *Mean value properties of polyharmonic functions.* Preliminary report.

By the use of Almansi's theorem expressing a polyharmonic function in terms of harmonic functions, various mean value properties of harmonic functions are generalized for polyharmonic functions. Parts of the paper are related to earlier work of the author (Amer. J. Math. vol. 61 (1939) pp. 217–230) and to two papers by Poritsky (Amer. J. Math. vol. 54 (1932) pp. 667–691, and Trans. Amer. Math. Soc. vol. 43 (1938) pp. 199–225). (Received March 18, 1949.)

362*t*. Hans Rademacher and I. J. Schoenberg: *Helly's theorems on convex domains and Tchebycheff's approximation problem.*

The main purpose of the authors is to demonstrate the close connection between Helly's theorems on convex domains having a common point and the problem of best approximation in the sense of Tchebycheff. This new connection benefits both subjects thus brought into contact: One result is a direct geometric proof of Helly's theorems, which, unlike other proofs, makes no use of mathematical induction; another result is an application of Helly's theorems to derive properties of the polynomial of best approximation to discontinuous functions. As further examples of the versatility of Helly's ideas, simple proofs are given for a recent theorem of M. Dresher and T. E. Harris as well as for an interesting separation theorem of Paul Kirchberger (Dissertation, Göttingen, 1902). (Received March 16, 1949.)

363*t*. J. F. Ritt: *Abel's theorem and a generalization of one-parameter groups.*

For any positive integer p, an associative operation is defined which replaces any set of $p-1$ complex numbers by a set of p numbers. The operation is found to be described by relations analogous to those satisfied by the integrals of the first kind on a Riemann surface of genus p. (Received March 24, 1949.)

364. Arthur Sard: *Remainders: functions of several variables.*

Remainders in the theory of approximation are often functionals which are linear on certain function spaces and zero whenever their argument is a polynomial of degree $n-1$. For spaces of functions of one variable, known mass and kernel theorems, due to F. Riesz, Peano, and Rémès, give integral forms of such functionals in terms of nth derivatives. The author extends the mass and kernel theorems to spaces of functions of several variables. Among the results is the following. Let a, b, p, q be fixed: $0 \leqq a, b \leqq 1$; p, q non-negative integers. Put $n = p+q$. Denote by $B_{p,q}$ the space of functions $x = x(s, t)$ for which the nth partial derivatives: $x_{n-j,j}(s, b), j<q$; $x_{p,q}(s, t)$; $x_{i,n-i}(s, t), i<p$ exist and are continuous in s, (s, t), t, respectively, on $0 \leqq s, t \leqq 1$. (The order of differentiation in $x_{i,j}$ is restricted: If $i>p$ the last $i-p$ differentiations are to be with respect to s; likewise if $j>q$.) Define the norm in $B_{p,q}$ as $\|x\| = \max [|x_{i,j}(s, t)|, |x_{i',j'}(s, b)|, [x_{i'',j''}(a, t)|]$, where the maximum is taken over $0<s, t \leqq 1$, $i \leqq p, j \leqq q, i'+j' \leqq n, i'>p, i''+j'' \leqq n, j''>q$. Let Rx be a functional, additive and continuous on $B_{p-1,q-1}$, $p \geqq 1$, $q \geqq 1$, and zero whenever $x = x(s, t)$ is a polynomial of degree $n-1$ in (s, t). Then there exist kernels $f^j(s), j<q$; $g(s, t)$; $h^i(t), i<p$; such that $Rx = \sum_{j<q} \int_0^1 x_{n-j,j}(s, b) f^j(s) ds + \int_0^1 \int_0^1 x_{p,q}(s,t) g(s,t) ds dt + \sum_{i<p} \int_0^1 x_{i,n-i}(a, t) h^i(t) dt$ whenever $z \in B_{p,q}$. The kernels are functions of bounded variation and are given by explicit formulas. (Received March 16, 1949.)

365. I. J. Schoenberg and Anne M. Whitney: *On the positivity of translation determinants for Pólya frequency functions.*

A function $\Lambda(x)$, $-\infty < x < \infty$, is called a Pólya frequency function (P.f.f.) if it is measurable, with $\int_{-\infty}^{\infty} \Lambda(x) dx = 1$, and satisfies the condition: If $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_n$, the $D = \det \|\Lambda(x_i - y_k)\| \geqq 0$, $n = 1, 2, \cdots$. It was shown (Proc. Nat. Acad. Sci. U. S. A. vol. 33 (1947) pp. 11–17) that every P.f.f. $\Lambda(x)$ has a Laplace transform $\int_{-\infty}^{\infty} \exp(-xs) \Lambda(x) dx = 1/F(s)$, where $F(s) = \exp(-\gamma s^2 + \delta s) \cdot \prod_{\nu=1}^{\infty} (1+\delta_\nu s) \exp(-\delta_\nu s)$ with all constants real, $\gamma \geqq 0$, $0 < \gamma + \sum \delta_\nu^2 < \infty$. The authors solve the problem of deciding when $D>0$. The results are as follows: 1. If $\gamma > 0$, then always $D>0$. 2. If $\gamma = 0$ there are two cases: 2a. If $\sum |\delta_\nu|$ diverges then again always $D>0$. 2b. If $\sum |\delta_\nu|$ converges, then by a shift of origin assume that $\int_{-\infty}^{\infty} \exp(-xs) \Lambda(x) dx = 1/\sum (1+\delta_\nu s)$. Let there be k positive δ_ν and h negative δ_ν $(0 \leqq k, h \leqq \infty)$; then $D>0$ if and only if the n inequalities (1) $x_{j-k} < y_j < x_{j+h}$ $(j = 1, \cdots, n)$ hold, where $x_r = -\infty$ if $r \leqq 0$ and $x_r = +\infty$ if $r > n$. The trivial case $k+h=1$, when $\Lambda(x)$ becomes a truncated exponential, is exceptional in that equality is allowed on one side of (1) depending on how $\Lambda(x)$ is defined at its discontinuity. These results contain as special cases all known positive determinants and many new ones. (Received March 21, 1949.)

366. Daniel Shanks: *Mathematical sequences treated as transients.* Preliminary report.

Given a sequence A_n, pass a curve through $2K+1$ points (n, A_n) of the sequence, $N-K \leqq n \leqq N+K$, of the form $A_n = B_{KN} + \sum_{i=1}^{K} a_i e^{\alpha_i n}$ with the $2K+1$ unknowns B_{KN}, a_i and α_i, and solve for B_{KN}. B_{KN} is the ratio of two $K+1$ order determinants whose elements are the A's and their differences. Two types of nonlinear sequence-to-sequence transforms are $A_N \rightarrow B_{NN}$ and $A_N \rightarrow B_{KN}$ (with K fixed). Both types may be iterated. The author shows the first-order iterated transform, $A_N \rightarrow B_{1N} \rightarrow C_{1N} \rightarrow \cdots$, to be a useful summation process. Applied to the first nine terms of the series $\sum_0^\infty (-1)^n (n+1)^{-1}$, $\sum_0^\infty (-1)^n n!$, $1/2 + \sum_0^\infty B_{2n} (2n)^{-1}$ the transform gives ln 2, $\int_0^\infty e^{-t} (1+t)^{-1} dt$, and Euler's C, correct to 9, 6, and 7 places, respectively. The transform $A_N \rightarrow B_{NN}$, when applied to the power series of analytic functions, yields known rational approximations to, and continued fraction representations for, these functions. Applications to other types of sequences, to iterative techniques, and to differential equations will be published later. (Received March 17, 1949.)

367. C. F. Stephens: *Nonlinear difference equations containing a parameter.*

This paper contains a quick and simple method of existence proof for solutions of the special class of nonlinear difference systems: $y_i(x+1)=x^{-1}\sum_{j=1}^{n} b_{ij}(x)y_j(x)+x^{-1}A_{i1}(x)p+f_i(y_1(x), \cdots, y_n(x); p; x)$, where $f_i(0, \cdots, 0; 0; x)=0$. The functions $b_{ij}(x)$ and $A_{i1}(x)$, where $A_{i1}(x)$ can conceivably be identically zero, are any bounded and continuous functions of x; the f_i are bounded, contain terms in the constant parameter p alone, are continuous in x, begin with terms of the second degree in $y_i(x)$ and p, can be developed into convergent power series in the $y_i(x)$, and p is the domain: $|y_i(x)| \leqq r_i$, $|p| \leqq \rho$, $|x| \geqq K$. The proof that the given system has a solution which is continuous in x and analytic in p is based on a lemma which is similar to a result obtained previously by the author (Trans. Amer. Math. Soc. vol. 64 (1948) pp. 277–278) and can be proved by similar methods. The transformation $y_i(x)=pz_i(x)$ is used to transform the above system into a system such that the result of the lemma is easily applied. (Received February 15, 1949.)

368*t*. Walter Strodt: *On a class of nonlinear difference equations in the complex domain.*

For any positive $\beta \leqq \pi/2$ the function $g(x)$ is called *β-admissible* if for each positive γ less than β there is a $\xi \geqq 0$ such that $g(x)$ is analytic and bounded in the sector $|\arg (x-\xi)| < \gamma$, and if $g(x)$ approaches a limit as x becomes infinite through positive values. Let $\Delta(x, y_1, \cdots, y_n)$ be a polynomial in y_i with β-admissible coefficients. Let $\omega_1=0$, and $\omega_2, \cdots, \omega_n$ be complex numbers with $|\arg \omega_j| < \beta$. Let $\mathfrak{S}$ be the set of all β-admissible solutions of the difference equation (1) $\Delta[x, y(x+\omega_1), \cdots, y(x+\omega_n)]=0$. If $\Delta(+\infty, \sigma, \cdots, \sigma)$ is not identically zero, if its zeros $\sigma_1, \cdots, \sigma_c$ are all simple, and if $\sigma\Delta(+\infty, \sigma_k, \cdots, \sigma_k)/\partial y_1$ is nonzero for every k, then $\mathfrak{S}=\mathfrak{S}_1+\cdots+\mathfrak{S}_c$, where $\mathfrak{S}_k$ is the set of all β-admissible solutions $y(x)$ of (1) such that $y(+\infty)=\sigma_k$. $\mathfrak{S}_k$ is a nonempty finite-parameter family of functions, the number q_k of complex parameters being the number of zeros of the exponential polynomial $\sum_{j=1}^{n} [\partial\Delta(+\infty, \sigma_k, \cdots, \sigma_k)/\sigma y_j] \exp (\omega_j z)$ lying in the complement of the set $|\arg z| < \beta+\pi/2$. (In terms of a suitable metric $\mathfrak{S}_k$ is a topological *manifold* of $2q_k$ dimensions.) This is proved by successive approximations using solutions of linear difference equations with constant coefficients. (Received February 9, 1949.)

369*t*. D. B. Sumner: *An integral inversion formula for certain convolution transforms. Preliminary report.*

The nucleus of the convolution transform $f(x)=\int_{-\infty}^{\infty}g(x-t)\phi(t)dt$ is defined in terms of the entire function $E(s)=s\cdot L(s)\cdot M(s)$ by $1/E(s)=\int_{-\infty}^{\infty} \exp (-st)g(t)dt$, where $L(s)=\prod_1^{\infty} [1-s/\lambda_n]e^{s/\lambda_n}$, $M(s)=\prod_1^{\infty} (1+s/\mu_n)e^{-s/\mu_n}$, $\{\lambda_n\}_1^{\infty}$, $\{\mu_n\}_1^{\infty}$ are increasing sequences of positive numbers, and $\sum\lambda_n^{-2}$, $\sum\mu_n^{-2} < \infty$. Under these assumptions Widder has shown that $E(D)\cdot f(x)$, interpreted as $\lim_{n\to\infty} E_n(D)\cdot f(x)$, inverts the transform. The canonical product $L(s)\cdot M(s)$ has finite order less than 2, so that the function $\exp (-kx^2)C(x) \in L(-\infty, \infty)$ for some positive k, and $C(x)=\lim_{\alpha\to 0} (2\pi)^{-1}\int_{-\infty}^{\infty} C(t)dt\int_{-\infty}^{\infty} \exp [-\alpha^2u^2+iu(x-t)]du$. The operation $E(D)\cdot f(x)=C(D)\ f'(x)$ is interpreted as $\lim_{\alpha\to 0} (2\pi)^{-1}\int_{-\infty}^{\infty} C(t)dt\int_{-\infty}^{\infty} \exp [-\alpha^2u^2-iut]f'(x+iu)du$, and it is shown that this formula inverts the convolution transform. (Received March 21, 1949.).

APPLIED MATHEMATICS

370*t*. Milton Abramowitz: *Laguerre functions expressed in terms of Bessel-Clifford functions.*

Expansions are derived from the Laguerre functions $L_\nu^{(\mu)}(x)$ in terms of the Bessel-Clifford functions $F_n(\nu x) = (\nu x)^{-n/2} J_n(\nu x)$. These expansions are valid for large values of ν. It is also shown that the Hermite polynomials $H_n(x)$ of large order may be expressed in terms of spherical Bessel functions. (Received March 16, 1949.)

371*t*. Carl Cohen: *Integration of the generalized dilogarithm.*

The generalized dilogarithm $\mathrm{dl}^{(k)}x = \int_0^x \mathrm{dl}^{(k-1)}t \cdot t^{-1}dt$, $\mathrm{dl}^{(0)}x = -\log(1-x)$, plays a role in the theory of iterated integrals of expressions $R(x, \log(x-a))$; R = rational. For $m>0$, $\int_0^x \mathrm{dl}^{(k)}t \cdot (t-c)^m dt$, c a constant, may be expressed elementarily by x, $\log x$ and $\mathrm{dl}^{(i)}x$, $0 \leqq i \leqq k$. Integrals involving the first dilogarithm only and $(t-c)^{-m}$, $m \neq 1$, can also be expressed elementarily by $\mathrm{dl}^{(1)}x$, $\log x$, and x. In this paper, it is shown that while by a classical result $\int_0^x \mathrm{dl}^{(1)}t \cdot (t-c)^{-1}dt$ involves a rational function of the second dilogarithm, it is impossible for $i \geqq 2$ to integrate $R(\mathrm{dl}^{(i)}(x-b)(x-a)^{-1})$ in terms of higher dilogarithms and elementary functions, except the cases $a=b$, $a=0$, though it is possible to express $\int \mathrm{dl}^{(2)}(1-ax) \cdot x^{-1}dx$ and $\int \mathrm{dl}^{(2)}x \cdot (1-ax)^{-2}dx$ as $R(\mathrm{dl}^{(2)}x, \mathrm{dl}^{(1)}x, \log(1-x), x)$. (Received March 15, 1949.)

372. A. H. Copeland: *A cybernetic model of memory and recognition.*

The purpose of the author is to produce a consistency proof for certain assumptions concerning neural networks and to offer direction to future experimentation concerning the nervous system. To accomplish this a linear graph (neural network) is constructed which is capable of storing information, recognizing similarities between new and old experiences, reminiscing, and reasoning. The graph's memory is by association and it is shown that this association gives rise to a Boolean algebra which forms the basis of the reasoning mechanism. The Boolean algebra of reasoning is only indirectly related to the binary number system although this system plays an important role in the construction of the graph. In this model, inhibition is not produced by special inhibitory fibres or special inhibitory impulses but rather by differences in firing thresholds and by timing. The assumption concerning differential thresholds has some experimental justification and its consistency is established by the model. Incidentally with a moderate alteration the graph becomes the wiring diagram of an ultra rapid electronic device to do the work of an I. B. M. sorting machine. (Received March 18, 1949.)

373. A. E. Heins: *A boundary value problem in the theory of elastic stiffeners.*

This problem at hand concerns the solution for the semi-infinite plane $-\infty < x < \infty$ of the two-dimensional Laplace equation $\phi_{xx} + \phi_{yy} = 0$ in which $y \geqq 0$ and $\phi(x, y)$ is subject to the following boundary conditions: (i) $\phi(x, y) = 0$, $x<0$, $y=0$; (ii) $\phi(x, y) - \alpha\phi_y(x, y) = \beta$, $x>0$, $y=0$. Here α and β are given physical constants. It is shown that the solution of $\psi_{xx} + \psi_{yy} - k^2\psi = 0$ is subject to the same boundary conditions as may be formulated as a Wiener-Hopf integral equation and its solution may be given in integral form. From this the behavior of $\psi(x, 0)$ may be determined for $|x| \to \infty$ or ∞. It is further shown that $\psi(x, y)$ considered as a function of parameter k is simply $\phi(x, y)$ when $|k| \to 0$. This problem has been recently considered by E. L. Buell who solved the problem by other methods (Journal of Mathematics and

Physics vol. 26, (1948) p. 223). The mathematical problem arose in the study of the effect of a semi-infinite elastic stiffener on the boundary of a semi-infinite thin plate. (Received March 2, 1949.)

374. C. A. Truesdell: *The kinematics of vorticity.* II.

The kinematic foundation of the Lagrange-Cauchy velocity-potential theorem is examined in detail. The proofs of Lagrange, Cauchy, Power, Stokes, Kelvin, and St. Venant are examined in the light of the criticisms of Poisson, Stokes, Boussinesq, and Duhem. The Cauchy, Stokes and Kelvin proofs do not require analyticity, but apply only to circulation-preserving motions. The Lagrange, Power, and St. Venant proofs apply to a more general class of motions, but require analyticity. The former yield the true velocity-potential theorem. The latter yield theorems which when applied to certain types of motions which are not circulation-preserving prove that those motions cannot be represented altogether by analytic functions of the time. (Received February 18, 1949.)

375*t*. A. H. Van Tuyl: *On the evaluation of some definite integrals which occur in hydrodynamics and elasticity.*

Integrals of the types $I_1=\int_0^\infty e^{-at}J_n(bt)J_n(ct)t^m dt$ and $I_2=\int_0^\infty e^{-at}J_{n-1}(bt)J_n(ct)t^{m+1}dt$, $a\geqq 0$, $2n+m\geqq 0$, occur in potential theory, hydrodynamics, and elasticity for $n=1$, 2, $m=0$, -1 in I_1, and $m=-1$, -2 in I_2 (A. Weinstein, Quarterly of Applied Mathematics vol. 5 (1948) pp. 429–444, and Courant Anniversary Volume, 1948, pp. 451–460). It is shown that for every integer $n\geqq 0$ and all permissible positive and negative integers m, I_1 and I_2 can be expressed in terms of elliptic integrals. Both complete and incomplete elliptic integrals of the first and second kinds are present for $m<0$, while for $m\geqq 0$, only complete elliptic integrals occur. The case $m\geqq 0$ has been known implicitly (G. N. Watson, *Bessel functions*, p. 389, and W. M. Hicks, Philos. Trans. Roy. Soc. London Ser. A vol. 172 (1881) pp. 609–652), but the results involving the incomplete elliptic integrals are new. Further, recurrence relations are obtained by means of which every I_1 and I_2 for $m<0$ can be expressed in terms of the integrals $\int_0^\infty e^{-at}J_\mu(bt)J_\mu(ct)t^m dt$ and $\int_0^\infty e^{-at}J_n(bt)J_n(ct)dt$, where $\mu=[(1-m)/2]$. Numerical calculations have been made using the preceding results, and are applied to the problems mentioned above. (Received March 4, 1949.)

376. Alexander Weinstein: *On the transonic flow around a wedge.*

Using the results previously obtained in the paper *Generalized potential theory* (A. Weinstein, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 342–354) an explicit solution involving Weber-Schafheitlin integrals is given of the Tricomi equation $3y(\psi_{xx}+\psi_{yy})+\psi_y=0$ in the domain lying above the characteristic $x^2+(4/9)y^3=0$ and bounded by the verticals $x=0$ and $x=1$. The solution vanishes on the verticals $x=0$, $y>1$ and $x=1$, $y>-(3/2)^{2/3}$ and is equal to one on the segment $x=0$, $0\leqq y<1$. $\psi(x, y)$ represents, in a distorted hodograph plane, the stream function of a transonic flow around a wedge in a channel which is rectilinear in the subsonic part of the flow. (Received March 18, 1949.)

GEOMETRY

377*t*. Edward Kasner and John DeCicco: *Physical curves in space of n dimensions.*

A system S_k of ∞^{2n-1} curves in a given positional field of force in space of n dimensions consists of curves along which a constrained motion is possible such that the osculating plane at each point contains the force vector F, and the pressure P along the principal normal to the curve is proportional to the normal component N of F. Consider the ∞^1 curves of a system S_n passing through a given point in a given direction. The initial osculating plane and three-flat are fixed. The osculating m-flats, for $m=4, 5, \cdots, n-1$, vary and form one-parameter families. The first curvature k_1 varies inversely as the square of the speed v. The rate of variation $d\rho_1/ds$ of the first radius of curvature ρ_1 is a linear integral function of ρ_1. The second curvature k_2 is constant. The remaining curvatures $k_3, k_4, \cdots, k_{n-1}$ are algebraically dependent on k_1. The locus of the centers of the osculating m-spheres for $m=3, 4, \cdots, n-1$ is an algebraic curve. For $m=2$, the locus of the centers of the osculating spherical surfaces is a straight line just as in the case for $n=3$. The interrelationships between the rest trajectories and the lines of force are also discussed. (Received February 15, 1949.)

378*t*. Edward Kasner and John DeCicco: *Scale curves in general surface mapping.*

Let a point transformation T be defined between two surfaces S and S_1 of Euclidean space such that points correspond if they are represented by the same curvilinear coordinates (x, y). The scale function $\sigma=ds/ds_1$ is the ratio of the differentials of the arc lengths. Only nonconformal maps are studied so that σ is dependent on dy/dx explicitly. The curves $\sigma=$const. are called the scale curves of the mapping T. These may be represented on S or S_1. There are ∞^3 scale curves. At a point (x, y) there are five inflectional directions and two cuspidal directions for the scale curves. The cuspidal directions on S and S_1 are always orthogonal, and they coincide with the characteristic directions on S and S_1. The central locus corresponding to a given point (x, y) is an algebraic curve of the fifth degree. The associated scale quartic is studied. (Received December 27, 1948.)

379*t*. Edward Kasner and Don Mittleman: *Second order differential equations of rank* 2.

Second order differential equations of rank 2 are of the form $A_0y''^2+(B_0+B_1y'+B_2y'^2+B_3y'^3)y''+(C_0+C_1y'+C_2y'^2+C_3y'^3+C_4y'^4+C_5y'^5+C_6y'^6)=0$ where the coefficients are arbitrary functions of x, y. Kasner had shown that under an arbitrary point transformation an equation of rank 2 is transformed into an equation of rank 2. At a fixed point, the locus of the centers of the osculating circles of the integral curves of the equation lie on a special sextic, which may degenerate, in a nontrivial way, to a special quartic or a conic. Differential equations of rank 2 arise in the study of scale curves, certain two-parameter families of curves which are quadratic in the parameters, as the images of the ∞^2 circles of unit radius under an arbitrary point transformation, as extremals of the variational problem $\delta\int[\nu(xy)ds+(Adx+Bdy)]=0$, in the study of the motion of an electric particle in a static electromagnetic field, and in the restricted problem of three bodies in the plane. (Received March 8, 1949.)

380*t*. Wouter van der Kulk: *The univalent algebraic transformations of the projective plane.*

A univalent function on the projective plane P onto P whose graph is an algebraic variety in $P\times P$ over a field Δ is called a univalent algebraic transformation of P over Δ (uat). If Δ has characteristic 0 then it is well known that the group G of uat's coincides with the group of projective transformations of P. This is no longer true if the characteristic of Δ is p $(p>0)$. In that case it is possible to construct a set of generators of G, called semi-projective transformations of P over Δ (spt). A spt can be characterized as a uat which maps either every straight line in P or every straight line of some linear pencil in P onto a straight line in P. The uniqueness of the representation of a uat as product of spt's is investigated. The proofs of the preceding results are based on the theory of algebraic curves over a perfect field (cf. W. L. Chow, *Die geometrische Theorie der algebraischen Funktionen für beliebige vollkommene Körper*, Math. Ann. vol. 114 (1937)). (Received March 14, 1949.)

381*t*. Jack Levine: *Classification of two-dimensional affinely connected spaces admitting groups of collineations.*

By using Lie's classification of continuous groups in two variables (*Theorie der Transformationsgruppen*, vol. 3, pp. 71–73) all two dimensional affinely connected spaces (A_2) are obtained which admit groups G_r of projective collineations. It is found that all G_r are possible for $1 \leq r \leq 8$, with the exception of $r=7$. There are 25 types of non-equivalent groups represented. The A_2's corresponding to 22 of the groups are projectively flat. The 3 groups corresponding to non-projectively flat spaces are $[p]$, $[p, xp+yq]$, $[p, 2xp+yq, xp+xyq]$. As special cases, the A_2 admitting affine and translatory collineations are found. The largest affine collineations group is a G_4 if the A_2 is not flat. The Riemannian V_2 admitting projective and affine collineations are also determined. The only V_2 of non-constant curvature admitting a G_r $(r>1)$ of affine collineations is $ds^2=y^m(adx^2+2dxdy+bdy^2)$ with a, b, m constants, and $m \neq -2$. The associated group is $[p, xp+yq]$, and it contains a one-parameter subgroup of motions. (Received March 7, 1949.)

382*t*. Don Mittleman: *Velocity families of rank* 2.

A two-parameter family of plane curves is called a velocity family of rank 2 if, at each point p, the centers of the osculating circles of those members of the family which pass through p lie on a conic for which p is a focus. An arbitrary nonsingular point transformation T will convert a velocity family into a velocity family if and only if T is conformal (either direct or reverse). Velocity families of rank 2 arise in the study of the plane motion of an electric particle in a static electro-magnetic field, as the conformal images of Γ^* curves (a two-parameter family whose curvature is a point function), as extremals of a variational problem derived from Hamilton's principle for non-conservative fields, and in the motion of a particle in a positional field of force in a rotating plane. This last case includes the restricted problem of three bodies in the plane. These various problems are not all equivalent under a conformal transformation. The conformal covariants of a velocity system are determined. Thus, problems which give rise to velocity families of rank 2 may be classified as being conformally equivalent or distinct. (Received March 8, 1949.)

383*t*. W. G. Wolfgang. *A representation of complex-two space.*

Define *complex-two space* C_2 to be the space whose points may be put into one-to-one correspondence with the set of all pairs (A, B), where A and B are complex numbers. The purpose of the author is to give a construction of a representation of space C_2 in Euclidean three-space R_3. From the geometric construction the representation is defined by means of the mapping: $f: C_2 \rightarrow R_3$ where $f(X, Y)=(x, y, z)$ is defined by $x=x'$, $y=x''+y'$, $z=y''$, where $X=x'+x''i$, $Y=y'+y''i$. This mapping is not one-to-one and some time is spent in characterizing the points of C_2 which map into the same point of R_3. By means of this mapping the representations of several of the familiar "curves" of analytic geometry as configurations in this space are discussed and constructed. A completely general analytic definition of the representation, of which the one previously presented is a specific form, is also developed. (Received March 17, 1949.)

384. W. G. Wolfgang. *Metric considerations in complex-two space.*

Define *complex-two space* to be the space whose points may be put into one-to-one correspondence with the set of all pairs (A, B), where A and B are complex numbers. The usual definition of a distance function, $\rho(P, Q) = ((X_1 - X_2)^2 + (Y_1 - Y_2)^2)^{1/2}$, $P:(X_1, Y_1)$, $Q:(X_2, Y_2)$, defined in this space is not a true metric. The purpose of the author is to show that the space is metrizable in the usual sense and that the distance function $\rho(P, Q) = (|X_1 - X_2|^2 + Y_1 - Y_2|^2)^{1/2}$ is a true metric for the space. (Received March 17, 1949.)

LOGIC AND FOUNDATIONS

385. Ilse L. Novak: *A construction of models for consistent systems of axioms.*

The methods in the author's paper *The relative consistency of von Neumann's and Zermelo's axioms for set theory* (Bull. Amer. Math. Soc. abstract 55-3-201)are here generalized to apply to any consistent system S which (i) contains a finite number of primitive functors, (ii) is based on a finite number of axioms and axiom-schemata and (iii) has rules of deduction based on first order functional calculus. The author describes the construction of a model of S in the syntax of S, based only on the hypothesis that S is consistent and assumptions equivalent to the 5 Peano axioms for arithmetic. This construction makes possible a purely arithmetical proof of the Skolem-Loewenheim Theorem. A model of S obtained by this method can be enlarged to one of a larger system S' related to S as the von Neumann-Bernays-Gödel system is to the Zermelo-Fraenkel system. The methods described therefore also provide a purely arithmetical proof of the relative consistency of S and S'. (Received April 6, 1949.)

STATISTICS AND PROBABILITY

386. K. L. Chung: *Fluctuations of sums of independent random variables.*

Let S_n be the sum of n independent random variables having the same non-lattice distribution $F(x)$ with mean 0, variance 1, and finite third absolute moment. Let c be any real number and T_n denote the number of k's not exceeding n for which $S_k > c$, $S_{k+1} < c$. Then $\lim P(T_n \leqq 2^{-1}\beta_1 n^{1/2}x) = (2/\pi)^{1/2}\int_0^x e^{-y^2/2}dy$ where $\beta_1 = \int_{-\infty}^{\infty} |x|\,dF(x)$. There is a corresponding theorem for a lattice distribution, when the factor $2^{-1}\beta_1$ in the above formula is replaced by a slightly different number. In particular if $c=0$, the limiting distribution of the number of changes of sign of S_n is obtained. The method of proof is that of convergence of moments. (Received March 11, 1949.)

387. D. A. Darling: *A theorem on stable distributions.*

Let X and Y be positive independent random variables distributed according to a stable law with exponent α so that $E(\exp\, isx) = \exp\, (-c_1(is)^\alpha)$, $E(\exp\, isy) = \exp\, (-c_2(is)^\alpha)$, $0<\alpha<1$, $c_1>0$, $x_2>0$, $s>0$, $i^\alpha = \exp\, (\pi i\alpha/2)$. Then $\Pr\{(1+x/y)^{-1} < \beta\} = F(\beta) = 1/2\alpha - (1/\pi\alpha)\, \tan^{-1}\, \{(\beta^{-1}(1-\beta))^\alpha + c\, \cos\, \pi\alpha/c\, \sin\, \pi\alpha\}$, $0 \leqq \beta \leqq 1$,

$c = c_1/c_2$. This expression gives, under certain circumstances, the limiting distribution for the proportion of time a Markoffian variable spends in a given state. For example, when $\alpha = 1/2$, $c = 1$, $F(\beta) = (2/\pi) \sin^{-1} \beta^{1/2}$. (Received March 17, 1949.)

388. D. A. Darling: *Sums of symmetrical random variables.*

Let $X_1, X_2, \cdots,$ be independent, symmetrical, identically distributed random variables with a continuous distribution, but otherwise arbitrary. The principal result is that the probability that the first n partial sums are positive is independent of the distribution of the X's. Specifically, letting $S_n = X_1 + X_2 \cdots + X_n$, $S_0 = 0$, $\Pr\{s_1 > 0, s_2 > 0, \cdots, s_n > 0\} = u_n = 2^{-2n} C_{2n,n}$. From this fact several theorems follow: (1) Let M_n be that value of j for which S_j attains its maximum ($j = 0, 1, \cdots, n$), then $\Pr\{M_n = k\} = u_k u_{n-k}$. (2) Let L_n be the smallest value of j for which S_{j+1}, $S_{j+2}, \cdots, S_n$ are of constant sign ($j = 1, 2, \cdots, n-1$), $L_n = 0$ if all S_j are $\leqq 0$, and $L_n = n$ if all S_j are $\geqq 0$, then $\Pr\{L_n = k\} \backsim u_k u_{n-k}$, for large n, k. (3) Let N_n be the number of positive S_j for $j = 1, 2, \cdots, n$, then $\Pr\{N_n = k\} = u_k u_{n-k}$. Limiting expressions for these theorems are as follows: $\lim_{n\to\infty} \Pr\{T_n < nx\} = 2\pi^{-1} \sin^{-1} x^{1/2}$ where T_n is any one of the three random variables M_n, L_n or N_n. The limiting distribution for N_n is in a certain way a generalization of an earlier result of Erdős and Kac (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1101–1120). (Received April 4, 1949.)

389*t*. Mark Kac and Harry Pollard: *The distribution of the maximum of sums of independent random variables.*

Let $X_1, X_2, \cdots$ be independent random variables each having the same Cauchy density function $\pi^{-1}(1+x^2)^{-1}$. Let $s_k = X_1 + \cdots + X_k$ and consider the probability $P_n(a)$ that $|s_n| < an$ for $k = 1, 2, \cdots, n$. In this paper the limit of $P_n(a)$, as $n \to \infty$, is calculated in terms of the eigenvalues and eigenfunctions of the kernel $K(x, y) = 4^{-1}\{\log(1 - xy + (1-x^2)^{1/2}(1-y^2)^{1/2}) - \log(1 - xy - (1-x^2)^{1/2}(1-y^2)^{1/2}\}$, $1 \leqq x, y \leqq 1$. For the sake of comparison the method is also applied to the case of normally distributed X_j and the known answer is rederived. (Received February 14, 1949.)

TOPOLOGY

390. A. L. Blakers and W. S. Massey: *Homotopy groups of triads.*

A triad $(X; A, B)$ consists of a topological space X and two subspaces A and B for which $A \cap B \neq 0$. Let E^n be an n-cell, E_+^{n-1} and E_-^{n-1} the upper and lower "hemispheres" of its boundary sphere S^{n-1}, $p_0 \in E_+^{n-1} \cap E_-^{n-1}$, and $x_0 \in A \cap B$. An addition is defined between homotopy classes of maps $(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \to (X; A, B, x_0)$ in such a way that these homotopy classes form a group $\pi_n(X; A, B)$ ($n \geqq 3$). These groups fit into two exact sequences each similar to the homotopy sequence of a pair. In case $X = A \cup B$, the triad homotopy groups $\pi_n(X; A, B)$ are a measure of the amount by which the triad $(X; A, B)$ fails to satisfy the excision axiom (see R. H. Fox, Ann. of Math. vol. 49 (1948) pp. 484–485). The triad homotopy groups are used to give a new interpretation and proof of the Freudenthal "Einhängung" theorems (Compositio Mathematica vol. 5 (1937) p. 299), and to study the effect produced on the homotopy groups of a complex when a new cell is adjoined. (Received March 17, 1949.)

391. W. A. Blankinship: *Generalization of a construction by Antoine.*

In 1921 L. Antoine constructed a compact, zero-dimensional set in R^3 whose complement is not simply connected. In the present paper Antoine's construction is generalized to give such a point set in Euclidean space of arbitrary dimension. As a corollary to the construction, one is also enabled to construct a set in R^n, $n \geqq 3$, homeomorphic to a q-cell, $0 < q \leqq n$, whose complement is not simply connected. It is further shown that the complement of any compact set in Hilbert space is contractible, and hence sets of the type above mentioned cannot exist in Hilbert space. (Received March 14, 1949.)

392. J. H. Blau: *The space of measures on a given set.*

Let R be an abstract set with a class of subsets called "open," satisfying only the condition that R is open. A *measure* is a Carathéodory outer measure which is regular with respect to the open sets, and for which open sets are measurable. A neighborhood topology is introduced in the set M of all measures, under which M becomes a T_1-space. This topology specializes to weak convergence under certain conditions. Suitable restrictions on the "open" sets of R imply topological properties of M. Necessary and sufficient conditions are found under which M is separable, Hausdorff, bicompact, and metrizable. (Received March 17, 1949.)

393*t*. D. G. Bourgin and Abraham Charnes: *On some normed rings.*

The introduction of multiplications into Banach spaces and the structures of the topological algebras so obtained is studied from a Banach space standpoint. Semi-simplicity, existence and equivalence of multiplications are characterized in terms of Banach isomorphisms. In another direction the ring of continuous functions $C(S)$ on a completely regular space S wtih F metric is studied. If T is an isometry of $C(S_1)$ on $C(S_2)$ and the maximum free ideals of $C(S_i)$ are not closed, and if pointwise convergence implies convergence of the T maps, then S_1 and S_2 are homeomorphic. (Received April 11, 1949.)

394. E. E. Floyd: *Some characterizations of interior maps.*

Several characterizations of light interior maps are given, each of which is one-dimensional in nature. Two results are the following. Let X and Y be compacta, and let f be a light map of X onto Y. If X is locally connected and connected, then f is interior if and only if for each region V in Y and each component U of $f^{-1}(V)$, $f^*: H^1(\overline{V}, \overline{V}-V) \rightarrow H^1(\overline{U}, \overline{U}-U)$ is an isomorphism into, where $H^1(X, A)$ denotes the one-dimensional Čech cohomology group of X relative to A with integer coefficients. Also f is interior if and only if for each map $h: I \rightarrow Y$, I the unit interval, and for each $x \in f^{-1}h(0)$ there exists a map $g: I \rightarrow X$ with $fg = h$ and $g(0) = x$. (Received March 10, 1949.)

395. R. H. Fox: *The order of the homology group of a cyclic covering of a knot.*

Let S_g denote the g-fold cyclic covering of spherical 3-space S, branched over a tame simple closed curve k. It is known (Alexander, Trans. Amer. Math. Soc. vol. 30 (1928) pp. 275–306 or Reidemeister, *Knotentheorie*) that the first homology group H_g of S_g is determined by the positive integer g and the fundamental group of $S - k$, and is therefore an invariant of the knot-type of k. It is shown that the *order* of the group H_g is determined by g and Alexander's polynomial $\Delta(t)$ alone; in fact the order of H_g is equal to the resolvent of the two polynomials $t^g - 1$ and $\Delta(t)$. (Received March 17, 1949.)

396. C. N. Reynolds: *On the problem of coloring maps in four colors.*

This is an application of recent papers by the author on a calculus of topologically defined finite differences. (Received February 23, 1949.)

397*t*. G. T. Whyburn: *On open and closed mappings.*

If A and B are Hausdorff spaces, a mapping $f(A)=B$ is quasi-compact if the image of every open (or closed) inverse set in A is open (closed) and is compact if the inverse of every compact set in B is compact. Any two factorizations $f=h\phi$, where h is 1-1 and ϕ is quasi-compact, are topologically equivalent as are also any two monotone-light factorizations of f whose monotone factors are quasi-compact. A mapping f is open (closed) if and only if it is quasi-compact and generates a l.s.c. (u.s.c.) decomposition of A. In order that f generate a (1) l.s.c. (2) u.s.c. (3) continuous, decomposition of A it is necessary and sufficient that it factor into the topologically unique form $f=h\phi$ where h is a 1-1 mapping and ϕ is (1) open (2) closed or (3) both open and closed, respectively. If f is closed it factors in the form $f=hrg$ with g compact, r a closed retraction and h a homeomorphism. If A is locally compact separable and metric and components of the sets $f^{-1}(y)$, $y\in B$, are compact, f has a topologically unique monotone-light factorization with closed monotone factor. Normality and perfect separability are invariant under closed mappings. (Received March 17, 1948.)

T. R. HOLLCROFT,
Associate Secretary

THE APRIL MEETING IN LAWRENCE

The four hundred forty-seventh meeting of the American Mathematical Society was held at the University of Kansas on Friday and Saturday, April 29–30, 1949. There was a joint session with the Biometric Society on Friday afternoon.

The total attendance was about 150 including the following 84 members of the Society:

L. W. Akers, E. S. Allen, E. W. Anderson, R. W. Babcock, M. A. Basoco, P. O. Bell, Arthur Bernhart, Herman Betz, R. H. Bing, Florence Black, H. D. Block, L. M. Blumenthal, P. B. Burcham, E. D. Cashwell, F. M. Clarke, H. E. Clarkson, H. J. Cohen, M. E. Daniells, Robert Davies, R. F. Deniston, A. H. Diamond, Paul Eberhart, J. H. Engel, G. M. Ewing, W. C. Foreman, R. E. Fullerton, Albert Furman, R. E. Gaskell, R. N. Goss, Cornelius Gouwens, L. Z. Greene, Edison Greer, Edwin Halfar, O. H. Hamilton, P. C. Hammer, W. L. Hart, F. F. Helton, I. N. Herstein, J. J. L. Hinrichsen, D. L. Holl, A. S. Householder, H. K. Hughes, W. C. Janes, L. H. Kanter, M. Z. Krzywoblocki, W. G. Leavitt, H. M. Lieberstein, Anna Marm, M. E. Martinson, E. J. Mickle, J. M. Mitchell, W. K. Moore, D. J. Morrow, Zeev Nehari, J. I. Northam, G. K. Overholtzer, S. T. Parker, M. M. Pihlblad, G. B. Price, O. M. Rasmussen, L. M. Reagan, P. G. Robinson, R. G. Sanger, J. W. Sawyer, Robert Schatten, B. R. Seth, F. B. Sloat, G. W. Smith, H. W. Smith, R. G. Smith, C. E. Springer, W. L. Stamey, R. H. Stark, E. B. Stouffer, W. T. Stratton, H. P. Thielman, R. M. Thrall, J. W. Tukey, Gilbert Ulmer, Bernard Vinograde, M. A. Woodbury, J. L. Yarnell, P. M. Young, J. W. T. Youngs.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor E. J. Mickle of The Ohio State University spoke at 11:00 A.M. Friday on *Some properties of the Lebesgue area*; and Professor R. M. Thrall of the University of Michigan spoke at 9:00 A.M. Saturday, the title of his address being *On classes of algebras with radical.*

The joint session with the Biometric Society was at 2:00 P.M. Friday. By arrangement with the program committee the following addresses were given: Dr. A. S. Householder, Oak Ridge National Laboratory, *Diffusion of isotopes in biological systems of several compartments*; Professor M. A. Woodbury, University of Michigan, *Prediction and estimation*; Professor J. W. Tukey, Princeton University, *The consultant and the biologist.*

Sessions for contributed papers were held at 10:00 A.M. Friday and 10:15 A.M. Saturday.

Presiding officers for the various sessions were Professors L. M. Blumenthal, D. L. Holl, E. J. Mickle, G. B. Price, E. B. Stouffer, R. M. Thrall, and Bernard Vinograde.

On Friday evening there was a dinner for the Society in the Memorial Union Building. At the conclusion of the dinner the Society witnessed an excellent production of *Hamlet* by the Department of Speech and Drama of the University.

Abstracts of all papers presented at the Meeting are given below. Papers read by title are indicated by the letter "*t*." Paper number 418 was presented by Mr. Fisher and paper number 419 by Dr. Woodbury. Mr. Steel was introduced by Professor Bernard Vinograde.

ALGEBRA AND THEORY OF NUMBERS

398. Arthur Bernhart: *Roots of polynomials and their derivatives.*

The average product of the roots taken r at a time is the same for the polynomial and all its derivatives. In particular if the roots of a cubic are at the vertices of a triangle, then the quadratic roots are found to be at the foci of the inscribed Steiner ellipse. (Received March 21, 1949.)

399. F. Marion Clarke: *On the factorization of polynomials in n variables.* III.

For a field F and a polynomial $f(x_1, \cdots, x_n)$ in the polynomial domain $F[x_1, \cdots, x_n]$ *a splitting field* P is defined by the properties (i) P is an algebraic extension of F, (ii) f is factorable into absolutely irreducible factors over P, (iii) there exists no field P' properly contained in P and having properties (i) and (ii). It is shown by means of a Kronecker substitution that if F has characteristic zero, P is unique to within relative isomorphism. (Received March 15, 1949.)

400. J. H. Engel: *Contributions to the solution of the word problem for groups.*

The word problem for the lower central series of the free group (called hereafter hypo-abelian groups) is solved by a method (distinct from the methods used by Magnus, Fouxe-Rabinovitch, or R. H. Fox) which utilizes invariants connected with a lattice representation of these groups. There is demonstrated for each word in each hypo-abelian group the existence of a unique canonical form, expressed as a product, in canonical order, of powers of generators and powers of commutators. An expansion theorem is developed, proving the known result that elements equalling the identity in all hypo-abelian groups of a free group equal the identity in the given free group. The word problem is solved for hypo-abelian groups of the first order with any number of generators and one additional defining relation, and for hypo-abelian groups of the first or second order with two generators and any number of additional defining relations. These results yield new necessary conditions for the equality of two elements of a group. (Received March 16, 1949.)

401. I. N. Herstein: *The radical of the group ring of a p-group.*

The radical of the group ring of a group G of order p^n over a field of characteristic P has been shown to contain all the elements $g-1$ where g is in G, by both Lombardo-Radice and Jennings. In this paper a short proof of this theorem, using elementary concepts of group and ring theory, is given. (Received March 16, 1949.)

402. R. G. D. Steel: *A problem on minimum principal minors.*

The problem is to minimize a principal minor of a symmetrically partitioned correlation matrix, one element of the minor from each block, under certain *internal* congruences, namely, those which preserve the unit diagonal coefficients and which are effected by matrices whose nonzero coefficients are in the blocks corresponding to the partition of the correlation matrix. In order to obtain a system of equations whose solution gives the desired minimum, the procedure is to reduce the problem to orthogonal internal congruences, form a suitable compound and use Cayley's parametrization of the orthogonal group. The result may be considered a generalization of canonical correlations (see H. Hotelling, *Relations between two sets of variates*, Biometrika vol. 28 (1936) pp. 321–377). (Received March 10, 1949.)

403t. A. F. Strehler: *The generalization of the Moebius μ-function to a partially-ordered set.*

The familiar Moebius μ-function, originally defined in the set of positive rational integers, is here extended to every partially-ordered set which has the property that for each pair of elements a, b in the set, there is at most a finite number of elements x in the set such that $a \subseteq x \subseteq b$, where $\subseteq$ denotes the relation which orders the set. An explicit formula is found for the function in terms of the internal structure of the set. (Such an extension was made by Weisner, in Trans. Amer. Math. Soc. vol. 38 (1935) pp. 474–484, to a partially-ordered system in which were postulated, in addition to the above condition, the existence of the g. c. d. and l. c. m., relative to $\subseteq$, for each pair of elements of the system.) This extended μ-function is then applied to the inversion of finite series of functions defined over the partially-ordered set; and a generalized Dedekind inversion formula is evolved. In particular, the structure of the μ-function and some applications of the inversion formula are examined in such examples as the set of subclasses of a given finite class, the algebra of statements under the "implication" relation, an arbitrary set of positive rational integers, and integral algebraic domains. (Received March 15, 1949.)

404. Bernard Vinograde: *Algebras generated by every residue system.*

The opposite of a ring which is a direct sum of its radical and semi-simple subring is one which is without this property modulo any proper ideal in the radical. It can be shown that (1) for a ring with minimum conditions the property is equivalent to every complete semi-simple residue system generating the ring, (2) in rings with unit which are primary or commutative or of index less than 4 the property is inherited by subrings eRe where e is any idempotent, (3) for algebras of finite rank the property can be translated into a test for the rank of a matrix easily derived from any multiplication table of the algebra modulo the square of the radical. (Received March 10, 1949.)

ANALYSIS

405. E. D. Cashwell: *The asymptotic solutions of an ordinary differential equation in which the coefficient of the parameter is singular.*

The asymptotic solutions of the differential equation $u''(z)-[\rho^2\phi^2(z)/(z-z_0)^2+\chi(\rho, z)]u(z)=0$ are derived. The variable z ranges over a bounded, simply-connected region of the complex plane which contains the fixed point z_0, and in which $\phi^2(z)$ is bounded from zero. The complex parameter ρ is assumed to be large in absolute value but otherwise unrestricted, while $\chi(\rho, z)$ is analytic in z and uniformly bounded in ρ. The procedure depends upon the determination of a differential equation whose solutions are known, and which approximates the original equation in a specific sense for large ρ. Then, for z in a suitably defined closed region containing z_0 and arg ρ in each of four nonoverlapping subregions covering the ρ-plane, a pair of independent solutions of the initial equation are obtained in terms of the solutions of the approximating equations. The methods used are similar to those of Langer (Trans. Amer. Math. Soc. vol. 37 (1935) pp. 397–416). (Received March 10, 1949.)

406. G. M. Ewing: *Surface integrals of the Weierstrass type*. Preliminary report.

For each oriented Fréchet surface $S:x(u)$, $u\in R$, of the type of the 2-cell, an area $A(S)$ and a vector $V(S)$ are supposed given, subject to certain postulates. Let τ be a triangulation of R into triangles t_j, let u_j be a point of t_j and let S_j be the piece of S represented by $x(u)$, $u\in t_j$. If $f(x, r)$ is continuous in (x, r) and homogeneous in r in the usual sense, and if S has suitable properties, the sum $\sum f(x(u_j), V(S_j))$ tends to a finite limit, as norm τ approaches zero, which is independent of the choice of the representation $x(u)$. Lower semi-continuity and convergence properties of the integral are investigated. Connections are found with work of Rado and of Cesari. (Received March 17, 1949.)

407. H. K. Hughes: *On the analytical continuation of generalized factorial series*. Preliminary report.

The series considered here is $\sum_{n=1}^{\infty} g(n)A_n(t, k)$, where $A_n(t, k)=k\Gamma(t+k)\cdot\Gamma(nk)/(t+k-1)\Gamma(k)\Gamma(t+nk)$. It reduces to an ordinary factorial series when $k=1$, and to a Dirichlet series when $k\rightarrow\infty$ through appropriate values. (For known results see T. Fort, Trans. Amer. Math. Soc. vol. 23, p. 26.) In the present paper it is assumed that the coefficient $g(n)$ becomes an analytic function $g(z)$ when n is replaced by $z=x+iy$, this function being analytic throughout the half-plane $x\geqq 1/2$. Two theorems on analytic continuation are obtained. If $g(n)$ has the form $(-1)^n h(n)$, where $h(z)=O\{\exp(\pi/2-\epsilon)|y|\}$; $\epsilon>0$, $|\arg z|\leqq\pi/2$, $|z|\rightarrow\infty$, then the function defined by the series can be expressed in a closed form which is analytic for all finite t except those of a certain sequence provided k is not zero or a negative real number. Again, if $g(z)=O\{\exp(\pi-\epsilon)|y|\}$; $\epsilon>0$, $|\arg z|\leqq\pi/2$, $|z|\rightarrow\infty$, a similar result holds. (Received March 12, 1949.)

408. W. G. Leavitt: *On vector differential equations*.

Let A be a matrix whose elements and characteristic roots are holomorphic; the vector differential equation $Y'=(\lambda A+B)Y$ with λ a parameter is then transformable into $W'=(\lambda P+Q)W$, where the elements of P below the main diagonal are all zero. For second order equations a further transformation gives diagonal elements of form $\pm\lambda$. Let the upper-right-hand element of P be ϕ; it is shown that a transformation exists replacing ϕ by any associate of any member of its residue class modulo μ. Restricting further by the assumption μ is an associate of z^m $(m\geqq 0)$, it then follows that ϕ is reducible to one of the $m+1$ canonical forms $0, z^j$ $(j=0, 1, \cdots, m-1)$. For certain special cases, forms to which the solutions are asymptotic for large $|\lambda|$ are shown to be obtainable. (Received March 9, 1949.)

409. Josephine M. Mitchell: *An example of a complete double orthogonal system defined on the unit circle*.

We define a system of functions $\{\phi_{mn}^{(k)}\}$ $(m, n=0, 1, \cdots ; k=1, \cdots , 2^{n-1} (n\neq 0))$ of class L^2 in the unit circle $\rho=1$ as follows: $\phi_{mn}^{(k)}(\theta, \rho)=(1/\pi^{1/2}) \cos m\theta\psi_n^{(k)}(\rho)$ (m even), $=(1/\pi^{1/2})\sin m\theta\psi_n^{(k)}(\rho)$ (m odd), and $=(1/(2\pi)^{1/2})\psi_n^{(k)}(\rho)$ $(m=0)$, where (θ, ρ) are polar coordinates. The functions $\psi_n^{(k)}(\rho)$ are defined analogously to the Haar orthonormal system, namely: $\psi_0(\rho)=2^{1/2}$ on $0\leqq\rho\leqq 1$, $\psi_n^{(k)}(\rho)=2^{n/2}$ on $i_{2k-1,n}=E[((2k-2)/2^n)^{1/2}<\rho<((2k-1)/2^n)^{1/2}]$, $=-2^{n/2}$ on $i_{2k,n}=E[((2k-1)/2^n)^{1/2}<\rho<(2k/2^n)^{1/2}]$, $=0$ on $I-(i_{2k-1,n}+i_{2k,n})$, where I is the unit interval $(0, 1)$, and at the end points of the intervals $i_{2k-1,n}$ $i_{2k,n}$ as equal to the arithmetic mean of the values which it takes in adjacent sub-intervals. It is proved that the system $\{\phi_{mn}^{(k)}\}$ is orthonormal and complete with respect to functions of class L^2 in the unit circle. Furthermore the Lebesgue function, $L_{mn}^{(k)}(\alpha,r)\cdot\int_0^1\int_0^{2\pi}\left|\sum_{\mu,\nu=1}^{m,n}\phi_{\mu\nu}^{(\kappa)}(\alpha,r)\phi_{\mu\nu}^{(\kappa)}(\theta,\rho)\right|\rho d\rho d\theta$ $(\kappa=1, \cdots , 2^{\nu-1})$ is $o(\log m)$ almost everywhere in the unit circle. The following convergence theorem is proved: if $\sum_{m,n=0}^{\infty} a_{mn}^{(k)} \log^2 (m+2)<\infty$, then the Fourier-Haar double orthogonal series $\sum_{m,n=0}^{\infty} a_{mn}^{(k)}\phi_{mn}^{(k)}(\theta, \rho)$, converges almost everywhere in the unit circle. (Received March 14, 1949.)

410. C. E. Springer: *Union curves and dynamical trajectories.*

In a manifold V_{n+1}, a totally geodesic surface S is considered with orientation of S determined by the tangent to a curve C in a hypersurface V_n of V_{n+1} and by the first curvature vector in V_{n+1} of the tangent to C. The curve C in V_n is a union curve with respect to a given congruence of geodesics in V_{n+1} if the surface S along C contains the geodesic of the congruence through each point of C. The differential equations of union curves in V_n are obtained. A relation connecting union curvature, geodesic curvature, and normal curvature is found. It is shown that the trajectories of a conservative dynamical system of n degrees of freedom correspond to the union curves of a Riemannian space of n dimensions. (Received March 16, 1949.)

411. H. P. Thielman: *On functional equations and generalized means.*

The mean $M_f[F]$, with respect to a strictly monotone function $f(x)$ and a finite distribution function $F(x)$ is defined as in the book *Inequalities* by Hardy, Littlewood and Pólya. Just as the powers of x lend themselves readily to the definition of a continuous set of means, it is shown that the Tschebyscheff polynomials can be used in a similar way. Means with respect to $\cos p \arccos x$ are defined for finite sets of numbers over subintervals of $0\leqq x\leqq 1$. Some theorems given in the above mentioned book for means with respect to x^p have their analogues in the theory given here. Necessary and sufficient conditions on a polynomial $F(x, y)$ are given in order that the functional equation $\phi[F(x, y)]=\phi(x)+\phi(y)$ can have monotone continuous solutions. (Received March 15, 1949.)

APPLIED MATHEMATICS

412. R. N. Goss: *Center of flexures of a triangular beam.*

In Saint-Venant's flexure problem the center of flexure is defined as that point of loading for which the local twist vanishes at the centroid of a section. Using Seth's solution of the flexure problem for a beam of isosceles-triangular section (Proc. London Math. Soc. (2) vol. 41 (1936) pp. 323–331), the center of flexure is determined for any isosceles triangle for any value of Poisson's ratio. Previously known results for the right isosceles and incompressible equilateral triangles are included as special cases. Excellent agreement is obtained with the experimental results of Duncan, Ellis and Scruton on a very narrow section (Philosophical Magazine (7) vol. 16 (1933) pp. 201–235). In the equilateral triangle the position of the center of flexure is found to depend on Poisson's ratio if resultant rather than exact boundary conditions are imposed on two of the sides. (Received April 25, 1949.)

413. M. Z. Krzywoblocki: *The invariant theory of isotropic, homogeneous turbulence in compressible fluid.*

Robertson's invariant theory of isotropic turbulence valid for an incompressible fluid was adjusted to a compressible flow. The correlation tensors are not any more solinoidal tensors. All the correlation tensors of the second and third rank are derived as well as the dynamical equations of motion and energy. (Received March 5, 1949.)

414. B. R. Seth: *Boundary conditions interpreted as conformal transformation.*

Solutions of two-dimensional problems in which a liquid is partly bounded by fixed plane walls and partly by surfaces of constant pressure are obtained by interpreting the boundary condition as a conformal transformation of a rectilinear area in the z-plane into a rectilinear area in another z-plane. Similarly all problems of irrotational flow either inside or outside a rectilinear polygonal cylinder can be solved by transforming the boundary condition into a form such that it amounts to only a conformal transformation from one plane to another (Seth, Philosophical Magazine vol. 20 (1935) pp. 632–640; Quarterly Journal of Mathematics vol. 5 (1934) pp. 161–171; Hodgkinson, Quarterly Journal of Mathematics vol. 6 (1935) pp. 171–183). This method has also been applied to a number of torsion and flexure problems in elasticity and can be easily extended to problems in viscous flow, electricity and magnetism and conduction of heat (Trefftz, Math. Ann. vol. 82 (1921) pp. 97–112; Seth, Proc. Cambridge Philos. Soc. vol. 33 (1934) pp. 392–403; Philosohical Magazine vol. 22 (1936) pp. 582–598; vol. 23 (1937) pp. 745–757). It has also been used for cylinders bounded by confocal parabolas (Seth, Proceedings of the Indian Academy of Sciences vol. 9 (1939) pp. 447–453). It is proposed to extend it to curvilinear boundaries. (Received March 17, 1949.)

415*t*. D. M. Studley: *The mathematics of language.*

As point of departure from conventional philosophizing the notion of Plato's Forms as sets is taken and raised to the level of a universal principle. Seen in that light ideas or words become fair game for set-theoretic manipulation. In a case such as that of the Form of Whiteness it seems fully natural to think of the class of white things and objects as typically representing the abstract form White. Extending the inspection to other colors, a set of primary colors can be selected. The union of red and blue is purple while the union of yellow and red is orange. Orange and purple intersect in red. Such application is truly endless and quite inescapable because thought with words has unique pattern. Since a word generally has a set of senses, some of which may intersect, a word-form is a set of classes. In event of ambiguity of meaning doubt is founded upon union of possible meanings. Clearly set addition is mostly non-commutative since meaning of words depends on their order although combinations sometimes occur where over-all meaning is not changed by change of order. A sentence is a complete set of words whereas a description is an incomplete set of words. Nevertheless a one sentence description is possible and occurs in the use of language with an indefinite frequency. Notion $\in$ words and meaning $\subset$ word $\subset$ sentence $\subset$ abstract.

The result of this method has been the construction of a logical semantics. (Received March 16, 1949.)

GEOMETRY

416. P. O. Bell: *On hypergeodesic triangles the ratio of those integral curvature to angular excess is constant.*

A new system of hypergeodesics G_k (generalized geodesics) is characterized on a general analytic surface S by the property that on any triangle T formed by arcs of the system G_k the ratio of the integral curvature of T to the "angular excess" of T is equal to k, a constant. The geodesics of S constitute the special system G_1. Let e denote a unit tangent vector at a point x of a curve C_λ of the system G_k and let ds denote the element of arc length of C_λ at x. Let C_1, C_2 denote the curves passing through x of an arbitrary orthogonal net N of S, and let e_1, e_2 and ds_1, ds_2 denote the unit tangent vectors at x and the elements of arc length C_1, C_2, respectively. The components of vector curvature de/ds normal to the surface and tangent to the surface are $k_n n$ and γe^*, respectively, where k_n is the normal curvature of S for the direction λ, γ is the geodesic curvature of C_λ at x, and e^* is the unit tangent vector perpendicular to e such that $e^* \times e = n$. The geometric characterization of the vector curvature de/ds is completed on establishing the following result: The geodesic curvature γ of a hypergeodesic of the system G_k is given by the scalar product $\gamma = (k-1) \cdot (de_1/ds_1 + de_2/ds_2) \cdot e^*$ in which de_1/ds_1 and de_2/ds_2 are the vector curvatures of C_1 and C_2, respectively. (Received March 16, 1949.)

417t. John DeCicco: *General geometry of second order elements.*

The group of arbitrary point transformations of the plane induces a certain eight parameter group G_8 on the differential elements of second order, all of which pass through a given point (x, y). In 1910, Kasner obtained all the invariants under this G_8 for n such elements. In the present paper, the effect of this G_8 upon the central locus C corresponding to a fixed point (x, y) of a second order differential equation is studied. It is proved that under G_8 an arc length s exists for C involving derivatives of fifth order. The curvature κ of C, invariant under G_8, is an expression involving derivatives of seventh order. Any two such curves whose curvatures K are the same functions of the arc length s are equivalent under G_8. (Received December 27, 1948.)

418. R. C. Fisher and P. O. Bell: *New geometric characterizations for the lines of curvature and the geodesics of a surface.*

Let π_1 and π_2 denote the normal planes to an analytic surface S in ordinary space at a generic point x in the directions of the u and v parametric curves respectively. The planes π_1 and π_2 generate developable surfaces as x moves along an arbitrary curve C_λ of S. Let P_λ and Q_λ denote the intersections of the normal to S at x with the characteristic lines of π_1 and π_2 for the direction λ. The points P_λ and Q_λ coincide for an arbitrary parametric net if and only if the curve C_λ is a line of curvature. Let π_λ denote the normal plane in the direction λ. The intersections of the characteristic lines of π_λ for the u and v directions with the normal (tangent plane) coincide if and only if the curve C_λ is a line of curvature (geodesic). If C_λ and $C_{\bar\lambda}$ are curves of a net on S, any two of the following conditions implies the third: (a) C_λ and $C_{\bar\lambda}$ intersect at a constant angle of S; (b) C_λ is a geodesic; (c) the characteristic line of $\pi_{\bar\lambda}$ for the direction λ is parallel to the tangent plane at x. (Received March 16, 1949.)

Statistics and Probability

419. C. L. Dolph and M. A. Woodbury: *An application of the Carleman singular integral equation theory to the spectral representation of stochastic processes.* Preliminary report.

Implicit in Carleman's work is a generalization of Mercer's theorem to positive definite Hermitian kernels of his class I (that is, where the associated integral operator $\int\rho(s, t)(\ )dt$ is hypermaximal) which reduces to a form of Bochner's theorem when $\rho(x, t)=\rho(s-t)$. This observation leads to a spectral representation of processes $\xi(x|t)$ whose correlation function $\rho(s, t)=E[\xi(x|s)\overline{\xi(x|t)}]$ is of Carleman's class I. This parallels and generalizes the Cramér-Karhunen representation of stationary processes. This representation does not seem to satisfy Karhunen's definition of a spectral representation but is similar in many respects. (Received February 19, 1949.)

420. J. I. Northam: *Single and double dependent rare events.* Preliminary report.

Let $n=n_1+2n_2$ where n_1 and n_2 are respectively the number of occurrences of certain single and double rare events, and where n_1 and n_2 are assumed to follow a bivariate Poisson distribution with parameters λ_{10}, λ_{01}, λ_{11}. The probability generating function for $P(n)$ is found by use of the probability generating function for a bivariate Poisson distribution. Previously, Cernuschi and Castagnetto (Ann. Math. Statist. vol. 17, no. 1) found $P(n)$ for $n=n_1+2n_2+\cdots+in_i+\cdots$ where the n_i were assumed to follow independent Poisson laws with $\lambda_s=\lambda_1 a^{s-1}/s!$, $a\geqq 0$. (Received March 7, 1949.)

Topology

421. R. H. Bing: *Metrization of topological spaces.*

A collection of point sets is discrete if the closures of these point sets are mutually exclusive and any subcollection of these closures has a closed sum. Generalizing the notion of perfect separability, we say that space is perfectly screenable if there exists a sequence $G_1, G_2, \cdots$ such that G_i is a discrete collection of domains and for each domain D and each point p in D there is an integer $n(p, D)$ such that $G_{n(p,D)}$ contains a domain which lies in D and contains p. A regular topological space is metrizable if and only if it is perfectly screenable. Space is screenable if for each open covering H of space, there is a sequence $H_1, H_2, \cdots$ such that H_i is a collection of mutually exclusive domains and $\sum H_i$ is a covering of space which is a refinement of H. While a screenable Moore space need not be normal, it is metrizable if it is normal. Space is collectionwise normal if for each discrete collection X of closed point sets, there is a collection Y of mutually exclusive domains covering X^* such that no element of Y intersects two elements of X. A Moore space is metrizable if it is collectionwise normal. (Received March 14, 1949.)

422. H. J. Cohen: *Homogeneous plane continua.*

In 1924, S. Manzurkiewicz proved that a homogeneous, bounded, plane continuum M which is locally connected must be a simple closed curve. The existence of a homogenous, bounded, plane continuum *other* than a simple closed curve has been established by R. H. Bing (1948). The present paper strengthens the result of Mazurkiewicz by showing, first, that it holds even if M is *locally homogeneous*—that is, for each pair of its points x, y, there exists a homeomorphism between two open subsets of M, one containing x, the other containing y, such that x maps into y. Alternatively, the requirement of local connectedness can be replaced by the condition that M contain a simple closed curve. It is incidentally shown that any bounded, plane continuum which is the sum of a collection of disjoint simple closed curves is topologically equivalent to an annulus. (Received March 9, 1949.)

423. O. H. Hamilton: *Transformations topologically equivalent to isometric transformations in Hilbert space.*

It is shown that if T is a pointwise periodic transformation of a point set in an n-dimensional Euclidean space, then T is topologically equivalent to a transformation of a point set in a Hilbert space which leaves the distance between two points invariant. (Received March 17, 1949.)

J. W. T. YOUNGS,
Associate Secretary

THE APRIL MEETING IN STANFORD UNIVERSITY

The four hundred forty-eighth meeting of the American Mathematical Society was held at Stanford University, Palo Alto, California, on Saturday, April 30, 1949. The attendance was approximately 85, including the following 71 members of the Society:

M. I. Aissen, T. M. Apostol, H. A. Arnold, H. M. Bacon, E. M. Beesley, R. L. Belzer, H. F. Bohnenblust, J. L. Botsford, F. A. Butter, Lamberto Cesari, C. L. Clark, F. G. Creese, P. H. Daus, A. C. Davis, E. A. Davis, M. R. Demers, S. P. Diliberto, Roy Dubisch, W. D. Duthie, J. M. G. Fell, G. E. Forsythe, A. L. Foster, R. A. Fuchs, P. R. Garabedian, M. A. Girshick, J. W. Green, Sarah Hallam, G. J. Haltiner, M. A. Heaslet, J. G. Herriot, M. R. Hestenes, R. E. Holdman, R. D. James, Samuel Karlin, J. L. Kelley, R. M. Lackness, Cornelius Lanczos, D. H. Lehmer, Hans Lewy, Eugene Lucacs, Rhoda Manning, W. A. Mersman, A. B. Mewborn, B. C. Meyer, E. D. Miller, F. R. Morris, Ivan Niven, A. M. Ostrowski, Anna Pell-Wheeler, R. S. Phillips, Edmund Pinney, Hugo Ribiero, J. B. Robinson, R. M. Robinson, H. L. Royden, Wladimir Seidel, Max Shiffman, D. C. Spencer, W. M. Stone, Irving Sussman, J. D. Swift, L. H. Swinford, Gabor Szegö, J. L. Ullman, F. A. Valentine, R. L. Vaught, Morgan Ward, D. V. Widder, V. A. Widder, A. R. Williams, František Wolf.

In the morning there was a general session for research papers and for the invited address, *Quadratic forms in the calculus of variations*, by Professor M. R. Hestenes of the University of California, Los Angeles. Professor Max Shiffman presided. In the afternoon there were two sections, pure and applied mathematics, at which Professors D. C. Spencer and Gabor Szegö presided.

After the meetings, those attending were guests at a tea in Stanford Union.

Abstracts of papers presented at the meetings follow. Papers whose abstract numbers are followed by "*t*" were presented by title. Paper 433 was presented by Mr. Shapley.

ALGEBRA AND THEORY OF NUMBERS

424. T. M. Apostol: *Dedekind sums and their generalizations.*

The Dedekind sums $s(h, k)$ which appear in Rademacher's convergent series for the partition function $p(n)$ satisfy a well known reciprocity law. The author obtains formulas which prove useful for computing these sums. The sums are generalized by considering $s_p(h, k) = \sum_{\mu=1}^{k-1} (\mu/k)\overline{B}_p(h\mu/k)$, where $\overline{B}_p(x)$ is the pth Bernoulli function and $s_1(h, k) = s(h, k)$. A reciprocity law for these sums is derived and the method gives a new arithmetic proof of the known law for $s(h, k)$. The generalized sums are related to the functions $G_p(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{-p}x^{mn}$ much in the same way that $s(h, k)$ is related to the modular form $\eta(\tau)$ where $\log \eta(\tau) = \pi i\tau/12 - G_1(e^{2\pi i\tau})$. Transformation formulas relating $G_p(e^{2\pi i\tau})$ to $G_p(e^{2\pi i\tau'})$ are obtained for odd p, $\tau' = (a\tau + b)/(c\tau + d)$ being a modular transformation. The $s_p(h, k)$ appear in these formulas. The $s_p(h, k)$ are also expressed as infinite series related to certain Lambert series and for odd $p \geqq 3$, $s_p(h, k)$ is the Abel-sum of a divergent series. (Received February 7, 1949.)

425. Roy Dubisch: *Remarks on isotopy of nilpotent algebras.*

Isotopy (see A. A. Albert, *Non-associative algebras* I, Ann. of Math. vol. 43 (1942) pp. 685–707) is applied to Ghent's classification of nilpotent algebras of order $\leqq 4$ (*A note on nilpotent algebras in four units*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 331–338) to yield a classification according to isotopy. It is also shown that the number of isotopy classes for any order is always less than the number of equivalence classes. Furthermore, it is shown that, for every order n, there exists a nilpotent algebra isotopic to a non-nilpotent algebra. On the other hand, an example is given of a nilpotent algebra, not a zero algebra, which is not isotopic to a non-nilpotent algebra. All of this is for associative algebras. (Received April 30, 1949.)

426. D. H. Lehmer: *The evaluation of certain exponential series.*

The series under consideration have nth terms of the form $f(n) \exp(-an\pi)$ where $a = 1$, $3^{1/2}$, 2 and where the numerical function f is any one of a large class of well known functions including $\sigma_k(n)$, the sum of the kth powers of the divisors of n, the number $p(n)$ of unrestricted partitions of n and other familiar partition functions, Ramanujan's function $\tau(n)$ and its generalizations (see Bull. Amer. Math. Soc. vol. 54 (1948) p. 829), numbers of representations by sums of squares, and so on. All series are evaluated in closed form in terms of π, $\Gamma(3/4)$, and $\Gamma(5/6)$ and their powers. Simple examples are $\sum\sigma_3(n) \exp(-2\pi n) = \pi^2\{[\Gamma(3/4)]^8\ 320\}^{-1} - 1/240$, $\sum(-1)^n n\sigma_7(n) \exp(-3^{1/2}\pi n) = 0$, and $\sum p(n) \exp(-\pi(24n-1)/12) = 2^{1/2}\pi^{-1/4}\Gamma(3/4)$. These results are derived from a consideration of the harmonic and equianharmonic cases of elliptic functions. (Received March 18, 1949.)

427. Ivan Niven: *A function associated with $\phi(n)$.*

S. S. Pillai (*On a function connected with $\phi(n)$*, Bull. Amer. Math. Soc. vol. 35 (1929) pp. 837–841) has investigated the arithmetic function $R(n)$ which is defined as the least integer r such that $\phi^r(n) = 1$, where $\phi^r(n)$ denotes the rth iterate of the Euler

ϕ-function. This function is investigated in slightly altered form: define $g(n)$ as $R(n)-1$ for $n\geqq 3$, and $g(2)=g(1)=0$. This alteration is to enable one to state the fundamental theorem: $g(ab)=g(a)+g(b)$, unless a and b are both even, in which case $g(ab)=g(a)+g(b)+1$. Various other properties of the function are obtained. (Received March 17, 1949.)

428. R. L. Vaught: *The continuous real-valued functions on a completely regular space.* Preliminary report.

The linear algebra $R(X)$ of all continuous real-valued functions on a completely regular topological space X is made into a topological algebra by the introduction of the topology in which convergence means uniform convergence on compact subsets of X. It is shown that this topological algebra characterizes X, in fact that X is homeomorphic to the set of all continuous homomorphisms of $R(X)$ into the real numbers (with the usual product topology). Certain subalgebras and ideals of $R(X)$ are discussed. It is hoped to use these results to study the possible uniform structures over a completely regular space. (Received April 30, 1949.)

429. Morgan Ward: *Divisibility sequences with a rank function.*

Let (ϕ): $\phi_0=0$, $\phi_1=1, \cdots, \phi_n, \cdots$ be an integral-valued divisibility sequence. (ϕ) is said to admit a rank function if for every modulus m there exists an integer $r=r(m)$ such that $\phi_n\equiv 0 \pmod{m}$ if and only if $n\equiv 0 \pmod{r}$. Two new necessary and sufficient conditions for (ϕ) to admit a rank function are obtained. The simpler of these is as follows: If p, q are any two primes, then for every n, the greatest common divisor of ϕ_{pn} and ϕ_{qn} is ϕ_n. (Received March 18, 1949.)

ANALYSIS

430. E. M. Beesley. *Concerning total differentiability of functions of class P.*

Since 1900, some eight or more writers have formulated conditions under which a function of two real variables shall be said to be of bounded variation. Adams and Clarkson (Trans. Amer. Math. Soc. vol. 35 (1933) pp. 824–854; vol. 36 (1934) pp. 711–730; vol. 46 (1939) p. 468) have examined and compared these definitions and have catalogued many of the properties of the functions which satisfy the various definitions. In the terminology of Adams and Clarkson, P is the class of functions which are of bounded variation in the sense of Pierpont (see, for example, Adams and Clarkson, loc. cit., or H. Hahn, *Theorie der Reelen Funktionen*, Berlin, 1921, p. 539). One of the questions left open by these writers is the following: Do there exist functions of class P which are not totally differentiable almost everywhere? It is shown that there are continuous functions of class P which are nowhere totally differentiable. A specific example is exhibited and several questions concerning the category of the set of these functions in various function sets are considered. (Received March 18, 1949.)

431. Lamberto Cesari: *On Geocze's problem.*

Given a Fréchet surface S: $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$, $(u,v)\in A$ (A a Jordan region), the Lebesgue area $L(S)$ of S is the lim inf of the elementary areas $a(\Sigma)$ of all polyhedral surfaces Σ for which $\|S,\Sigma\|\to 0$. $L^*(S)$ is the corresponding lim inf when Σ is restricted to those Σ inscribed in S. $L^*(S)\geqq L(S)$; Geocze's problem is to prove

$L^*(S)=L(S)$. If the proof is developed only in terms of the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ of the given representation of the surface S, this is the so-called strong form of the Geocze problem (Radó, Huskey); if it is permitted to use in the proof other and convenient representations of the same surface S, this is the weak form of the problem (Mambriani). The author has proved, in the weak form and utilizing his previous research on representation of surfaces, that for each Fréchet surface S, $L^*(S)=L(S)$. (Received April 30, 1949.)

432. R. D. James: *A generalized integral.* II.

The definition and some of the properties of what may be called a Perron second integral (P^2-integral) were given in a previous paper (Transactions of the Royal Society of Canada (3) vol. 40 (1946) pp. 25–35). In the present paper the definition is changed slightly and further properties are deduced. It is shown, for example, that the P^2-integral provides a complete solution to a problem of Denjoy which he discussed in five notes in the C. R. Acad. Sci. Paris in 1921 (vol. 172, p. 653, p. 833, p. 903, p. 1218; vol. 193, p. 127). It is also proved that if a trigonometric series converges in $(0, 2\pi)$ to a function $f(x)$, then $f(x)$ is P^2-integrable, and that, with a suitable modification of the definition of the Fourier coefficients for the P^2-integral, the trigonometric series is the Fourier series of $f(x)$. (Received March 17, 1949.)

433. Samuel Karlin and L. S. Shapley: *Moment spaces and polynomial games.*

The reduced moment space D^n is the set of points in E^n whose coordinates are the first n moments of some distribution function $\phi(t)$ over the interval $[0, 1]$. D^n is the convex closure of the skew curve $x_i=t^i$, $0\leqq t\leqq 1$, $i=1, \cdots, n$. Boundary points of D^n may be classified according to facial dimension: let $a(x)$ denote the dimension of the intersection $L(x)$ of all planes of support at x. The boundary has two faces of each dimension $a=0, \cdots, n-1$. Each x in the boundary is uniquely representable by extreme points and corresponds to a unique distribution ϕ_x. The number of points in the representation of x and in the spectrum of ϕ_x is the same, and is denoted by $b(x)$, or by $\bar{b}(x)$ if the points 0 and 1 are counted with half weight when they appear. Then $a(x)=2\bar{b}(x)-1$ and $b(x)=c(x)+1$, where $c(x)$ is the dimension of $L(x)\cap D^n$. The moment-space game with strategies $x\in D^m$, $y\in D^n$, and pay-off $\sum a_{ij}x_iy_j$ is equivalent to the polynomial game with strategies t, $u\in[0, 1]$ and pay-off $\sum a_{ij}t^iu^j$; the mixed strategies of the latter corresponding to the pure strategies of the former. The foregoing considerations reveal many facts about the solutions (best mixed strategies) of polynomial games. (Received April 30, 1949.)

434. Cornelius Lanczos: *An iteration method for solving the eigenvalue problem of linear differential operators.*

Let $Df(x)$ be a given linear differential operator of a given finite range, with linear homogeneous boundary conditions which establish an eigenvalue problem. Expand $y=f(x)$ into a finite series of linearly independent functions, such as the powers of x. Determine the coefficients of this expansion by the condition that not only y, but also $Dy, D^2y, \cdots, D^my$ shall satisfy the given boundary conditions. Then an approximate representation of the first k eigenfunctions of the operator Dy is obtainable as a given linear superposition of $y, Dy, \cdots, D^{k-1}y$; $(k<2m)$. With increasing m (k fixed) the error converges to zero. Compared with the Rayleigh-Ritz method the following advantages may be listed: (1) The self-adjointness of Dy is not required. (2) No integration is involved, only the solution of linear equations. (3) Instead of successive reductions, a whole set of eigenfunctions and eigenvalues can be obtained simultaneously. The convergence is satisfactory if proper care is taken in the choice of the class of functions into which y is expanded. (Received March 17, 1949.)

435*t*. Walter Leighton: *Principal quadratic functionals.*

A principal quadratic functional is defined to be a functional of the form $J=\int_0^b[r(x)y'^2-p(x)y^2]dx$, where $b>0$, $r(x)$ and $p(x)$ are continuous, and $r(x)>0$ on $0<x\leqq b$. The minimizing of J among various classes of admissible curves is studied, and necessary and sufficient conditions are obtained. This paper, which will appear in the Trans. Amer. Math. Soc., may be regarded as a continuation of investigations begun by Morse and Leighton (*Singular quadratic functionals*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 252–286). (Received March 7, 1949.)

436. W. A. Mersman: *A new form of solution of Hermite's equation.*

In a certain compressible laminar boundary-layer investigation it is found necessary to solve the differential equation $d^2y/dz^2+2zdy/dz-4ny=0$, n being any nonnegative integer, subject to the boundary condition that $y(z)$ vanish at infinity. Although this equation is readily identifiable with the Hermite equation of order $2n$ with imaginary independent variable, the standard form of the classical solution is of no practical use in satisfying the condition at infinity. In the present paper, a general solution is obtained as the sum of three terms: (1) a polynomial, (2) a polynomial multiplied by the error function, and (3) a polynomial multiplied by exp $(-z^2)$. This form of solution is amenable to numerical computations for all values of z, whether large or small, and the condition at infinity is easily satisfied. (Received March 14, 1949.)

APPLIED MATHEMATICS

437*t*. H. W. Becker: *A new proof and application of Kirchhoff's rules.*

Philip Franklin (Journal of Mathematics and Physics (1925) pp. 97–102) gave a determinantal proof of Kirchhoff's rules for assembling a network function, by excluding terms whose complementaries contain meshes. A proof is here given, by ultimate fracto decomposition of the circuit into nothing but opens and shorts, using all branches as keys. Duality yields another rule, excluding terms containing modes, proper or *improper* (pairs or sets of adjacent modes). The rules are applied to the Z solution of all 16 fundamental bridge networks of from 6 to 11 unit branches in R. M. Foster's enumeration (Transactions of the American Institute of Electrical Engineers (1932) pp. 309–317) and to all 54 fundamental passive bridge circuits therein imbedded. If $Z=(za+b)/(zc+d)$ is the driving point impedence of a circuit, and $Z_z=(za+b)/f$ is the transfer impedence from the source to branch z, then the transfer factor $f=(ad-bc)^{1/2}$ entails $(f)=a\cap d$, where (f) is the set of terms of f without regard to sign. That is, $a=(f)+\alpha$, $d=(f)+\delta$, so the source $\leftrightarrow z$ interchange factor $h=a-d$ is irreducibly $h=\alpha-\delta$. (Received March 7, 1949.)

438*t*. H. W. Becker: *Circuitoidal functions.*

These are quotients $Z=D/N$, where D is a partial sum of products of n variables μ at a time, and N is a p.s.p. of the n or less variables $\mu-1$ at a time. The number of such possible quotient forms $CF(n)=3[(n-1)!]^2$, $n=2, 3, 4$. Those C.F. which are not driving point or transfer impedences of any physical circuit (and for $n \geqq 4$, such are in the vast majority) have the status of imaginary circuits. The simplest is Cohn's i.c., $Z_C = z$, $(w+v)\overset{+}{+}(w\|v) = (zw+zv+wv)/(z+w+v)$. The i.c. may be a fracto imaginary (the fracto sum of physical circuits having consistent multiplicity but not topology) or a semi-imaginary (the quotient of physical semicircuits having c.m.n.t.) or both f.i. and s.i. or neither of them. Z_C is both f.i. and s.i., and so sometimes is ^-Z_n the dual of a nonplanar circuit. If S is a set and Ω an operation such that M and m being any two members of S, $m' = M\Omega m$ is *not* a member of S, then S is a exogroup (exogamous group) with respect to Ω. If S is the infinite class of proper series-parallel circuits (having no opened or shorted branches), then S is an exogroup with respect to fracto addition, $\overset{+}{+}$. (Received March 14, 1949.)

439. A. M. Ostrowski: *A problem of abstract algebra connected with Horner's rule.*

In applying Horner's rule for computing the values of polynomials to polynomials in several variables, $N-1$ multiplications and the same number of additions are to be carried out, where N is the number of terms in the corresponding "complete" polynomial. The question whether for polynomials in one variable this number is the best is discussed and for the first four degrees solved. (Received April 30, 1949.)

440. R. S. Phillips: *The electromagnetic field produced by a helix.*

Starting with the retarded vector potential produced by a sinusoidal current traveling along an infinitely-thin perfectly-conducting helix, the phase velocity of this current is determined so that the electric field along the helix is "essentially" zero. The phase velocity along the helix so obtained is equal to the free space velocity of propagation. A Bessel function expansion is found for the electromagnetic field from integral evaluations of the type $\int_{-\infty}^{\infty} e^{i\beta\zeta}(e^{iKR}/R)d\zeta = 4\pi\sum_{n=-\infty}^{\infty} e^{in\phi}(I_n(\Gamma_n a)K_n(\Gamma_n\rho))$ where $R = (\zeta^2+a^2+\rho^2-2a\rho\cos(\phi-\zeta/\alpha))^{1/2}$, $\Gamma_n = ((\beta-n/\alpha)^2-K^2)^{1/2}$ and $\rho > a$. In order to determine the phase velocity, it is shown that $\lim_{\rho\to a^+}\Omega(\beta, \rho) = \infty$ and $\lim_{\rho\to a^+}\Omega(\beta_1, \rho)/\Omega(\beta_2, \rho) = 1$ for any real β_1 and β_2, where $\Omega(\beta, \rho) = \sum_{n=-\infty}^{\infty} I_n(\Gamma_n a)K_n(\Gamma_n\rho)$. (Received February 16, 1949.)

441. Edmund Pinney: *Elastic vibrations due to a point source within a semi-infinite solid.*

P or S waves originating at a point within a semi-infinite solid are investigated. Surface accelerations are calculated. (Received March 21, 1949.)

GEOMETRY

442*t*. D. B. Dekker: *Hypergeodesic curvature and torsion.*

This paper deals with the differential equation of an arbitrary family of hypergeodesics on a surface and definitions of hypergeodesic curvature and hypergeodesic torsion relative to the family in such a manner that the definitions reduce to those for union curvature and union torsion when the family is a family of union curves. The differential equation is first expressed in a tensor form as the vanishing of a certain scalar for any curve of the family. Then this single differential equation is transformed into two equations stating that a certain vector shall have zero components for any curve of the family. Hypergeodesic curvature and the hypergeodesic curvature vector are defined as the above mentioned scalar and vector respectively. A study of the cusp cone provides a geometric interpretation of hypergeodesic curvature of an arbitrary curve on the surface. An expression for the torsion of a hypergeodesic yields a definition of hypergeodesic torsion of a curve and leads to the theorem that a hypergeodesic is a plane curve if and only if the elements of contact of the osculating planes with the corresponding cusp cones along the curve form a developable. (Received March 14, 1949.)

443. J. W. Green: *Sets subtending a constant angle on a circle.*

Let C be a closed circular area in the plane and C' its circumference. Let K be a convex set in C which subtends at each point of C' the constant angle, α, $0<\alpha<\pi$. The paper considers the possibility of the existence of non-circular sets K with this property. It is found that if α is an irrational multiple of π or if $\alpha=(m/n)\pi$, m and n relatively prime and odd, there is only the obvious concentric circle to C as solution; in all other cases there is a large class of solutions. In the case where non-circular solutions are possible, a number of extremal problems are solved, involving the length, width, diameter, and area of these sets. For example, among all convex sets in a circle subtending $\pi/2$ on its circumference (and there are many, including a one-parameter family of ellipses) the circle has the greatest length. The minimum length is not attained but the limiting case is a diameter of C. (Received March 17, 1949.)

444. F. A. Valentine: *Some properties of C-convex sets.*

A set is said to be C-convex if each of its components is convex. A C-convex set having n components is called a C_n-convex set. A set S is called an L_n set if each pair of points in S can be joined by a polygonal line in S having at most n segments. The author shows that in the plane an open, bounded C_n-convex set ($n>1$) is an L_{n+1} set. In order to prove this, use is made of the notion of maximal family. A family of n disjoint open convex sets M_n is said to be maximal if no member of the family is a proper subset of an open convex set which is disjoint with the remaining members of the family. The boundary of an open maximal family M_n is the sum of linear edges, and it may or may not be connected. In the plane, each component of the complement of the closure of an open maximal family M_n is a convex set bounded by a closed polygon. Further properties of such sets are obtained and applied to the study of C_n-convex sets. (Received March 17, 1949.)

J. W. GREEN,
Associate Secretary

BOOK REVIEWS

Regular polytopes. By H. S. M. Coxeter. London, Methuen, 1948; New York, Pitman 1949. 20+321 pp. $10.00.

The study of polytopes (that is, polygons and polyhedra of three or higher dimensions) appears to interest more different kinds of people than any other branch of mathematics with the possible exception of number theory. Its beauties inspired the rug merchant, P. S. Donchian, patiently to construct a remarkable set of models representing all varieties of polytopes. It enabled the housewife, Alicia Boole Stott, a daughter of George Boole, to capitalize on her unusual powers of geometrical visualization in spite of her meager mathematical education. It provided the struggling young lawyer, Thorold Gosset, with an amusing and constructive pastime during his long waits between clients. And equally well it has attracted the attention of many famous mathematicians such as Klein, Poincaré, Poinsot, Schläfli, Cayley, Euler, and Goursat, to mention only a few. Nor is it solely a "pure" discipline devoted to beauty but not utility, for it has been cultivated by a number of crystallographers such as Fedorov.

Coxeter has spent a major portion of his mathematical career digging out the obscure references in early works, in making personal contact with contemporary gifted amateurs, and in developing his own outstanding contributions to the field. In this book he has poured forth all his devotion and scholarship and has produced a work which will be the standard treatise in this field for many years. It is beautifully illustrated with photographs of Donchian's models and with numerous drawings. Its value as a reference book is greatly enhanced by historical material at the end of each chapter, by tables giving the essential combinatorial and metric properties of polytopes of many varieties, by an exhaustive bibliography, and by a carefully constructed index. It is a particular pleasure to record this last feature; for its omission in so many mathematical books published in England greatly detracts from their value.

In his preface Coxeter follows the lead of Birkhoff-MacLane and says: "Anyone familiar with elementary algebra, geometry, and trigonometry will be able to appreciate this book." In a literal sense this is true, but let no one be deceived—this is a serious book, full of advanced ideas, and worthy of careful study by professional mathematicians. In the elementary section Coxeter gives a brief résumé of the Platonic solids, and then discusses other solids related to these which, though not regular, have many regular features. Examples are the cuboctahedron, the rhombic dodecahedron, and the zonohedra.

The serious mathematics begins with the third chapter in which Coxeter introduces the symmetry groups of the Platonic solids. After a full discussion of this important topic, he turns to degenerate polyhedra such as tessellations and honeycombs and their groups. These lead to results of crystallographic importance. Under the heading "The Kaleidoscope" he then describes the discrete groups generated by reflections. The exposition is greatly illuminated by his own "graphical notation" which makes complicated relations self-evident. The treatment of three-dimensional solids closes with a chapter on regular solids which are not quite polyhedra in the strict sense. These are obtained from the Platonic solids by "stellating" (adding pointed solid pieces) or "faceting" (removing solid pieces). This process raises the number of regular polyhedra from five to nine.

The remaining two-thirds of the book is devoted to polytopes of higher dimensions. The general program is similar to that carried out for ordinary polyhedra. It is shown that in four-space there are six regular polytopes and that in n-space ($n>4$) there are only three regular polytopes. Explicit constructions are given for these and metrical properties are derived. There are even photographs of models of three-dimensional projections of some of these hypersolids. The methods used are increasingly analytic, but the underlying geometry is never lost among the formulae.

I have only one regret about this theory, and Coxeter should not be blamed for this. I refer to the formidable terminology, doubtless invented by mathematicians with a far better education in classical languages than myself. Dry-sounding words like "enantiomorphous" and "great stellated triacontahedron" tend to obscure the geometrical treasures of the subject. This is a place where a judicious use of American slang would greatly improve the situation.

C. B. ALLENDOERFER

Mathematical theory of human relations. An approach to a mathematical biology of social phenomena. By N. Rashevsky. Bloomington, Principia, 1947. 14+202 pp. $4.00.

The name Rashevsky is virtually synonymous with the term "mathematical biophysics," a term which he coined to designate a field created and developed by himself and under his own direction. In this field he is the author of two books, one recently reissued in revised and greatly expanded form, and the editor of a quarterly journal, the Bulletin of Mathematical Biophysics.

Professor Rashevsky entered this field by way of a theoretical study of the behavior of liquid drops within which constituents drawn from the surrounding medium undergo chemical transformation. The analogy to a metabolizing biological cell is obvious. From the "vegetative" cell, he passed quickly to the "sensitive" cell, which has lost the capacity to grow and divide, but undergoes characteristic reversible changes when subjected to external influences of certain types.

The next step was to consider an organized aggregate of cells of either type. Concentrating attention upon the vegetative constituents, he developed a theory of organic form. In considering the sensitive constituents one has a model of the central nervous system, whose structure imposes certain general patterns upon its own activity.

The mathematical theory of human relations applies a similar methodology to the consideration of interacting organisms. Much of the content has appeared over the past 10 years in a series of papers published in *Psychometrika*. The point of departure is the observation that the actions of an individual are dependent jointly upon his own desires and the influence of others. In qualitative terms the observation is as trite as the "law of supply and demand," but with the assignment of numerical parameters and specific functional dependences it yields a tool that can be worked with.

Of course, there are horrible complexities already present at the start. There are many types of activity; there are attitudes toward each ranging all the way from psychopathic aversion to equally psychopathic drive; and there are all degrees of influence, varying with both of the parties and with the activity. These complexities the author sidesteps in characteristic fashion by renouncing any claim to realism, indulging merely in the mathematical exercise of manipulating a few systems of the simplest conceivable type. He considers a single type of activity, usually economic, and generally considers two or three groups, each characterized by only a single coefficient of influence averaged over the group.

Herein is exhibited Rashevsky's cardinal methodological principle. Any approximation is better than none; any guess is better than none, if it is quantitative. Simplify equations until they become manageable, study these, and add complexities as you are able to take care of them. By progressing in this fashion one may hope ultimately to arrive at estimates that are reliable. In the present instance the dichotomies and abstractions accepted as temporary makeshifts are very little different from, and scarcely worse than, those commonly accepted in either the forum or the academy.

In the main course of the development, individuals are classed into two groups, the "actives," having a high coefficient of influence, and the "passives," having a low one. The actives may, themselves, be split on some issue, in which case the passives are found to drift, ultimately, all into the same party. This is a secular trend: presidential elections are not an example!

In the economic field, the actives are the organizing and directing group, whose skills can greatly enhance the productivity of the passives. For this reason the passives will accept their direction, receiving for their own labors a certain portion of the output. Or one may consider a governing group in addition to the two industrial groups, exacting payment (taxes) from both.

The development becomes most dramatic when consideration is given to the fact that once the social lines are drawn, membership in a social group may be hereditary while the characteristics are not. Over the generations there will be born into each class individuals who are members of this by birth but have characteristics that would place them in the other. Such a course of events may lead, as one illustration, to the eventual passage of control into the hands of the active subclass of the "passive" group, or, as another example, to the shift of control from one group of actives to another. The significant feature of the discussion is not the qualitative conclusion, which is, perhaps, obvious enough, but the possibility of estimating the duration of an epoch.

As the subtitle indicates, this is "an approach" to the problem, rather than a systematic development. The endeavor throughout is to suggest problems for study, to indicate the relevant variables and how they might be measured, rather than to elaborate any single line. A few graphs are presented (some should have been made as scatter plots or bar graphs) to show that the predicted variation goes in the right direction. But the success of the book must be measured ultimately by the extent to which others will pick up and carry on the suggestions in it. For this reason, since its incompleteness is repeatedly pointed out by the author himself, the book is scarcely a subject for legitimate criticism. One may feel that the approach is fruitless, and with this feeling there can be no argument until there are fruits to show.

The type of mathematics can be fairly well guessed in advance. The integral and integro-differential equations are generally pared down by simplifying assumptions to ordinary finite or differential equations. Statisticians may be startled to find essentially no statistical concepts whatever, but this is only an aspect of the early embryonic phase in which the subject is left.

ALSTON S. HOUSEHOLDER

NOTES

Mr. W. D. Lambert, formerly of the U. S. Coast and Geodetic Survey, has been elected to membership in the National Academy of Science. He has also been awarded the William Bowie Medal for leadership in the study of earth sciences by the American Geophysical Union.

Professor Saunders MacLane of the University of Chicago has been elected to membership in the American Philosophical Society and the National Academy of Science.

Assistant Professor Wilfred Kaplan of the University of Michigan, Professor S. C. Kleene of the University of Wisconsin, Associate Professor G. W. Mackey of Harvard University, and Associate Professor J. W. Tukey of Princeton University have been awarded Guggenheim Fellowships.

Professor G. Elfving of the University of Helsingfors has been appointed to a visiting professorship of mathematical statistics at Cornell University.

Dr. Nachman Aronszajn of the National Center for Scientific Research in France has been appointed to a research professorship at Oklahoma Agricultural and Mechanical College.

Dr. A. L. Blakers of Princeton University has been appointed to an assistant professorship at Lehigh University.

Dr. Mary L. Boas has been appointed lecturer at Wellesley College.

Dr. K. Chandrasekharan of the Institute for Advanced Study has been appointed to the position of reader in mathematics at the Tata Institute of Fundamental Research.

Associate Professor John DeCicco of the Illinois Institute of Technology has been appointed to a visiting professorship at DePaul University.

Professor J. L. Doob of the University of Illinois is on leave of absence and will spend the coming academic year at Cornell University.

Professor H. L. Dorwart of Washington and Jefferson College has been appointed to the Seabury professorship of mathematics and natural philosophy, Trinity College.

Dr. Arthur Erdélyi of the University of Edinburgh has been appointed to a professorship at the California Institute of Technology.

Mr. F. H. Fowler of Lessells and Associates, Brighton, Massachusetts, has accepted a position as associate editor with the McGraw-Hill Publishing Company.

Dr. S. W. Hahn of the University of Michigan has been appointed to an assistant professorship at Wittenberg College.

Dr. I. I. Hirschman of Harvard University has been appointed to an assistant professorship at Washington University.

Dr. Gilbert Hunt has been appointed to an assistant professorship at Cornell University.

Dr. Joseph Lehner of Hydrocarbon Research, Inc., New York City, has been appointed to an associate professorship at the University of Pennsylvania.

Miss Zung-nyi Loh of Wellesley College has been appointed to an assistant professorship at Wilson College.

Professor Dorothy McCoy of Belhaven College, Jackson, Mississippi, has been appointed to a professorship at Wayland College, Plainview, Texas.

Professor H. B. Mann of Ohio State University has accepted a position at the Applied Mathematics Laboratory of the National Bureau of Standards.

Mr. E. P. Miles of Duke University has been appointed to an associate professorship at Alabama Polytechnic Institute.

Dr. C. W. Moran of Wright Junior College has been appointed lecturer at DePaul University.

Assistant Professor F. M. Pulliam of the University of Kentucky has been appointed to an assistant professorship at the U. S. Naval Postgraduate School.

Dr. J. M. Sachs of Chicago Teachers College has been appointed lecturer at DePaul University.

Professor Emeritus E. W. Sheldon of the University of Alberta has been appointed to an interim professorship at Acadia University.

Mr. W. B. Stovall of the University of Florida has accepted a position as statistician with the Bureau of Vital Statistics, Florida State Board of Health.

Associate Professor G. W. Whitehead of Brown University has been appointed to an assistant professorship at Massachusetts Institute of Technology.

Dr. Daniel Zelinsky of the Institute for Advanced Study has been appointed to an assistant professorship at Northwestern University.

The following promotions are announced:

J. H. Bell, Michigan State College, to an associate professorship.

J. D. Hill, Michigan State College, to a professorship.

Leo Katz, Michigan State College, to an associate professorship.

Mary D. Kirby, Manhattanville College of the Sacred Heart, to a professorship.

Norman Levinson, Massachusetts Institute of Technology, to a professorship.

Eric Reissner, Massachusetts Institute of Technology, to a professorship.

J. R. Vatnsdal, State College of Washington, to a professorship.

The following appointment to an instructorship is announced: Pennsylvania State College, Miss Helen F. Story.

Professor A. Buhl of the University of Toulouse died March 24, 1949, at the age of seventy-one years.

Professor William Threlfall of the University of Heidelberg died on April 4, 1949, at the age of sixty years.

Professor Helen A. Merrill of Wellesley College died on May 2, 1949, at the age of eighty-six years. She had been a member of the Society for forty-six years.

NEW PUBLICATIONS

Albert, A. *Solid analytic geometry.* New York, McGraw-Hill, 1949. 10+162 pp. $3.00.

Birkhoff, G. *Lattice theory.* Rev. ed. (American Mathematical Society Colloquium Publications, vol. 25.) New York, American Mathematical Society, 1949. 14+280 pp. $6.00.

Blanche, E. E. *You can't win.* Washington, Public Affairs Press, 1949. 155 pp. $2.00.

Bloch, A., and Guillaumin, G. *La géométrie intégrale du contour gauche.* Paris, Gauthier-Villars, 1949. 6+141 pp. 1500 Fr.

Boyer, C. B. *The concept of the calculus.* Reprint. Foreword by R. Courant. New York, Hafner, 1949. 12+346 pp. $5.50.

Dadourian, H. M. *Introduction to analytic geometry and the calculus.* New York, Ronald, 1949. 10+246 pp. $3.25.

Dark, H. J. *The life and works of Herbert Elsworth Slaught.* Nashville, George Peabody College for Teachers, 1948. 8+152 pp. $2.25.

David, F. N. *Probability theory for statistical methods.* Cambridge University Press, 1949. 10+230 pp. $3.50.

Green, S. L. *The theory and use of the complex variable.* London, Pitman, 1949. 8+136 pp. 12s 6d.

Guillaumin, G. See Bloch, A.

Hardy G. H. *Divergent series.* Oxford University Press, 1949. 16+396 pp. $8.00.

Heffter, L. *Kurvenintegral und Begründung der Funktionentheorie.* Berlin, Springer, 1948. 4+48 pp. 5.40 DM.

van der Kulk, W. See Schouten, J. A.

Miller, F. H. *Analytic geometry and calculus: a unified treatment.* New York, Wiley, 1949. 658 pp. $5.00.

Schouten, J. A. and van der Kulk, W. *Pfaff's problem and its generalizations.* Oxford University Press, 1949. 16+542 pp. $12.50.

Sommerfeld, A. *Partial differential equations in physics.* New York, Academic Press, 1949. 329 pp. $5.80.

Tables of the Bessel functions of fractional order. Vol. 2. Prepared by the Computation Laboratory of the National Bureau of Standards. New York, Columbia University Press, 1949, 18+365 pp. $10.00.

Tables of inverse hyperbolic functions. (The Annals of the Computation Laboratory of Harvard University, vol. 20.) Cambridge, Harvard University Press, 1949. 20+290 pp. $10.00.

Weiss, M. J. *Higher algebra for the undergraduate.* New York, Wiley, 1949. 162 pp. $3.75.

Whitehead, A. N. *An introduction to mathematics.* Oxford University Press, 1948. 192 pp. 5 s.

SPLITTING CRITERIA AND EXTENSION TYPES[1]

REINHOLD BAER AND CHRISTINE WILLIAMS

Introduction. Following the lead given by Galois theory it appears natural to term extensions G and H of the group N equivalent extensions, if there exists an isomorphism of G upon H which leaves invariant every element in N. But if G is an extension of N, and if S is some group, then it will, in general, be impossible to assert the equivalence of the extensions G and $G\oplus S$ of N in spite of the fact that these two extensions of N do not differ in any interesting feature. Thus one may feel that the concept of equivalence provides too narrow a principle of classification of extensions. With this in mind we say that the extension type of the extension G of N is contained in the extension type of the extension H of N, if there exists a homomorphism of G into H which leaves invariant every element in N and which maps normal subgroups of G upon normal subgroups of H; and we say that the extensions G and H of N belong to the same extension type if the type of G is contained in the type of H and the type of H is contained in the type of G. In this way we obtain a partially ordered set of extension types of N (§3).

It is clear that the extensions G and $G\oplus S$ of N, mentioned before, belong to the same extension type of N; and more generally $G\oplus S$ and $G\oplus T$ belong to the same extension type of N, if G is an extension of N. In order to obtain criteria for the validity of the converse we need splitting criteria asserting the existence of complements which are normal subgroups. Since hardly any such criteria are available in the literature (see Baer [2][2] for references) we devote the first section to the derivation of such criteria which may be of interest independent of the applications in §3. In these applications there exists always a normal subgroup whose elements are left invariant by the endomorphisms under consideration. This makes it possible to weaken the hypotheses usually needed; and a systematic account of these phenomena is given in §2. In an appendix we show that finitely generated groups are not isomorphic to proper quotient groups, if their normality preserving endomorphisms split.

1. Normality and uniformity of splitting.

We consider M-groups G where the composition of elements in G is addition $x+y$ and where

---

Received by the editors May 15, 1948.

[1] This work has been done partially under a contract with the Office of Naval Research; however, ONR assumes no responsibility for the material in this paper.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

the operators in M are left-multipliers so that the effect of applying m in M on g in G is indicated by mg. Addition and multiplication meet the customary requirements. We note that no composition of the elements in M is defined.

Terms like M-subgroup and M-endomorphism and so on are used in the customary way. In particular we note that we have $m(g\eta) = (mg)\eta = mg\eta$ for m in M, g in G and η an M-endomorphism. The *kernel* of η will be denoted by $K(\eta)$ and the join of the ascending chain of kernels $K(\eta^i)$ is the *radical* $R(\eta)$. Kernel and radical are M-admissible normal subgroups of G.

An M-admissible subgroup C of G is called *a complement of the endomorphism* η *of* G, if $C = C\eta$ and if every coset of G modulo $R(\eta)$ contains one and only one element in C. Then η induces essentially the same automorphism in $G/R(\eta)$ and in the complement C. If it so happens that η possesses one and only one complement, then we designate the uniquely determined complement of η by $C(\eta)$.

We shall be interested in finding complements that are normal subgroups too, since then the group G is represented as the direct sum $R \oplus C$ of the radical and complement of η. That this is by no means a rare occurrence may be seen from the following criterion.

If the M-endomorphism η of G possesses one and only one complement $C(\eta)$, and if η commutes with every inner automorphism of G, then $C(\eta)$ is a normal subgroup of G.

PROOF. If g is an element in G, then $-g+R(\eta)+g=R(\eta)$, since $R(\eta)$ is a normal subgroup of G. Hence every coset of G modulo $R(\eta)$ contains one and only one element in $-g+C(\eta)+g$. But η commutes with inner automorphisms. Hence

$$[-g + C(\eta) + g]\eta = -g + C(\eta)\eta + g = -g + C(\eta) + g.$$

Thus $-g+C(\eta)+g$ is a complement of η. But η possesses only one complement. Hence $C(\eta) = -g+C(\eta)+g$. This proves the normality of $C(\eta)$.

In our applications in §3 we shall not be able to make use of this criterion, since then we shall not be assured of the hypothesis that η commutes with the inner automorphisms, as the endomorphisms considered there are products of homomorphisms. Thus we turn now to the derivation of another type of criterion which assures at the same time the existence of a complement.

A subgroup S of G is (M, η)-admissible, if $MS \leqq S$ and $S\eta \leqq S$. The subgroups L_i form *an ascending Loewy chain for* η, if

(a) $0 = L_0 \leqq L_1 \leqq \cdots \leqq L_i \leqq L_{i+1} \leqq \cdots$;

(b) every L_i is a normal (M, η)-admissible subgroup L of G;

(c) every L_{i+1}/L_i is the sum of (a finite or infinite number of) minimal normal (M, η)-admissible subgroups of G/L_i.

Here we note that J/L_i is a minimal normal (M, η)-admissible subgroup of G/L_i if, and only if, J is a normal (M, η)-admissible subgroup of G such that $L_i<J$ and such that $L_i<S\leqq J$ together with normality and (M, η)-admissibility of the subgroup S imply $S=J$.

At present it does not matter whether such a chain terminates after a finite number of steps or not. Since equality of successive terms is permitted, we may always assume these chains to be formally infinite.

LEMMA 1. *Let η be an M-endomorphism of the M-group G. If J is a minimal normal (M, η)-subgroup of G, then either*

(a) $J\eta=0$ *or*

(b) *η maps J isomorphically upon $J\eta$.*

PROOF. Since $K(\eta)$ is an (M, η)-subgroup which is normal in G and $K(\eta)\cap J\leqq J$, either

(a) $K(\eta)\cap J=J$ so that η maps every element of J into zero or

(b) $K(\eta)\cap J=0$ so that η induces an isomorphism of J onto $J\eta$.

A particular type of M-endomorphism will be of importance in our considerations. We shall say that the M-endomorphism η is a *normality preserving M-endomorphism of G* if η maps normal M-subgroups of G upon normal M-subgroups of G.

LEMMA 2. *If S is the sum of minimal normal (M, η)-subgroups of G, $S\cap R(\eta)=S\cap K(\eta)$; there exists one and only one normal (M, η)-subgroup C such that $S=C\oplus[S\cap R(\eta)]$. Furthermore, if η is a normality preserving M-endomorphism of G, $C=C\eta=S\eta$.*

PROOF. Define C to be the sum of all minimal normal (M, η)-subgroups J of G contained in S with $J\eta\neq 0$. Then since S is the sum of the minimal normal (M, η)-subgroups contained in it, $S=C+[S\cap K(\eta)]$.

If x is an element of $R(\eta)\cap C$, x is in some direct sum $J_1\oplus\cdots\oplus J_n$, where the J_i are minimal normal (M, η)-subgroups with $J_i\eta\neq 0$; hence $x=x_1+\cdots+x_n$ with x_i in J_i. Since x belongs to $R(\eta)$, there is a positive integer m such that $0=x\eta^m=x_1\eta^m+\cdots+x_n\eta^m$; but $x_i\eta^m$ is in J_i and the sum of the J_i is a direct sum so that $0=x_1\eta^m=\cdots=x_n\eta^m$. Since it follows from Lemma 1 that η (and hence η^m) induces an isomorphic mapping of J_i, this implies that $x_1=\cdots=x_n=0$ and therefore, $x=0$. Hence $C\cap R(\eta)=0$. Since $S\cap K(\eta)\leqq S\cap R(\eta)$, we deduce from the Dedekind law that $S\cap R(\eta)$

$= [S\cap R(\eta)]\cap[C+(S\cap K(\eta))] = [S\cap R(\eta)\cap C]+[S\cap K(\eta)] = S\cap K(\eta)$. Thus $S = C\oplus[S\cap R(\eta)]$.

That C is the only normal (M, η)-admissible subgroup such that $S = C\oplus[S\cap R(\eta)]$ is a consequence of the fact that C contains every minimal normal (M, η)-admissible subgroup which is not part of $S\cap R(\eta)$.

If η is normality preserving, $J\eta = J$ for a minimal normal (M, η)-subgroup J contained in C. Hence if $J\leqq C$, $J = J\eta\leqq C\eta$ so that $C\leqq C\eta$ and $C = C\eta$.

An example shows that for the last result, that is, that $S = S\eta\oplus[S\cap R(\eta)]$, it is necessary to assume that η is a normality preserving M-endomorphism of G.

EXAMPLE 1. Let S be a simple group which possesses an isomorphic proper subgroup R (for instance, S can be taken as the group of all finite even permutations of the positive integers, and R as the subgroup which consists of all permutations of S which leave 1 fixed; see Schreier-Ulam [1]). Let M be void, and let η be the endomorphism which maps S isomorphically onto R.

Then S is a minimal normal (M, η)-subgroup of itself and $K(\eta) = 0$ but $S\neq S\eta = R$.

PROPOSITION 1. *If the L_i form an ascending Loewy chain of (M, η)-admissible subgroups of G, then $L_i\cap R(\eta) = L_i\cap K(\eta^i)$ and η induces an isomorphism of $L_i\eta^i$ upon $L_i\eta^{i+1}$. If η is normality preserving, $L_i = L_i\eta^i\oplus[L_i\cap R(\eta)]$ and $L_i\eta^i$ is the one and only complement of the endomorphism which is induced in L_i by η.*

PROOF. We show first that $L_i\cap R(\eta) = L_i\cap K(\eta^i)$. By Lemma 2, $L_1\cap R(\eta) = L_1\cap K(\eta)$; assume that $L_{i-1}\cap R(\eta) = L_{i-1}\cap K(\eta^{i-1})$. Since L_i/L_{i-1} is the sum of minimal normal (M, η)-subgroups of G/L_{i-1}, by Lemma 2 $L_i/L_{i-1} = C_i/L_{i-1}\oplus R_i/L_{i-1}$, where R_i consists of all x in L_i such that $x\eta$ is in L_{i-1}; and R_i contains every x in L_i such that $x\eta^j$ is in L_{i-1} for some j. Suppose that x is in $L_i\cap R(\eta)$; then $x\eta^m = 0$ for some m and hence x is in R_i. But this implies that $x\eta$ is in $L_{i-1}\cap R(\eta)\leqq K(\eta^{i-1})$. Thus $x\eta^i = (x\eta)\eta^{i-1} = 0$ and x is in $K(\eta^i)$. Therefore, $L_i\cap R(\eta) = L_i\cap K(\eta^i)$.

It is clear that η maps $L_i\eta^i$ homomorphically upon $L_i\eta^{i+1}$. Suppose that $x_i\eta^i$ is in $L_i\eta^i$ with x_i in L_i and that $(x_i\eta^i)\eta = 0$; then x_i is an element of $L_i\cap R(\eta) = L_i\cap K(\eta^i)$ so that $x_i\eta^i = 0$. Hence η maps $L_i\eta^i$ isomorphically upon $L_i\eta^{i+1}$.

For the remainder of the proof we assume that η is a normality preserving M-endomorphism of G. By Lemma 2, $L_1\eta^2 = L_1\eta$; we use induction to show that $L_i\eta^{i+1} = L_i\eta^i$ for $i = 1, 2, 3, \cdots$.

Assume that $L_{i-1}\eta^i = L_{i-1}\eta^{i-1}$. Then $L_i/L_{i-1} = C_i/L_{i-1} \oplus R_i/L_{i-1}$, where R_i is the subgroup of all elements x in L_i such that $x\eta$ belongs to L_{i-1} and where $[L_i/L_{i-1}]\eta = [C_i/L_{i-1}]\eta = C_i/L_{i-1}$ by Lemma 2. Hence $C_i = L_i\eta + L_{i-1} = C_i\eta + L_{i-1}$, $C_i\eta^{i-1} = L_i\eta^i + L_{i-1}\eta^{i-1} = L_i\eta^i + L_{i-1}\eta^i = L_i\eta^i$; on the other hand, $C_i\eta^{i-1} = C_i\eta^i + L_{i-1}\eta^{i-1} = C_i\eta^i + L_{i-1}\eta^i = C_i\eta^i$. Hence $L_i\eta^i = C_i\eta^{i-1} = C_i\eta^i = L_i\eta^{i+1}$. Thus we have shown that η (and its powers) induce automorphisms in every $L_i\eta^i$.

We prove next that $L_i = L_i\eta^i \oplus [R(\eta)\cap L_i]$. $L_i\eta^i$, $R(\eta)\cap L_i$ are normal subgroups and $L_i\eta^i \cap [R(\eta)\cap L_i] = L_i\eta^i \cap K(\eta^i) = 0$, since η^i induces an automorphism of $L_i\eta^i$. Hence it suffices to show that $L_i = L_i\eta^i + [R(\eta)\cap L_i]$.

Let x_i be in L_i; since $L_i\eta^i = L_i\eta^{2i}$, there exists a y_i in L_i such that $x_i\eta^i = [y_i\eta^i]\eta^i$ and $x_i = y_i\eta^i + (-y\eta^i + x_i)$. Since$(-y\eta^i + x_i)\eta^i = -x_i\eta^i + x\eta^i = 0$, $-y\eta^i + x_i$ is in $R(\eta)\cap L_i$ and $L_i = L_i\eta^i \oplus [R(\eta)\cap L_i]$.

Finally we show that $L_i\eta^i$ is the one and only complement of η in L_i. We have already shown that $L_i\eta^i$ is a complement of η in L_i. Suppose C' is any complement of η in L_i. If x' is in C' then $x' = x_i\eta^i + z$, with x_i in L_i, and z in $R(\eta)\cap L_i = K(\eta^i)\cap L_i$; hence $x'\eta^i = x_i\eta^{2i}$ so that $C' = C'\eta^i \leqq L_i\eta^i$; but since C' has one element in each coset of L_i, modulo $L_i\cap R(\eta)$, $C' = L_i\eta^i$.

THEOREM 1. *If η is a normality preserving M-endomorphism of the M-group G, and if there exists an ascending Loewy chain of normal (M, η)-subgroups such that G is their join, then $G = C(\eta) \oplus R(\eta)$, where $C(\eta)$ is the unique normal complement of η. Furthermore, if N is a normal (M, η)-subgroup of G, $N = [N\cap C(\eta)] \oplus [N\cap R(\eta)]$.*

PROOF. Let $0 = S_0 \leqq S_1 \leqq \cdots \leqq S_i \leqq \cdots$ be an ascending Loewy chain of normal (M, η)-subgroups of G and let $N = \sum_{i=1}^{\infty} S_i$. By Proposition 1, $S_i = S_i\eta^i \oplus [S_i\cap R(\eta)]$. If x is an element of $N\cap R(\eta)$, x is some $S_i\cap R(\eta)$ so that $N\cap R(\eta) = \sum_{i=1}^{\infty} [S_i\cap R(\eta)]$. Let $Q = \sum_{i=1}^{\infty} S_i\eta^i$; since $S_i\eta^i\cap R(\eta) = 0$ for each i, $Q\cap R(\eta) = 0$. Furthermore, $S_i \leqq S_i\eta^i + [N\cap R(\eta)] \leqq Q + [N\cap R(\eta)] \leqq N$ for each i; therefore, $N = Q \oplus [N\cap R(\eta)]$. Also $S_i\eta^i = S_i\eta^{i+1} \leqq Q\eta$ for each i so that $Q\eta = Q$.

Now let $0 = L_0 \leqq L_1 \leqq \cdots \leqq L_i \leqq \cdots$ be an ascending Loewy chain of normal (M, η)-subgroups of G such that $G = \sum_{i=1}^{\infty} L_i$. Then if $C(\eta) = \sum_{i=1}^{\infty} L_i\eta^i$, $G = C(\eta) \oplus R(\eta)$ and $C(\eta)$ is a complement of η. Let N be a normal (M, η)-subgroup of G; then $0 = N\cap L_0 = L_0' \leqq N\cap L_1 = L_1' \leqq \cdots \leqq N\cap L_i = L_i' \leqq \cdots$ is an ascending chain of normal (M, η)-subgroups of G with $N = \sum_{i=1}^{\infty} L_i'$. Also $L_{i+1}'/L_i' = N\cap L_{i+1}/[N\cap L_{i+1}]\cap L_i \cong (L_i + [N\cap L_{i+1}])/L_i \leqq L_{i+1}/L_i$; this isomorphy between L_{i+1}'/L_i' and the normal (M, η)-subgroup

$L_i+[L_{i+1}\cap N]/L_i$ of G/L_i maps normal subgroups of G/L_i' upon normal subgroups of G/L_i. Hence L_{i+1}'/L_i' is the sum of minimal normal (M, η)-subgroups of G/L_i' and the L_i' form an ascending Loewy chain. Therefore, as we showed above, $N=\sum_{i=1}^{\infty}L_i'\eta^i \oplus [N\cap R(\eta)]$. But $\sum_{i=1}^{\infty}L_i'\eta^i=\sum_{i=1}^{\infty}[N\cap L_i]\eta^i \leqq N\cap \sum_{i=1}^{\infty}L_i\eta^i = N\cap C(\eta)$ and we deduce from the Dedekind law that

$$\begin{aligned} N \cap C(\eta) &= \left[\sum_{i=1}^{\infty} L_i'\eta^i + [N \cap R(\eta)]\right] \cap [N \cap C(\eta)] \\ &= \sum_{i=1}^{\infty} L_i'\eta^i + [[N \cap R(\eta)] \cap [N \cap C(\eta)]] \\ &= \sum_{i=1}^{\infty} L_i'\eta^i. \end{aligned}$$

Hence $N=[N\cap C(\eta)]\oplus[N\cap R(\eta)]$.

Assume that C' is a normal complement of G. Then $C'=C'\cap C(\eta)$, since $C'\cap R(\eta)=0$; but this implies that $C'\leqq C(\eta)$, which shows that $C'=C(\eta)$. Thus $C(\eta)$ is the unique normal complement of η.

If the M-endomorphism η of the group G possesses a complement, then η is said to *split* G. If S is an M-subgroup of G, and if $S\eta\leqq S$, then η induces an M-endomorphism in S. If this induced endomorphism splits S, for every (M, η)-admissible subgroup S of G, then we say that η is a ***uniformly splitting endomorphism*** of G. Elsewhere a characterization of uniformly splitting endomorphisms has been given (Baer [2, Theorem A]); and it has been shown there that *uniformly splitting endomorphisms possess one and only one complement* (Baer [2, Corollary to Theorem A]).

The criterion for splitting given in Theorem 1 assures splitting of normal subgroups only so that this result cannot be used; and we can prove only the uniqueness of normal complements.

REMARK 1. Elsewhere (Baer [2, p. 515]) one may find an example of an endomorphism of an abelian operator group which does not split in spite of the fact that this group is the $(\omega+1)$th term of its ascending Loewy chain. Thus our hypothesis on the length of the Loewy chain cannot be weakened in general.

We offer one application of the preceding results. If G is an M-group, then we define inductively the normal M-subgroups $L_i(G)$ of G by the rules:

$L_0(G)=0$, $L_{i+1}(G)/L_i(G)$ is the sum of all the minimal normal M-subgroups of $G/L_i(G)$.

It is easily seen that the $L_i(G)$ form a well defined ascending chain of characteristic and normal M-subgroups of G and that every normality preserving M-endomorphism of G maps every L_i into itself.

THEOREM 2. *Suppose that the M-group G meets the following re-*

quirements:

(a) *Every element in G belongs to at least one $L_i(G)$.*

(b) *$L_{i+1}(G)/L_i(G)$ is the sum of a finite number of minimal normal M-subgroups of $G/L_i(G)$.*

Then normality preserving M-endomorphisms of G possess unique normal complements.

Proof. The normal M-subgroups of G situated between $L_i(G)$ and $L_{i+1}(G)$ satisfy the double chain condition, by (b). If η is a normality preserving M-endomorphism of G, then $L_i(G)\eta \leqq L_i(G)$. Consequently we may construct, by intercalation of terms between $L_i(G)$ and $L_{i+1}(G)$, an ascending chain of (M, η)-admissible normal subgroups S_i of G with the following properties: S_{i+1}/S_i is either 0 or a minimal normal (M, η)-subgroup of G/S_i; every element in G belongs to at least one S_i. Then the S_i form an ascending Loewy chain of normal (M, η)-subgroups of G with join G; and now it is clear how to deduce Theorem 2 from the preceding results.

Remark 2. It is not difficult to verify that Theorem C of Baer [2], if applied only on operator groups whose admissible subgroups are normal, will be a special case of Theorem 2.

2. **Almost periodicity on a subgroup.** The M-endomorphism η of the M-group G is said to be *almost periodic on the M-subgroup S of G*, if

(a) $S\eta \leqq S$ and

(b) there exists to every element s in S a positive integer $n=n(s)$ such that $s=s\eta^n$.

If η is almost periodic on the normal M-subgroup N of G, then η induces an M-endomorphism in G/N. It is our object in the present section to relate the splitting properties of this induced endomorphism with those of η itself.

Lemma 1. *If the M-endomorphism η of the M-group G is almost periodic on the normal M-subgroup N of G, then*

$$N \cap R(\eta) = 0$$

and $[N \oplus R(\eta)]/N$ is the radical of the endomorphism induced by η in G/N.

Proof. If x belongs to $N \cap R(\eta)$, then there exist positive integers i and n such that $x\eta^i=0$ and $x\eta^n=x$. Then $x=x\eta^{in}=0$. Thus $N \cap R(\eta) =0$.

It is clear that $[N+R(\eta)]/N$ is part of the radical R/N of the endomorphism which η induces in G/N. If furthermore x belongs to

R, then there exists a positive integer i such that $(N+x)\eta^i=0$; and this is equivalent to saying that $x\eta^i$ belongs to N. Consequently there exists a positive integer $n=n(x\eta^i)$ such that $x\eta^i=x\eta^{i+n}$. This implies $x\eta^i=x\eta^{i+in}$. Let $y=x-x\eta^{in}$. Then $y\eta^i=0$ so that y belongs to $R(\eta)$. Furthermore $x\eta^{in}=x(\eta^i)^n$ belongs to N, since $x\eta^i$ belongs to N, and since N admits η. Hence $x=y+x\eta^{in}$ belongs to $R(\eta)+N$. This shows that $R=R(\eta)+N$; and this completes the proof.

PROPOSITION 1. *Suppose that the M-endomorphism η of the M-group G is almost periodic on the normal M-subgroup N of G. Then C is a complement of η if, and only if, $N\leqq C$ and C/N is a complement of the endomorphism induced by η in G/N.*

PROOF. Assume first that C is a complement of η. If x is an element in N, then there exists one and only one element c in C such that $x\equiv c$ modulo $R(\eta)$. Consequently there exists a positive integer i such that $x\eta^i=c\eta^i$. Since x is in N, there exists a positive integer n such that $x=x\eta^n$. Hence $x=x\eta^{ni}=c\eta^{in}$ belongs to C, since C admits η. Thus we have shown that $N\leqq C$.

From $N\leqq C$ we infer by Dedekind's law that

$$C\cap[N+R(\eta)]=N+[C\cap R(\eta)]=N,$$

since the complement C meets $R(\eta)$ in 0. It follows now from Lemma 1 that 0 is the cross cut of C/N and the radical of the endomorphism which η induces in G/N.

From $C+N+R(\eta)=C+R(\eta)=G$ we infer that G/N is the sum of C/N and of the radical of the endomorphism induced in G/N by η; and now it is easily verified that C/N is a complement of the endomorphism induced by η in G/N.

Assume now conversely that $N\leqq C$ and that C/N is a complement of the endomorphism which is induced by η in G/N. Since complement and radical have only 0 in common, it follows from Lemma 1 that

$$N=C\cap[N+R(\eta)]=N+[C\cap R(\eta)]$$

by Dedekind's law. Hence $C\cap R(\eta)\leqq N$ and it follows from Lemma 1 that

$$C\cap R(\eta)=C\cap R(\eta)\cap N=0.$$

Since the sum of radical and complement is the full group, it follows from Lemma 1 that

$$G=C+[N+R(\eta)]=C+R(\eta),$$

since $N\leqq C$. Thus we have shown that every coset of G modulo

$R(\eta)$ contains one and only one element in C. Since the endomorphism induces an automorphism in the complement, it follows that

$$C\eta + N = C.$$

Thus there exist to every element x in C elements y and z in C and N respectively such that $x = y\eta + z$. Since z is in N, there exists a positive integer n such that $z = z\eta^n$. Since N admits η, and since z is in N, $z\eta^{n-1}$ belongs to N too. Since $N \leqq C$, this implies that $y + z\eta^{n-1} = c$ is an element in C. Hence

$$x = y\eta + z = y\eta + z\eta^n = c\eta$$

belongs to $C\eta$. Thus we have shown that

$$C\eta \leqq C\eta + N = C \leqq C\eta \quad \text{or} \quad C = C\eta;$$

and this completes the proof of the fact that C is a complement.

For a convenient enunciation of the next proposition we say that an endomorphism is *normally splitting* if one of its complements is a normal subgroup; and that the endomorphism is *uniquely splitting* if it possesses one and only one complement.

PROPOSITION 2. *Suppose that the M-endomorphism η of the M-group G is almost periodic on the normal M-subgroup N of G. Then*

(a) *η is a normally splitting endomorphism of G if, and only if, η induces a normally splitting endomorphism in G/N.*

(b) *η is a uniquely splitting endomorphism of G if, and only if, η induces a uniquely splitting endomorphism in G/N.*

(c) *η is a uniformly splitting endomorphism of G if, and only if, η induces a uniformly splitting endomorphism in G/N.*

The statements (a) and (b) are immediate consequences of Proposition 1, and a proof of (c) may be constructed without too much difficulty by using Proposition 1 and Baer [2, Corollary to Theorem A, p. 512]. We note that this theorem constitutes a generalization of Baer [2, Theorem B].

3. The partially ordered set of extension types. It will be convenient to express the fact that N is a normal and M-admissible subgroup of the M-group G by saying: *the M-group G is an extension of the M-group N.* This makes it possible to begin with some M-group N and to consider all the extensions of N. For comparison of extensions we need the following concept.

DEFINITION 1. The M-homomorphism η of the extension G of N into the extension H of N is a normality preserving N-M-homomorphism, if

(a) $x = x\eta$ for every x in N; and

(b) η maps normal M-subgroups of G upon normal M-subgroups of H.

Using this concept we are able to construct the partially ordered set of extension types of the M-group N. If the M-groups G and H are extensions of the M-group N, and if there exists a normality preserving N-M-homomorphism of G into H, then we say that the extension type of the extension G of N is less than or equal to the extension type of the extension H of N; in symbols: $\langle N<G\rangle \leqq \langle N<H\rangle$. It is clear that $\langle N<A\rangle \leqq \langle N<B\rangle$ and $\langle N<B\rangle \leqq \langle N<C\rangle$ imply $\langle N<A\rangle \leqq \langle N<C\rangle$. If we supplement this definition by the further definition: $\langle N<G\rangle = \langle N<H\rangle$ if, and only if, $\langle N<G\rangle \leqq \langle N<H\rangle$ *and* $\langle N<H\rangle \leqq \langle N<G\rangle$ then we see easily that *the extension types of N form a partially ordered set.* (See Baer [3] for a discussion of these concepts in a concrete situation.)

In order to be able to apply the results of the preceding sections to the problem of characterizing the extension types we are going to restrict our attention to those extensions which meet the following requirement:

(S) *Every normality preserving N-M-endomorphism of the extension G of the M-group N is a normally splitting endomorphism (possesses a complement which is a normal subgroup).*

Remark 1. It is an immediate consequence of §2, Proposition 2 that the extension G of the M-group N meets requirement (S), if

(S′) *every normality preserving M-endomorphism of G/N is a normally splitting endomorphism of G/N.*

We note furthermore that, as a consequence of §1, Theorem 2, Condition (S′) will be satisfied whenever G/N meets the following requirement:

(S″) *Every element in G/N belongs to at least one $L_\iota(G/N)$; and every $L_{\iota+1}(G/N)/L_\iota(G/N)$ is the sum of a finite number of minimal normal M-subgroups of $[G/N]/L_\iota(G/N)$.*

Consequently the subsequent results will be applicable whenever we restrict our attention to extensions by groups G/N meeting requirement (S″).

Theorem 1. *Suppose that G and H are extensions of the M-group N and that Condition* (S) *is satisfied by the extension G of N. Then $\langle N<G\rangle = \langle N<H\rangle$ if, and only if, there exist direct decompositions of the M-groups G and H:*

$$\text{(D)} \qquad G = G' \oplus G'' \quad \text{and} \quad H = H' \oplus H''$$

such that $N \leqq G'$ and there exists an M-isomorphism of G' upon H' which leaves invariant every element in N.

REMARK 2. The existence of an M-isomorphism of G' upon H' which leaves invariant every element in N implies that $N \leqq H'$ and that, in the usual terminology borrowed from Galois theory, *G' and H' are equivalent extensions of the M-group N.*

PROOF. If there exist decompositions (D) which meet our requirements, then we denote by γ the normality preserving N-M-endomorphism of G which leaves invariant every element in G' and which maps G'' upon 0; and we denote by ρ an M-isomorphism of G' upon H' which leaves invariant every element in N. Then $\gamma\rho$ is a normality preserving N-M-homomorphism of G into H so that $\langle N<G\rangle \leqq \langle N<H\rangle$; and $\langle N<H\rangle \leqq \langle N<G\rangle$ may be shown likewise. Hence $\langle N<G\rangle = \langle N<H\rangle$. Thus our conditions are sufficient.

Assume conversely that $\langle N<G\rangle = \langle N<H\rangle$. Then there exists a normality preserving N-M-homomorphism α of G into H and a normality preserving N-M-homomorphism β of H into G. It is clear that $\alpha\beta$ is a normality preserving N-M-endomorphism of G. Thus we may apply Condition (S) on $\alpha\beta$. Consequently there exists a complement C of $\alpha\beta$ which is a normal M-subgroup of G. Furthermore, since $\alpha\beta$ leaves invariant every element of N, it follows from §2, Proposition 1 that $N \leqq C$. This shows that

(1) $G = C \oplus R(\alpha\beta)$, $N \leqq C$ and $\alpha\beta$ induces an automorphism of C.

If x is in C and $x\alpha = 0$, then $x\alpha\beta = 0$. But $\alpha\beta$ induces an automorphism in C. Hence $x = 0$. Consequently

(2) α induces an isomorphism of C upon $D = C\alpha$ which leaves invariant every element in N.

Since C is a normal subgroup of G, and since α maps normal M-subgroups of G upon normal M-subgroups of H, D is a normal M-subgroup of H. If x belongs to the cross cut of D and $R(\beta\alpha)$, then $x\beta$ belongs to the cross cut of $D\beta = C\alpha\beta = C$ and $R(\beta\alpha)\beta \leqq R(\alpha\beta)$, as is easily seen. But the latter cross cut is 0 so that $x\beta = 0$. Since x is in $D = C\alpha$, there exists an element y in C such that $x = y\alpha$. Hence $y\alpha\beta = x\beta = 0$. But $\alpha\beta$ induces an automorphism in C. Hence $y = 0$ and this implies $x = 0$. Consequently $D \cap R(\beta\alpha) = 0$. If h is an element in H, then $h\beta$ belongs to $G = C \oplus R(\alpha\beta)$. Consequently there exist elements h', h'' in C and $R(\alpha\beta)$ respectively such that $h\beta = h' + h''$. Since $\alpha\beta$ induces an automorphism in C, we have $C = C\alpha\beta$. Hence there exists an element c in C such that $h' = c\alpha\beta$. Then $(-c\alpha + h)\beta = h''$ is an element in $R(\alpha\beta)$; and this implies clearly that $-c\alpha + h$ belongs to $R(\beta\alpha)$. Since $c\alpha$ belongs to $C\alpha = D$, it follows that h belongs to $D + R(\beta\alpha)$. Hence we have shown that $H = D + R(\beta\alpha)$. Combining the results of this paragraph we obtain:

$$H = D \oplus R(\beta\alpha). \tag{3}$$

But it follows from (2) that C and D are equivalent extensions of the M-group N. Thus these direct decompositions $G=C\oplus R(\alpha\beta)$ and $H=D\oplus R(\beta\alpha)$ are of the desired type (D). This completes the proof.

REMARK 3. Theorem 1 shows that extensions of the M-group N which belong to the same extension type really have all the essential features of the extension in common. For if $G=G'\oplus G''$ and $N\leqq G'$, then G'' is clearly a trivial component of the extension whereas the problematic part of this extension is already contained in the step from N to G'. This leads us to the following concept.

DEFINITION 2. The extension G of the M-group N is a minimal extension of N, if $G=G'\oplus G''$ and $N\leqq G'$ imply that $G=G'$ and $G''=0$.

It is a rather obvious consequence of Theorem 1 that minimal extensions of the M-group N which meet requirement (S) are equivalent extensions.

THEOREM 2. *Suppose that the extension G of the M-group N satisfies Condition* (S); *and that*

$$G = E' \oplus J' = E'' \oplus J''$$

are direct decompositions of the M-group G such that E' and E'' are minimal extensions of the M-group $N\neq 0$. Then

(a) *E' and E'' are equivalent extensions of the M-group N;*

(b) *J' and J'' are M-isomorphic M-groups;*

(c) *there exists an M-automorphism of G which leaves invariant every element in N, maps E' upon E'' and J' upon J''; and*

(d) $G=E'\oplus J''=E''\oplus J'$.

PROOF. We note first that Condition (S) is satisfied by E' and E'', since every normality preserving N-M-endomorphism of E' (E'') is induced by one and only one endomorphism of G which maps J' (J'') upon 0, and since the latter is necessarily a normality preserving N-M-endomorphism of G.

Denote now by α, β, γ, δ the decomposition endomorphisms satisfying $J'\alpha=E'\beta=J''\gamma=E''\delta=0$ and $x=x\alpha$ for x in E', $x=x\beta$ for x in J', $x=x\gamma$ for x in E'', $x=x\delta$ for x in J''.

Then $\alpha\gamma$ is a normality preserving N-M-endomorphism in E''. Since (S) holds in E'', there exists a direct decomposition $E''=C\oplus D$ such that D is the radical of the endomorphism $\alpha\gamma$ of E'' and C a complement. It follows from §2, Proposition 1 that $N\leqq C$. But E'' is a minimal extension of N. Hence $C=E''$. But $\alpha\gamma$ induces automorphisms in its complements. Thus we have shown that

(1″) $\alpha\gamma$ induces an automorphism in E'';

and we see likewise that

(1′) $\gamma\alpha$ induces an automorphism in E'.

It is clear that $E'\gamma \leqq E''$ and $E''\alpha \leqq E'$. Hence we have

$$E' = E'\gamma\alpha \leqq E''\alpha \leqq E' \quad \text{or} \quad E' = E''\alpha;$$

and $E'' = E'\gamma$ is shown likewise. Now one deduces readily from (1′) and (1″) that

(2) γ induces an M-isomorphism of E' upon E'' and α induces an M-isomorphism of E'' upon E'; and both these isomorphisms leave invariant every element in N.

The statement (a) is contained in (2). Next we prove

$$G = E' \oplus J'' = E'' \oplus J'. \tag{3}$$

If x belongs to $E' \cap J''$, then $x\gamma = 0$. But γ induces, by (2), an M-isomorphism in E'. Hence $x = 0$. Thus we have shown that

$$E' \cap J'' = 0.$$

If x is an element in E', then $x = x\gamma + x\delta$. But $x\delta$ belongs to J''; and so $x\gamma$ is in $E' \oplus J''$. Hence it follows from (2) that

$$E'' = E'\gamma \leqq E' \oplus J''.$$

Hence $G = E'' \oplus J'' \leqq E' \oplus J''$ or $G = E' \oplus J''$, and the other equation (3) is shown likewise.

From (3) we deduce (b) in the stronger form:

(b*) *there exist central M-isomorphisms between J' and J''*; and (a) may be strengthened likewise.

The statement (c) is a fairly obvious consequence of (a) and (b), and (d) is just a restatement of (3).

Remark 4. To insure the existence of direct decompositions of G, as considered in Theorem 2, one may make for instance either of the following hypotheses:

(A) Every ascending chain of direct summands of G terminates after a finite number of steps; or

(D) every descending chain of direct summands of G which all contain N terminates after a finite number of steps.

Remark 5. Combining Theorems 1 and 2 one sees that the extension type of an extension G of N is essentially characterized by the "essential component" E of G which is a minimal extension of N, provided this extension meets suitable requirements, like (S) and (A) on (D). The "inessential component" J is uniquely determined too (up to central isomorphisms) and is needed to characterize the equivalence class of G within its extension type.

Appendix: Q-groups. Groups which are not isomorphic to proper quotient groups have been termed elsewhere Q-groups (Baer [1]). It is an important problem to decide whether all finitely generated groups are Q-groups. This problem is as yet unsolved. An example purporting to be a counter-example contains a fairly obvious mistake (Baer [1, p. 273, line 12]). Thus the following criterion appears to be interesting.

Theorem. *If G is a finitely generated M-group, and if every normality preserving M-endomorphism of G splits G, then G is a Q-group.*

Proof. If N is a normal M-subgroup of G such that G and G/N are M-isomorphic, then there exists an M-endomorphism η such that N is the kernel of η and such that $G = G\eta$. It is clear that η maps normal M-subgroups of G upon normal M-subgroups of G, since every subset of G is the image under η of some subset of G. Thus η is a normality preserving M-endomorphism of G; and we infer from our hypothesis the existence of a complement C of η.

Since G is finitely generated, there exists a finite number of elements $g_1, \cdots, g_n$ which generate the M-group G. Since every coset of $G/R(\eta)$ contains one and only one element in the complement C, there exist uniquely determined elements c_i in C such that $g_i \equiv c_i$ modulo $R(\eta)$. Thus every $g_i - c_i$ belongs to $R(\eta)$. Since $g_i - c_i$ are finite in number, there exists a positive integer m such that

$$(g_i - c_i)\eta^m = 0 \text{ for } i = 1, \cdots, n \quad \text{or} \quad g_i\eta^m = c_i\eta^m \text{ for } i = 1, \cdots, n.$$

If t is an element in M, then

$$(tg_i)\eta^m = t(g_i\eta^m) = t(c_i\eta^m) = (tc_i)\eta^m \qquad \text{for } i = 1, \cdots, n.$$

Thus g_i and Mg_i are mapped by η^m into $C\eta^m = C$, since C as a complement is η-invariant. But the M-group G is generated by the elements g_i.

Hence it follows from $G = G\eta$ that

$$G = G\eta^m \leqq C \leqq G \text{ or } G = C, N \leqq G \cap R(\eta) = C \cap R(\eta) = 0 \text{ or } N = 0.$$

Hence G is a Q-group, as we claimed. It follows furthermore that η is an automorphism of G; and thus one deduces readily from the preceding arguments the validity of the following proposition.

Corollary. *The M-endomorphism η of the finitely generated M-group G is an automorphism if, and only if, $G = G\eta$ and η splits G.*

Remark. An attempt to combine the preceding theorem with §1, Theorem 2 leads to trivial results, since finitely generated M-groups G, satisfying the conditions (a) and (b) of §1, Theorem 2, satisfy the double chain condition for normal M-subgroups. But an M-group is easily (and trivially) seen to be a Q-group, if it satisfies the ascending chain condition for normal M-subgroups.

BIBLIOGRAPHY

REINHOLD BAER

1. *Groups without proper isomorphic quotient groups*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 267–278.
2. *Splitting endomorphisms*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 508–516.
3. *Extension types of Abelian groups*, Amer. J. Math. vol. 71 (1949) pp. 461–490.

J. SCHREIER AND S. ULAM

1. *Über die Permutationsgruppe der natürlichen Zahlenfolge*, Studia Math. vol. 4 (1933) pp. 134–141.

UNIVERSITY OF ILLINOIS AND
CORNELL UNIVERSITY

A NOTE ON LOCALLY COMPACT GROUPS

A. M. GLEASON

In this note we shall prove that every locally compact group can be embedded as a closed subgroup in a unimodular group. If the original group is locally Euclidean, the enlarged group will be also, hence the fifth problem of Hilbert is reduced to the unimodular case.

We shall use certain results concerning Haar measure whose proof may be found in A. Weil, *L'integration dans les groupes topologiques et ses applications*, Paris, Hermann, 1938, Chap. II, which we shall refer to as AW.

Let $\int$ stand for a left Haar integral. There is (AW) a continuous function Δ defined on G such that $\int_G f(xs^{-1})dx = \Delta(s)\int_G f(x)dx$ for all integrable functions f and all $s \in G$.

LEMMA. *Let G be a locally compact topological group and let K be a closed normal unimodular subgroup such that G/K is also unimodular. If $s \in G$ the automorphism of K defined by $k \to s^{-1}ks$ multiplies the Haar measure of K by $\Delta(s)$.*

PROOF. Let s be fixed and let μ be a left Haar measure on K. Because of unicity properties of Haar measure, there is a constant c such that, for all measurable sets E, $\mu(s^{-1}Es) = c\mu(E)$; we must show that $c = \Delta(s)$. The first formula implies that, for any integrable function f, $\int_K f(k)d(s^{-1}ks) = c\int_K f(k)dk$, which may be abbreviated by the formula $d(s^{-1}ks) = cdk$.

Let f be a non-negative continuous function on G which vanishes outside a compact set but is not everywhere zero. If the integrals are suitably normalized, we have (AW) $\int_G f(x)dx = \int_{G/K} F(X)dX$, where $x \to X$ is the natural map of G onto G/K and $F(X) = \int_K f(xy)dy$, the latter being independent of the choice of representative x of X. Let $f^*(x) = f(xs^{-1})$. Putting $F^*(X) = \int_K f^*(xy)dy = \int_K f(xys^{-1})dy = \int_K f(xs^{-1}k)d(s^{-1}ks) = c\int_K f(xs^{-1}k)dk = cF(XS^{-1})$, we have

$$\int_G f(xs^{-1})dx = \int_G f^*(x)dx = \int_{G/K} F^*(X)dX = c\int_{G/K} F(XS^{-1})dX$$

$$= c\int_{G/K} F(X)dX = c\int_G f(x)dx$$

(recalling that G/K is unimodular), which shows that $c = \Delta(s)$ since $\int_G f(x)dx$ is not zero.

Received by the editors June 11, 1948.

THEOREM. *A locally compact group G is isomorphic to a closed subgroup of a unimodular locally compact group whose space is $G\times R$, where R is the real line.*

PROOF. The function Δ is a homomorphism of G into the positive real numbers under multiplication since $\Delta(st)=\Delta(s)\Delta(t)$. The kernel K of this homomorphism is a unimodular normal subgroup of G (AW) and G/K is abelian and hence unimodular. Applying the lemma we obtain $d(gkg^{-1})=\Delta(g^{-1})dk$ for $k\in K$ and $g\in G$.

Let $H=G\times R$ as a topological space; then H is locally compact. Define a binary operation in H by setting $(g_1, r_1)(g_2, r_2)=(g_1g_2, r_1\Delta(g_2^{-1})+r_2)$. This is easily verified to be associative; there is a unit element $(e, 0)$; the element (g, r) has the inverse $(g^{-1}, -r\Delta(g))$; and both the operation and its inverse are continuous: hence H is a locally compact topological group. Since $G\times\{0\}$ is a closed subgroup of H isomorphic to G, the proof will be complete if we show that H is unimodular.

The set L of all pairs (k, r), where $k\in K$, is a closed subgroup of H. Since $\Delta(k)=1$ for all $k\in K$, L is the direct product of the groups K and R. The Haar measure of L is the product of the Haar measures of K and R, which we may symbolize by the formula $d(k, r)=dkdr$. Since both these groups are unimodular, L is unimodular. Let (g, x) be a fixed element of H, and let $(k, r)\in L$. Then $(g, x)(k, r)(g, x)^{-1}=(gkg^{-1}, r\Delta(g))\in L$. Hence L is normal. Moreover, the automorphism of L induced by (g, x) is measure preserving since $d(gkg^{-1}, r\Delta(g))=d(gkg^{-1})d(r\Delta(g))=\Delta(g^{-1})dk\Delta(g)dr=dkdr=d(k, r)$. The group H/L $(\approx G/K)$ is abelian and hence unimodular. Applying the lemma, we see that H is unimodular.

HARVARD UNIVERSITY

ON THE EXISTENCE OF THE INVERSE OPERATION IN ALTERNATION GROUPOIDS[1]

MARLOW SHOLANDER

Introduction. An Abelian quasigroup S may be defined as a set of elements $a, b, c, \cdots$ for which the following postulates hold:

Postulate I. There is an equivalence relation in S, denoted by "$=$" —that is, equality of elements is reflexive, symmetric, and transitive. There is a binary operation in S, denoted by # or, when convenient, by the notation of multiplication. It is understood that this implies S is closed with respect to #, and that # is uniquely defined in S—that is, if $a=b$ and $c=d$, then $ac=bd$.

Postulate II. # is an alternation—that is, if a, b, c, and d are elements in S, $(ab)(cd)=(ac)(bd)$.

Postulate III. Each element in S is proper (see §2 for definitions).

D. C. Murdoch [1, p. 516][2] pointed out that an Abelian quasigroup is a natural generalization of an Abelian group. The same author [2, Theorem 11] proved that in an Abelian quasigroup it is always possible to define a new operation under which the elements of the quasigroup form an Abelian group. R. H. Bruck [3, Theorem 12] in a sense completed the theory by showing how all Abelian quasigroups may be derived from Abelian groups.

In this paper a set of elements S for which postulates I and II hold is called an alternation groupoid. A series of extensions of an alternation groupoid is described which for certain groupoids leads to an Abelian quasigroup (Corollary 5.5). This imbedding process has as a special case the well known procedure for imbedding commutative semigroups in groups or, more generally, the procedure for imbedding a space S, in which there is a commutative and associative operation, in a space S' in such a way that each "regular" element in S has an inverse in S' (see, for example, [4, p. 24]). It should be noted that A. Malcev [5, §2] has given an example of a noncommutative semigroup which cannot be imbedded in a group.

Some interesting examples of Abelian quasigroups and alternation groupoids are given in §1.

1. **Examples of alternation groupoids.** The examples of Abelian

---

Presented to the Society, November 30, 1946; received by the editors June 12, 1948.

[1] A part of this paper was written while the author was receiving support from the Office of Naval Research.

[2] Numbers in brackets refer to references cited at the end of the paper.

quasigroups that have appeared in print fall mainly in two classes—those based on a finite set of elements and those based on a properly chosen set of numbers by defining $x \# y$ as $ax+by+c$, or cx^ay^b, where a, b, and c are constants (in particular, $\#$ can be ordinary addition, subtraction, multiplication, division, or the operation of taking an arithmetic or geometric mean). These examples give little indication of the variety of form found in alternation groupoids.

Moreover, the geometric interest of alternation groupoids can be given greater emphasis. For instance, each Abelian quasigroup furnishes an analogue of the theorem that the midpoints of a quadrilateral determine a parallelogram. To see this, let S be the set of points in the plane and let $\#$ be the operation of taking the midpoint. Consider the associated group operation defined by Murdoch—for fixed elements r and s in S, $x \circ y$ is defined as uv where $x=ur$ and $y=sv$. Under $\circ$, the elements of S form a commutative group which has sr as its unit element. Thus $ur \circ sv = uv \circ sr$ and sr, sv, uv, and ur are the vertices of a parallelogram. In the case of example E below, this process gives an interesting construction for an ordinary parallelogram. From example F, we obtain an odd construction for projective addition by following the steps in the definition of operation $\circ$.

A. Let S be the set of positive integers. Let α, β, γ, and δ be integers such that α, γ, $\alpha+\beta$, and $\gamma+\delta$ are positive and $\alpha\delta+\beta=\gamma\beta+\delta$. Define $x \# y$ as max $[\alpha x+\beta, \gamma y+\delta]$.

B. Let a and b be points in the projective plane. Let S be the set of points of the plane not on the line ab. Define $x \# y$ as the point of intersection of the lines xa and yb.

C. Let a be a point in the complex plane and S the set of points in the plane excluding a and ∞. Let $x \# y = \overline{x-a}(y-a)/\overline{y-a}+a$, the reflection of x with respect to the line ay.

D. Let a and S be chosen as in the preceding example but let $x \# y = (y-a)\overline{y-a}/\overline{x-a}+a$, the inverse of x with respect to the circle of center a which passes through y.

E. Let S be the set of points in the Euclidean plane and a, b, and c fixed elements of S. Let $x \# y$ be defined as the point z which determines a triangle xyz strictly similar to the triangle abc in such a way that vertices x, y, z correspond respectively to vertices a, b, c.

F. Let S be the set of points on the projective line, s a fixed element of S, and r a fixed real number. Let $x \# y$ be defined as the point z for which the double ratio $(sxyz)=r$. Defined in numerical terms $x \# y = [xy - s\{x(1-r)+yr\}]/[xr+y(1-r)-s]$ and when $r=1/2$ we see that the arithmetic and harmonic means are special cases of this alternation.

2. Definitions and notation. In the pages following S denotes an alternation groupoid. It is well known that if S has a unit element the alternation is commutative and associative and that, conversely, if an operation is commutative and associative it is an alternation.

An element a in S is called left regular if, for all elements x and y in S, $ax=ay$ implies $x=y$. In much of the material which follows we consider only "left" conditions and their consequences. We shall omit statements with corresponding "right" or "left and right" conditions. Here, for example, right regularity of an element has an obvious definition and to say an element is regular means that it is both left and right regular.

An element a in S is called left proper if, for all elements b in S, $ax=b$ has a unique solution, x, in S. It follows that each element left proper in S is left regular in S. An element θ in S is called left null if $\theta x=\theta$ for all elements x in S. If S has more than one element, a left null element is not left regular.

An element, $\phi(x)$, is a power of x if it belongs to the set of elements $\Phi(x)$ defined recursively by the rules:

(i). x belongs to $\Phi(x)$,

(ii). If y and z belong to $\Phi(x)$, yz belongs to $\Phi(x)$.

Murdoch has shown that in an alternation groupoid a power of a power of x is a power of x and that the following laws hold:

$$\phi(xy) = \phi(x)\phi(y), \qquad \phi_1\phi_2(x) = \phi_2\phi_1(x).$$

An element of S is said to be left regular* if all of its powers are left regular. We denote by L, L^*, and L^{**} respectively the sets of elements in S which are left regular, left regular*, and left proper in S. The sets of elements with the corresponding right properties are denoted by R, R^*, and R^{**} respectively. Finally, we let $P=L^{**}\cap R^{**}$ and $Q=L^*\cap R^*$.

3. Relations between subsets of S.

THEOREM 3.1. *L^* is an alternation groupoid.*

PROOF. It is sufficient to show that L^* is closed under $\#$. Let $\phi(ab)$ be a power of ab where a, $b\in L^*$. Assume that for some x, $y\in S$, $\phi(ab)x=\phi(ab)y$. Then

$$\begin{aligned}[\phi(aa)\phi(aa)][\phi(bb)x] &= [\phi(aa)\phi(bb)][\phi(aa)x] \\ &= \phi\{(aa)(bb)\}[\phi(aa)x] = \phi\{(ab)(ab)\}[\phi(aa)x] \\ &= [\phi(ab)\phi(aa)][\phi(ab)x] = [\phi(ab)\phi(aa)][\phi(ab)y].\end{aligned}$$

Similarly,

$$[\phi(aa)\phi(aa)][\phi(bb)y] = [\phi(ab)\phi(aa)][\phi(ab)y].$$

We may cancel to obtain $\phi(bb)x=\phi(bb)y$ and $x=y$. Hence L^* is closed under $\#$.

COROLLARY 3.2. *L is an alternation groupoid if and only if $L=L^*$.*

THEOREM 3.3. *L^{**} is an alternation groupoid.*

PROOF. We must show that L^{**} is closed under $\#$. For h and $k\in L^{**}$ and $a\in S$, we find a solution x of the equation $(hk)x=a$ by choosing y, z, w and x so that $hy=a$, $hz=h$, $kw=y$, and $x=zw$.

COROLLARY 3.4. *$L^{**}\subseteq L^*\subseteq L$.*

PROOF. The second inclusion relation follows from the definitions of L and L^*. Since the theorem implies that all powers of a left proper element are left proper and since a left proper element is left regular, the first inclusion relation is valid.

THEOREM 3.5. *If $ab\in L$, then $b\in L$. If in addition $R\neq 0$, $a\in L$.*

PROOF. If we have $bx=by$, $(ab)[(ab)x]=[a(ab)](bx)=[a(ab)](by)$ $=(ab)[(ab)y]$ and $x=y$. If $r\in R$ and $ax=ay$, $(ab)(xr)=(ab)(yr)$, and $x=y$.

COROLLARY 3.6. If $ab\in L^*$, $b\in L^*$. *If in addition $R\neq 0$, $a\in L^*$.*

THEOREM 3.7. *If $L^*\neq 0$ and $R^*\neq 0$, then $L^*=R^*=Q$.*

PROOF. Consider elements l and r in L^* and R^* respectively and p, a power of r. We first prove that $lp\in L$. Suppose $(lp)x=(lp)y$. Then

$$\begin{aligned}[(ll)x][(pp)(pp)] &= [(lp)x][(lp)(pp)]\\ &= [(lp)y][(lp)(pp)] = [(ll)y][(pp)(pp)]\end{aligned}$$

and $x=y$. By Theorem 3.5, $lp\in L$ implies $p\in L$. Hence $r\in L^*$ and $R^*\subseteq L^*$. Similarly, we prove $L^*\subseteq R^*$.

LEMMA 3.8. *If $hk\in L^{**}$, $h(hk)\in L$.*

PROOF. Let $hk=l$. If $(hl)x=(hl)y$ choose z and w so $lz=x$, $lw=y$. Then

$$\begin{aligned}[(ll)(ll)][l(lz)] &= [(ll)(ll)][(hl)(kz)]\\ &= [(ll)(hl)][(lk)x] = [(ll)(lk)][(hl)y].\end{aligned}$$

Similarly,

$$[(ll)(ll)][l(lw)] = [(ll)(lk)][(hl)y\ .$$

We cancel to obtain $z=w$. Hence $x=y$.

THEOREM 3.9. *If* $hk\in L^{**}$, $k\in L^{**}$.

PROOF. Let $l=hk$. To solve the equation $kx=a$ choose y and x so $ly=(hl)a$ and $lx=y$. Then $(hl)(kx)=(hk)(lx)=ly=(hl)a$. By the previous lemma, $kx=a$.

COROLLARY 3.10. *All elements left proper in* S *are left proper in* L^{**}.

COROLLARY 3.11. *If* $L^{**}\neq 0$ *and* $R^{**}\neq 0$, *then* $L^{**}=R^{**}=P$.

PROOF. To prove, say, $R^{**}\subseteq L^{**}$, consider $r\in R^{**}$. Since there exists an element x such that $xr=l$, it follows from the theorem that $r\in L^{**}$.

LEMMA 3.12. *If* $L^{**}\neq 0$, $a\in L^*$, $b\in L$, *and* $c\in R$, *then* $ab\in L$ *and* $bc\in R$.

PROOF. If $l\in L^{**}$, l and $al\in L^*$. If $(ab)x=(ab)y$, let $lz=x$ and $lw=y$. Then $(al)(bz)=(al)(bw)$, $z=w$, and $x=y$. Thus $ab\in L$. It follows as a particular case of this result that $lb\in L$. Hence if $x(bc)=y(bc)$, $(lb)(zc)=(lb)(wc)$, $z=w$, and $x=y$.

LEMMA 3.13. *If* $L^{**}\neq 0$ *and* $R\neq 0$, $L\subseteq R$.

PROOF. Let $b\in L$ and $c\in R$. By Lemma 3.12, $bc\in R$. By the "right" dual of Theorem 3.5, $bc\in R$ implies $b\in R$.

LEMMA 3.14. *If* $R\neq 0$, $L^{**}\subseteq R^*$.

PROOF. Let $r\in R$ and let $\phi(l)$ be a power of $l\in L^{**}$. By Lemma 3.12, $\phi(l)r\in R$. Hence $\phi(l)\in R$ and $l\in R^*$.

THEOREM 3.15. *If* $L^{**}\neq 0$ *and* $R\neq 0$, $L^{**}\subseteq Q\subseteq L\subseteq R$.

PROOF. The inclusion relations are consequences of Corollary 3.4, Lemmas 3.13 and 3.14, and Theorem 3.7.

THEOREM 3.16. *If* $P\neq 0$, $P\subseteq Q=L=R$.

PROOF. If $P\neq 0$, neither L nor R^{**} is null. By the dual of Theorem 3.15, $R^{**}\subseteq Q\subseteq R\subseteq L$. From these relations, from Theorem 3.15, and from Corollary 3.11, the conclusion of this theorem follows.

As an immediate consequence of our definitions, we have the following relations.

THEOREM 3.17. *If* S_1 *and* S_2 *are alternation groupoids and* $S_1\subseteq S_2$, *then* $L_1\supseteq S_1\cap L_2$ *and* $L_1^*\supseteq S_1\cap L_2^*$.

4. The imbedding theorem. This section is devoted to a proof of Theorem 4.1. If S has no left regular* element we may satisfy the theorem by choosing S_∞ as S. Hence throughout the section we assume $L^* \neq 0$. Notations such as L^*_∞, the set of left regular* elements in S_∞, have an obviously meaning. We show in Theorem 4.11 that the following theorem cannot be made stronger by replacing left regularity* with mere left regularity.

THEOREM 4.1. *An alternation groupoid S can be imbedded in an alternation groupoid S_∞ with the following properties*:

(A). *L^* is isomorphic to a subset of L^*_∞.*

(B). *Left regular* elements in S_∞ are left proper in S_∞.*

(C). *S_∞ is a minimal extension of S into an alternation groupoid with properties* A *and* B.

DEFINITION. S_1, the first left extension of S, is a groupoid whose elements are ordered pairs (a, b), where $a \in L^*$ and $b \in S$. In S_1 we define a binary operation $\#$ and the equality of elements as follows:

(i). $(a, b) \# (c, d) = (ac, bd)$.

(ii). $(a, b) = (c, d)$ if and only if, in S, $(aa)d = (ac)b$.

When the notation (a, b) is used it is assumed a is left regular*. By $\phi(a, b)$ we shall mean $\phi((a, b))$. This power obviously has the value $(\phi(a), \phi(b))$.

LEMMA 4.2. *$\#$ is a binary operation in S_1.*

PROOF. S_1 is clearly closed with respect to $\#$. To show the operation is uniquely defined, we assume $(a, b) = (c, d)$ and $(e, f) = (g, h)$. It follows that

$$[(ae)(ae)](dh) = [(aa)d][(ee)h] = [(ac)b][(eg)f] = [(ae)(cg)](bf),$$

$(ae, bf) = (cg, dh)$, and $(a, b)(e, f) = (c, d)(g, h)$.

LEMMA 4.3. *If $(a, b) = (c, d)$, then $(xa)d = (xc)b$ for all x in S. Conversely, if $x \in L^*$ and $(xa)d = (xc)b$, then $(a, b) = (c, d)$.*

PROOF. Since $(aa)d = (ac)b$,

$$[(aa)a][(xc)b] = [(ax)a][(ac)b] = [(ax)a][(aa)d] = [(aa)a][(xa)d].$$

Since $(aa)a$ is left regular, the first statement is proved. The second statement has a similar proof.

LEMMA 4.4. *The relation "=" is an equivalence relation in S_1.*

PROOF. That the equality is reflexive is trivial; that it is symmetric follows from the previous lemma. To prove transitivity, assume $(a, b)=(c, d)$ and $(c, d)=(e, f)$. It follows that

$$\begin{aligned}[(cc)(cc)][(aa)f] &= [(cc)(aa)][(cc)f] \\ &= [(ca)(ca)][(ce)d] = [(ca)(ce)][(ca)d] \\ &= [(cc)(ae)][(cc)b] = [(cc)(cc)][(ae)b].\end{aligned}$$

Cancellation of $(cc)(cc)$ gives the condition that $(a, b)=(e, f)$.

We have shown that S_1 satisfies postulate I. We can show without difficulty that postulate II is also satisfied, and we state the following theorem by way of summary.

THEOREM 4.5. *S_1 is an alternation groupoid.*

THEOREM 4.6. *$(a, x)\in L_1^*$ [resp. R_1^*] if and only if $x\in L^*$ [resp. R^*].*

PROOF. Proofs of both statements are similar—that the conditions are sufficient is easily seen. To prove necessity assume, say, $x\notin L^*$. There exist elements y and z and a power of x such that $y\neq z$ but $\phi(x)y=\phi(x)z$. Then $\phi(a, x)(a, y)=(\phi(a)a, \phi(x)y)=(\phi(a)a, \phi(x)z)=\phi(a, x)(a, z)$. If $\phi(a, x)\in L_1$, $(a, y)=(a, z)$ and $y=z$. Hence $\phi(a, x)\notin L_1$ and $(a, x)\notin L_1^*$.

COROLLARY 4.7. *$R_1^*=0$ if and only if $R^*=0$.*

LEMMA 4.8. *$(a, ax)=(b, by)$ if and only if $x=y$.*

THEOREM 4.9. *If $e\in L^*$, the correspondence $x\sim(e, ex)$ defines an isomorphism of S to a subset of S_1 under which L^* is isomorphic to a subset of L_1^*.*

PROOF. That the correspondence determines an isomorphism follows from Lemma 4.8. From Theorems 3.1 and 4.6 and from Corollary 3.6, $(e, ex)\in L_1^*$ if and only if $x\in L^*$.

Whenever it proves convenient, we identify S with the subset of S_1 to which it is isomorphic.

We have shown that S_1 satisfies property (A) of Theorem 4.1. That we cannot expect it to satisfy properties (B) and (C) follows from consideration of the example where S is the set of real numbers on a finite interval and where $x \# y$ is defined as $(x+y)/2$. S_1 does, however, satisfy a weakened form of Theorem 4.1 given below.

THEOREM 4.10. *S_1 can be uniquely characterized as a minimal extension of S into an alternation groupoid which satisfies property* (A) *of Theorem* 4.1 *and the following property: If $a\in L^*$ and $b\in S$, then $ax=b$ has a solution x in S_1.*

PROOF. Since $(e, ea)(a, b) = (e, eb)$, (a, b) is the required solution. If S is imbedded in a space T with this property and property (A), let T_1 be the subset of T made up of elements which are solutions of the equations $ax = b$ where $a \in L^*$ and $b \in S$. Each element of T_1 can be represented as an ordered pair $[a, b]$. It is a routine matter to show that the correspondence $(a, b) \sim [a, b]$ maps S_1 isomorphically on T_1.

DEFINITION. S_n, the nth left extension of S, is defined as the first left extension of S_{n-1}, $n = 2, 3, \cdots$. S_∞, the left extension of S, is defined as the union of the spaces S_n.

Thus by repeating the imbedding process just discussed, we obtain $S \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_\infty$. It is evident that S_∞ is an alternation groupoid and that $L^* \subseteq L^*_\infty$. If $\alpha \in L^*_\infty$ and $\beta \in S_\infty$, there exists (see Theorem 4.9) an integer n such that $\alpha \in L_n^*$ and $\beta \in S_n$. By Theorem 4.10, $\alpha x = \beta$ has a solution x in $S_{n+1} \subseteq S_\infty$ and thus $\alpha \in L^{**}_\infty$. Finally it is not difficult to show S_∞ is the minimal extension described in Theorem 4.1, and the proof of that theorem is now complete.

If we measure the worth of an extension T of S by the number of elements of S which are left proper in T, the following theorem shows that the left extension of S, described in this section, is the best possible. The theorem is a direct consequence of Theorem 3.17 and Corollary 3.4.

THEOREM 4.11. *There exists no extension of an alternation groupoid S into an alternation groupoid T such that an element not left regular* in S is left proper, or even left regular*, in T.*

THEOREM 4.12. *The properties of being right regular*, right proper and left proper are preserved under left extension.*

PROOF. That right regularity* is preserved follows from Theorems 3.7 and 4.6 (it can be shown that if $L^{**} \neq 0$, even right regularity is preserved). To show right properness is preserved it will be sufficient to find an element of S_1 which when multiplied on the right by (e, er) gives (a, f), where $r \in R^{**}$, e and $a \in L^*$, and $f \in S$. Choose g and d so $f = g(rr)$ and $a = dr$. Applying Theorem 3.7 and the dual of Corollary 3.6, we have $a \in R^*$, $d \in R^*$, and $d \in L^*$. Hence (d, g) is an element in S_1. It proves to be the element we sought. That left properness is preserved is proved similarly.

Before leaving this section we note the inclusion relations which have been established by Theorems 3.17, 4.1, 4.12, and Corollary 3.4.

THEOREM 4.13. $L^{**} \subseteq L^{**}_\infty = L^*_\infty$, $R^{**} \subseteq R^{**}_\infty$, $L^* = S \cap L^*_\infty$, *and* $R^* = S \cap R^*_\infty$.

5. **General extensions of S.** It is necessary to introduce new notation. Let S_0 be S. Let $S_{\infty 0}$ be S_∞ and let S_{n0} be S_n, the nth left extension of S. Let $S_{0\infty}$ and S_{0n} be respectively the right extension and the nth right extension of S. Finally let S_{mn} be $(S_{m0})_{0n}$ where m and n are any non-negative integers or where either m and n or both are replaced by the symbol ∞. It is evident that $S_{n+1\,0}=(S_{n0})_{10}$ and $S_{0\,n+1}=(S_{0n})_{01}$. It is consistent with what we have done in §4 to define $S_{10}=S$ $[S_{01}=S]$ in case $L^*=0$ $[R^*=0]$.

LEMMA 5.1. *S_{11} is isomorphic to $(S_{01})_{10}$.*

PROOF. If $L^*=0$, $L^*_{01}=0$ by the dual of Corollary 4.7, and $(S_{01})_{10}=S_{01}=(S_{10})_{01}=S_{11}$. Similarly if $R^*=0$, $S_{11}=(S_{01})_{10}$. We now assume $L^*=R^*\neq 0$ (see Theorem 3.7) and let $x_n\in S_{11}$. We define a correspondence $x_n\sim y_n$ where $y_n\in(S_{01})_{10}$ by means of six equations. By Theorem 4.10 and its dual, there exist elements r_n in R^*_{10}, p_n in S_{10}, a_n and b_n in L^*, and c_n and g_n in S, such that $x_nr_n=p_n$ in S_{11}, $a_nr_n=c_n$ in S_{10}, and $b_np_n=g_n$ in S_{10}. Since a_n and r_n are regular* in S_{10}, c_n is regular* in S_{10} and hence in S (see Theorems 3.1, 3.17, and 4.12). Let s_n and q_n be elements in S_{01} such that $s_na_n=b_n$ and $q_nc_n=g_n$ in S_{01}. By the dual of Corollary 3.6, s_n is right regular* and hence regular* in S_{01}. There exists an element y_n in $(S_{01})_{10}$ such that $s_ny_n=q_n$ in $(S_{01})_{10}$. That a different choice of elements r_n, p_n, a_n, b_n, c_n and g_n will lead to the same element y_n will follow from our proof that $x_m=x_n$ if and only if $y_m=y_n$. We apply repeatedly Lemma 4.3, Theorem 4.10, and their duals. Assume, say, $x_1=x_2$. By the definition of equality (in the 1st right extension of S_{10}), $p_2(r_1r_1)=p_1(r_2r_1)$ in S_{10}. But

$$[b_1(a_2a_1)][p_1(r_2r_1)] = g_1(c_2c_1)$$

and

$$[b_2(a_1a_1)][p_2(r_1r_1)] = g_2(c_1c_1).$$

Hence, in S,

$$[\{b_1(a_1a_1)\}\{b_1(a_2a_1)\}]\{g_2(c_1c_1)\} = [\{b_1(a_1a_1)\}\{b_2(a_1a_1)\}]\{g_1(c_2c_1)\}$$

or, restated,

$$\{(b_1b_1)g_2\}[\{(a_1a_2)c_1\}\{(a_1a_1)c_1\}] = \{(b_1b_2)g_1\}[\{(a_1a_1)c_2\}\{(a_1a_1)c_1\}].$$

This equality implies that the solutions α and β of the equations

$$\alpha[(a_1a_1)c_2] = (b_1b_1)g_2,$$
$$\beta[(a_1a_2)c_1] = (b_1b_2)g_1$$

are equal in S_{01}. These solutions are $(s_1s_1)q_2$ and $(s_1s_2)q_1$. Their equality implies that $y_1=y_2$ in $(S_{01})_{10}$. Since these steps are reversible, the correspondence of x_n to y_n is one to one. It remains to prove that the correspondence is an isomorphism. If $x_1\sim y_1$ and $x_2\sim y_2$, let $x_1x_2=x_3$, $r_1r_2=r_3$, $p_1p_2=p_3$, and so on. It follows that $x_1x_2=x_3\sim y_3=y_1y_2$.

For convenience, we identify S_{11} with $(S_{01})_{10}$ and use equality below instead of an isomorphism notation.

LEMMA 5.2. $(S_{0n})_{m0}=(S_{m0})_{0n}$ *for* $m, n=0, 1, 2, \cdots$.

PROOF. As in Lemma 5.1 we may assume $L^*=R^*\neq 0$. Clearly the equality holds if either m or $n=0$. We assume $m\neq 0$ and $n\neq 0$ and proceed by induction from the result of Lemma 5.1. If $S_{h1}=(S_{01})_{h0}$, then $S_{h+1\ 1}=(S_{h+1\ 0})_{01}=[(S_{h0})_{10}]_{01}=[(S_{h0})_{01}]_{10}=[(S_{01})_{h0}]_{10}=(S_{01})_{h+1\ 0}$, and $S_{m1}=(S_{01})_{m0}$ for $m=1, 2, \cdots$. Similarly, if $S_{mh}=(S_{0h})_{m0}$, $S_{m\ h+1}=(S_{0\ h+1})_{m0}$, and the lemma is proved.

THEOREM 5.3. *The right extension of the left extension of S is isomorphic to the left extension of the right extension of S.*

PROOF. It is easy to validate the two equalities below which have not yet been established. $S_{\infty\infty}=(S_{\infty 0})_{0\infty}=\bigcup_n(\bigcup_m S_{m0})_{0n}=\bigcup_n\bigcup_m(S_{m0})_{0n}$ $=\bigcup_m\bigcup_n(S_{0n})_{m0}=\bigcup_m(\bigcup_m S_{0n})_{m0}=(S_{0\infty})_{\infty 0}$.

The following theorem follows principally from Theorems 4.1, 4.13, and their duals.

THEOREM 5.4. *In Theorem* 4.1 *we may replace* S_∞ *by* $S_{\infty\infty}$ *and left regularity* properties by the corresponding regularity* properties.*

COROLLARY 5.5. *An alternation groupoid S may be imbedded in an Abelian quasigroup if and only if all elements of S are regular.*

We may classify alternation groupoids according to whether or not they have left regular* or right regular* elements. Members in different classes, as we have seen, have distinctly different types of extensions.

I. $L^*=R^*=0$.

Here $S_{\infty\infty}=S$ has no left or right proper element. An example: example A of §2 with $\alpha=5, \beta=-2, \gamma=3$, and $\delta=-1$. In this example, 1 is regular and $2\in R$.

II. $L^*\neq 0$, $R^*=0$.

Here $S_{\infty\infty}=S_\infty$ and in S_∞ we have $L_\infty^{**}=L_\infty^*\subseteq L_\infty$ and $R_\infty=0$. Example: let S be the set of points (x, y) in the square $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$ and define $(x_1, y_1)\ \#\ (x_2, y_2)$ as $(x_1/2+x_2/2, y_2)$.

III. The situation dual to that of II.

IV. $L^* = R^* \neq 0$.

Here $S_{\infty\infty}$ is such that $P_{\infty\infty} = Q_{\infty\infty} = L_{\infty\infty} = R_{\infty\infty}$. (If $S_\infty \neq S_{\infty\infty}$, $L_\infty^{**} = Q_\infty \subseteq L_\infty \subseteq R_\infty$ and $R_\infty^{**} = 0$.) Example: example A of §2 with $\alpha = 3$, $\beta = -2$, $\gamma = 2$, and $\delta = -1$. In this example, $1 \in Q$ and $2 \in R$.

6. Conditions which insure that $L = L^*$. Since a necessary and sufficient condition that S have an extension with left proper elements is that $L^* \neq 0$, it is of interest to find what conditions insure that left regular elements are left regular* or, in other words, that the left regular elements are closed under $\#$. This closure property holds in many special alternation groupoids—for example, an idempotent groupoid or a groupoid in which $\#$ is associative. The following theorem shows in general how badly an alternation groupoid must behave if $L \neq L^*$.

THEOREM 6.1. *If S is an alternation groupoid in which $L \neq 0$ and $L \neq L^*$, then*:

(1). *S is infinite.*

(2). *There is no proper element in S.*

(3). *If S has a right proper element, S has left regular elements which are not right regular.*

(4). *If S has a left proper element, then either S has no right regular element or S has right regular elements which are not left regular.*

(5). *S has left regular elements whose squares are not left regular.*

(6). *There are elements in S neither left regular nor left null.*

PROOF. One proves property (1) just as one proves every finite integral domain is a field. Assume S contains a finite number of elements $x_1, x_2, x_3, \cdots, x_n$ where x_1, say, $\in L$. Since no two elements in the set $x_1x_1, x_1x_2, \cdots, x_1x_n$ are equal, this set consists of the elements of S in some order. Hence $x_1 \in L^{**}$, and by Corollary 3.4, $x_1 \in L^*$.

Property (2) is a direct consequence of Theorem 3.16. To prove property (5) we show that if all squares of left regular elements are left regular, L is closed under $\#$. Let a and $b \in L$ and let $(ab)x = (ab)y$. Then $[(aa)(aa)][(bb)x] = [(ab)(aa)][(ab)x] = [(aa)(aa)][(bb)y]$. Since bb, aa, and $(aa)(aa) \in L$, $x = y$.

We prove property (3) by showing that if it does not hold property (5) does not hold. Let $a \in L$ and assume $L \subseteq R$. Then, by the dual of Lemma 3.12, $aa \in L$. Property (4) is proved similarly. Property (6) is a consequence of Theorem 6.6 to a proof of which the remainder of this section is devoted.

Let N, N_1 and N_2 denote respectively the sets of null, left null, and right null elements in S. The following lemma holds for any groupoid satisfying postulate I of the introduction.

LEMMA 6.2. *If* $N_1 \neq 0$ *and* $N_2 \neq 0$, *there exists a null element* θ *and* $N = N_1 = N_2 = (\theta)$.

PROOF. If $\theta_1 \in N_1$ and $\theta_2 \in N_2$, $\theta_1 = \theta_1\theta_2 = \theta_2$. Thus each left null element equals every right null element and, in particular, all null elements are equal.

LEMMA 6.3. *If an alternation groupoid* S *has more than one element and if* $ab \in L$, *then neither* a *nor* $b \in N_1$.

PROOF. If $a \in N_1$, $ab = a \in N_1$. If $b \in N_1$, $(ab)[(ab)a] = [a(ab)](ba) = [a(ab)](bb) = (ab)[(ab)b]$ and $a = b \in N_1$.

LEMMA 6.4. *If* $S = L \cup N_1$, *if* $N_1 \neq 0$, *and if* $N_1 \neq S$, *there exists a null element* θ *in* S.

PROOF. Let $x \in S$, $a \in L$, θ_1 and $\theta_2 \in N_1$. By Lemma 6.3, $x\theta_1$ and $x\theta_2 \in N_1$. Hence $a\theta_1 = (a\theta_1)(\theta_2 a) = (a\theta_2)(\theta_1 a) = a\theta_2$, and $\theta_1 = \theta_2$. Thus all left null elements are equal, say, to θ. In particular $x\theta$, being left null, equals θ.

As a corollary to Lemmas 6.2 and 6.4 we have:

LEMMA 6.5. *If* $S = L \cup N_1$, *if* $L \neq 0$, *and if* $L \neq S$, *there exists a null element* θ *such that* $L \cup (\theta) = S$.

THEOREM 6.6. *If* $S = L \cup N_1$, $L = L^*$.

PROOF. If $L = 0$ or $L = S$, the theorem holds. If neither equality holds, $N_1 = (\theta)$ by Lemma 6.5. Then a and $b \in L$ implies $ab \in L$, for if $ab = \theta$, $ab = a\theta$, and $b = \theta$.

REFERENCES

1. D. C. Murdoch, *Quasi-groups which satisfy certain generalized associative laws*, Amer. J. Math. vol. 61 (1939) pp. 509–522.

2. ———, *Structure of abelian quasigroups*, Trans. Amer. Math. Soc. vol. 49 (1941) pp. 392–409.

3. R. H. Bruck, *Some results in the theory of quasigroups*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 19–52.

4. N. Bourbaki, *Algèbre*, Actualités Scientifiques et Industrielles, no. 934, Paris, 1942.

5. A. Malcev, *On the immersion of an algebraic ring into a field*, Math. Ann. vol. 113 (1936) pp. 686–691.

BROWN UNIVERSITY AND
WASHINGTON UNIVERSITY

A THEOREM ON QUADRATIC FORMS OVER THE RING OF 2-ADIC INTEGERS[1]

BURTON W. JONES AND WILLIAM H. DURFEE

1. **Introduction.** The theory of the equivalence of quadratic forms over the ring of 2-adic integers is considerably more difficult than the corresponding theory for forms over the p-adic integers, p an odd prime, and has only recently been worked out to a comparable degree. A problem still not completely solved is Witt's cancellation theorem: namely, if f, g and h are forms such that g and h have no variables in common with f, then $f+g$ equivalent to $f+h$ implies g equivalent to h. This theorem, though true for p-adic integers when p is odd, does not always hold when p is even. We give here a case when it does hold (see theorem below). While this theorem follows almost immediately from results in a paper by Jones [1, Theorems 2 and 6][2] it seems worthwhile in view of the rather long arguments of the theorems there to give an independent proof, especially since it in turn can be used to shorten some of the proofs of that paper.

2. **Proof of the theorem.** We denote by capital italic letters matrices over $R(2)$, the ring of 2-adic integers, while small italic letters with the exception of f, g and h will stand for numbers in $R(2)$. We shall consider only forms whose symmetric matrices have elements in $R(2)$. A matrix is ***unimodular*** if its elements are in $R(2)$ and its determinant a unit of $R(2)$. A form is called unimodular if its symmetric matrix is unimodular.

THEOREM. *Let f_1 and f_2 be two equivalent unimodular forms over $R(2)$ in $x_1, x_2, \cdots, x_n$, g a nonsingular form over $R(2)$ in $x_{n+1}, x_{n+2}, \cdots, x_{n+s}$ and h a nonsingular form over $R(2)$ in $x_{n+1}, x_{n+2}, \cdots, x_{n+t}$. If there is a matrix over $R(2)$ taking f_1+2g into f_2+2h, then there is one over $R(2)$ taking g into h; if $s=t$ and the former matrix is unimodular, then so is the latter.*

PROOF. Since f_1 and f_2 are equivalent we may take f_1+2g into f_2+2g by a unimodular transformation and thus we set $f_1=f_2=f$. By [1, Lemma 1] f is equivalent to either a diagonal form or a sum of binary forms. If the latter is the case, then x^2+f will be equivalent to

---

Presented to the Society, September 10, 1948; received by the editors June 15, 1948.

[1] This paper was written while the first author was on sabbatical leave from Cornell University with the aid of a grant from the Research Corporation.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

a diagonal form and since $x^2 \dotplus f \dotplus 2g$ is equivalent to $x^2 \dotplus f \dotplus 2h$ we shall assume that f is a unimodular diagonal form. Let f, g and h have respectively F, G and H as their symmetric metrices over $R(2)$ and let Q be the $n+s$ by $n+t$ matrix taking $F \dotplus 2G$ into $F \dotplus 2H$, where $\dotplus$ denotes direct sum, that is,

$$F \dotplus 2G = \begin{pmatrix} F & 0 \\ 0 & 2G \end{pmatrix}.$$

Set

$$Q = \begin{pmatrix} T & S_2 \\ S_3 & S_4 \end{pmatrix},$$

where T is an n by n matrix, S_2 is n by t, S_3 is s by n and S_4 is s by t. Then $Q'(F \dotplus 2G)Q = F \dotplus 2H$ yields the following equations:

$$T'FT + S_3' 2GS_3 = F, \tag{1}$$

$$T'FS_2 + S_3' 2GS_4 = 0, \tag{2}$$

$$S_2'FS_2 + S_4' 2GS_4 = 2H. \tag{3}$$

We show in the following lemma that, since equation (1) implies $T'FT \equiv F \pmod 2$, there is an automorph D of F over $R(2)$ such that $2(T+D)^{-1}$ is in $R(2)$. (An automorph of F is a matrix D such that $D'FD = F$.) Furthermore equation (2) and the fact that $|T'F|$ is a unit implies that $S_2 \equiv 0 \pmod 2$ and hence that

$$S = S_4 - S_3(T+D)^{-1}S_2$$

has its elements in $R(2)$. We shall show that $S'2GS = 2H$. Now

$$\begin{aligned} S'2GS &= [S_4' - S_2'(T+D)'^{-1}S_3']2G[S_4 - S_3(T+D)^{-1}S_2] \\ &= S_4' 2GS_4 - S_2'(T+D)'^{-1}S_3' 2GS_4 - S_4' 2GS_3(T+D)^{-1}S_2 \\ &\quad + S_2'(T+D)'^{-1}S_3' 2GS_3(T+D)^{-1}S_2. \end{aligned}$$

Using (1) we have

$$\begin{aligned} &S_2'(T+D)'^{-1}S_3' 2GS_3(T+D)^{-1}S_2 \\ &\qquad = S_2'(T+D)'^{-1}\{F - T'FT\}(T+D)^{-1}S_2. \end{aligned} \tag{4}$$

Since $F - T'FT = (D+T)'F(D+T) - T'F(D+T) - (T+D)'FT$, the right side of (4) equals

$$S_2'FS_2 - S_2'(T+D)'^{-1}T'FS_2 - S_2'FT(T+D)^{-1}S_2,$$

which, using (2), becomes

$$S_2'FS_2 + S_2'(T+D)'^{-1}S_3' 2GS_4 + S_4' 2GS_3(T+D)^{-1}S_2.$$

Hence $S'2GS=S_4'2GS_4+S_2'FS_2=2H$ from equation (3).

If $s=t$ and Q is unimodular, we have not only $S'GS=H$, but also by symmetry $S^{*\prime}HS^*=G$ for some s by s matrix S^* over $R(2)$. By substitution we see that, since $|G|$ is not zero, $|S|$ is a unit and S is unimodular. *The condition that g and h be nonsingular is not essential*, as it can be shown with little difficulty that the theorem is still true without this restriction.

We now prove the basic lemma.

LEMMA. *If F is a unimodular diagonal matrix over $R(2)$ and T a matrix over $R(2)$ such that $T'FT\equiv F$ (mod 2), then there is an automorph D of F over $R(2)$ such that $2(T+D)^{-1}$ is in $R(2)$.*

PROOF. Let $T=(t_{ij})$, $i, j=1, 2, \cdots, n$ and $F=\alpha_1\dotplus\alpha_2\dotplus\cdots\dotplus\alpha_n$. Since permuting the rows and *the same* columns of T and F does not alter the properties we desire, we shall do this at will. Note that $T'T\equiv I$ (mod 2).

First, suppose that t_{ii} is a non-unit for some i. Permute rows and columns, if necessary, to make t_{11} a non-unit, choose $D=1\dotplus D_1$ and set $u=t_{11}+1$, which will be a unit. If we set

$$P=\begin{pmatrix}1 & -u^{-1}T_2\\ 0 & I\end{pmatrix} \quad\text{and}\quad T=\begin{pmatrix}t_{11} & T_2\\ T_3 & T_4\end{pmatrix},$$

where T_2 is a 1 by $n-1$ matrix, T_3 is $n-1$ by 1 and T_4 is $n-1$ by $n-1$, we have

$$(T+D)P=\begin{pmatrix}t_{11}+1 & 0\\ T_3 & T_1+D_1\end{pmatrix}, \tag{5}$$

where $T_1=T_4-T_3u^{-1}T_2$ and D_1 is to be chosen an automorph of $F_1=\alpha_2\dotplus\alpha_3\dotplus\cdots\dotplus\alpha_n$. By adding appropriate multiples of the first row of (5) to the later rows we can replace T_3 by 0 without altering T_1+D_1. Referring now to the proof of the theorem and there replacing T, S_2, S_3, S_4 by t_{11}, T_2, T_3, T_4, respectively, we see that $T'T\equiv I$ (mod 2) implies that $T_1'T_1\equiv I$ (mod 2).

Secondly, suppose that t_{ii} is a unit for every i and $t_{ij}t_{ji}$ is a unit for some i and j, $i\neq j$. Permute rows and columns, if necessary, to make $t_{12}t_{21}$ a unit, and choose $D=I_2\dotplus D_2$, where I_2 is the 2 by 2 identity matrix. Then

$$U=\begin{pmatrix}t_{11}+1 & t_{12}\\ t_{21} & t_{22}+1\end{pmatrix}$$

is unimodular and if we set

$$P = \begin{pmatrix} I_2 & -U^{-1}T_2 \\ 0 & I \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} T_0 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where T_0 is a 2 by 2 matrix, T_2 is 2 by $n-2$, and so on, we have

$$(T + D)P = \begin{pmatrix} T_0 + I_2 & 0 \\ T_3 & T_1 + D_2 \end{pmatrix},$$

where $T_1 = T_4 - T_3U^{-1}T_2$ and D_2 is to be chosen an automorph of $F_2 = \alpha_3 + \alpha_4 + \cdots + \alpha_n$. As in the preceding case we can replace T_3 by 0 and have $T_1'T_1 \equiv I \pmod{2}$.

Thirdly, suppose T has the property that t_{ii} is a unit for every i, t_{ij} is a unit for some $i \neq j$ and $t_{ij}t_{ji}$ is a non-unit for every $i \neq j$. Now $T'T \equiv I \pmod{2}$ implies that each row and column of T contains an odd number of units and that for each i and j, $i \neq j$, there is an even number of values of k such that $t_{ik}t_{jk} \equiv 1 \pmod{2}$. Thus by a permutation of rows and columns we may assume the leading 3 by 3 minor of T to be congruent (mod 2) to

$$\begin{Bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{Bmatrix}.$$

Furthermore, two of the first three diagonal elements of F are congruent (mod 4) and by a permutation of rows and columns of F and T we may assume that α_1 and α_2 are congruent (mod 4), that $t_{11}t_{12}t_{22}$ is a unit and t_{21} a non-unit. We can complete the proof along the lines of the previous case if we can find an automorph D_0 of $\alpha_1x_1^2 + \alpha_2x_2^2$, where

$$D_0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2},$$

since then

$$U = D_0 + \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$$

will be unimodular. Suitable matrices D_0 are

$$\begin{pmatrix} 0 & \sigma^{-1} \\ \sigma & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & 3\tau^{-1} \\ \tau & 2 \end{pmatrix}$$

according as $\alpha_1 \equiv \alpha_2 \pmod 8$ or $-3\alpha_1 \equiv \alpha_2 \pmod 8$, where σ and τ are p-adic units satisfying the equations $\alpha_1 = \alpha_2\sigma^2$, $-3\alpha_1 = \alpha_2\tau^2$.

Finally, suppose $T \equiv I \pmod 2$. Choose $\lambda_1 = \pm 1$ so that $t_{11} + \lambda_1$ is twice a unit and set $D = \lambda_1 \dotplus D_1$. By adding appropriate multiples of the first column of $T+D$ to the other columns and then similarly for rows we can reduce the first row and column of $T+D$ to $(t_{11}+\lambda_1, 0, \cdots, 0)$. The remaining elements still will have the property that those on the diagonal are units and the non-diagonal elements are non-units.

By continuing the above reductive process we can reduce $T+D$ to a direct sum of matrices T_i+D_i, where each T_i+D_i is one of three types: a unit, twice a unit, or a 2 by 2 unimodular matrix. Hence $2(T+D)^{-1}$ will be in $R(2)$.

BIBLIOGRAPHY

1. B. W. Jones, *A canonical quadratic form for the ring of 2-adic integers*, Duke Math. J. vol. 11 (1944) pp. 715–727.

CORNELL UNIVERSITY AND
DARTMOUTH COLLEGE

ABSOLUTE-VALUED ALGEBRAIC ALGEBRAS

A. A. ALBERT

1. Introduction. An *algebra* $\mathfrak{A}$ over a field $\mathfrak{F}$ is a vector space over $\mathfrak{F}$ which is closed with respect to a product xy which is linear in both x and y. The product is not necessarily associative. Every element x of $\mathfrak{A}$ generates a subalgebra $\mathfrak{F}[x]$ of $\mathfrak{A}$ and we call $\mathfrak{A}$ an algebraic algebra if every $\mathfrak{F}[x]$ is a finite-dimensional vector space over $\mathfrak{F}$.

We have shown elsewhere[1] that every absolute-valued real finite-dimensional algebra has dimension 1, 2, 4, or 8 and is either the field $\mathfrak{R}$ of all real numbers, the complex field $\mathfrak{C}$, the real quaternion algebra $\mathfrak{Q}$, the real Cayley algebra $\mathfrak{D}$, or certain isotopes without unity quantities of $\mathfrak{Q}$ and $\mathfrak{D}$. In the present paper we shall extend these results to algebraic algebras over $\mathfrak{R}$ showing that every algebraic algebra over $\mathfrak{R}$ with a unity quantity is finite-dimensional and so is one of the algebras listed above. The results are extended immediately to absolute-valued algebraic division algebras, that is, to algebras without unity quantities whose nonzero quantities form a quasigroup.

2. Quadratic algebras. Let $\mathfrak{F}$ be a field whose characteristic is not two and $\mathfrak{A}$ be an algebra over $\mathfrak{F}$ with a unity quantity 1. The scalar multiples $\alpha 1$ defined for α in $\mathfrak{F}$ form a subalgebra of $\mathfrak{A}$ isomorphic to $\mathfrak{F}$ and we may assume that $\mathfrak{F}$ is actually a subalgebra of $\mathfrak{A}$ whose unity element coincides with that of $\mathfrak{A}$. Then $\mathfrak{F}$ is a subalgebra of the center[2] of $\mathfrak{A}$. We shall call the elements of $\mathfrak{A}$ which are in $\mathfrak{F}$ the *scalars* of $\mathfrak{A}$ and all other elements of $\mathfrak{A}$ *nonscalars*.

In an algebra of degree two over $\mathfrak{F}$ every x is a root of an equation

$$f(\xi, x) \equiv \xi^2 - 2\xi\tau(x) - \nu(x) = 0, \tag{1}$$

where $\tau(x)$ and $\nu(x)$ are functions on $\mathfrak{A}$ to $\mathfrak{F}$. Then every nonscalar x of $\mathfrak{A}$ determines a commutative associative algebra $\mathfrak{F}[x] = \mathfrak{F} + x\mathfrak{F}$ of order two over $\mathfrak{F}$. It is seen trivially that $f(\xi, x)$ is unique for every

---

Presented to the Society, September 10, 1948; received by the editors June 21, 1948.

[1] See the author's *Absolute valued real algebras*, Ann. of Math. vol. 48 (1947) pp. 495–501.

[2] The center of a finite-dimensional algebra was defined in the author's *Nonassociative algebras* I, Ann. of Math. vol. 43 (1942) on page 707. The same definition was given later for rings by T. Nakayama, *Über einfache distributive Systeme unendlicher Range*, Proc. Imp. Acad. Tokyo vol. 20 (1944) p. 62 and by N. Jacobson, *Structure theory of rings without finiteness assumptions*, Trans. Amer. Math. Soc. vol. 57 (1945) p. 239.

nonscalar x of $\mathfrak{A}$ and that $\mathfrak{F}[x]=\mathfrak{F}[y]=\mathfrak{F}+y\mathfrak{F}$ where $y=x-\tau(x)$, $\tau(y)=0$,

$$y^2 = \nu(y) = \nu(x) + [\tau(x)]^2. \tag{2}$$

It will be convenient to call an element y a *square root* if y is a nonscalar and $\tau(y)=0$ so that $y^2=\nu(y)$ is in $\mathfrak{F}$.

An algebra $\mathfrak{A}$ will be called a *quadratic* algebra if $\mathfrak{A}$ has a unity quantity and $\mathfrak{F}[x]$ is a quadratic field over $\mathfrak{F}$ for every nonscalar x of $\mathfrak{A}$. Then $\mathfrak{A}$ has degree two over $\mathfrak{F}$ and every $f(\xi, x)$ is irreducible for x a nonscalar, $\nu(x)\neq 0$. Conversely an algebra $\mathfrak{A}$ with a unity quantity and degree two over $\mathfrak{F}$ is a quadratic algebra provided that $\nu(x)\neq 0$ for every nonscalar x. For if $f(\xi, x)$ were reducible we would have $f(\xi, x)=(\xi-\alpha)(\xi-\beta)$, $y(y-\beta+\alpha)=0$ for $y=x-\alpha$, y is a nonscalar such that $\nu(y)=0$. Assume throughout the remainder of this section that $\mathfrak{A}$ is a quadratic algebra.

The following lemma is due to L. E. Dickson.[3]

LEMMA 1. *Let u and v be square roots in $\mathfrak{A}$ such that* 1, u, v *are linearly independent in $\mathfrak{F}$. Then $uv+vu$ is a scalar.*

The proof is made by the computations $u^2=\alpha$, $v^2=\beta$, $(u+v)^2=\lambda(u+v)+\mu$, $(u+2v)^2=\rho(u+2v)+\sigma$ for α, β, λ, μ, ρ, σ in $\mathfrak{F}$. Then $t=uv+vu=\lambda(u+v)+\mu-(\alpha+\beta)$, $2t=\rho(u+2v)+\sigma-\alpha-4\beta$ so that $2\lambda=\rho=2\rho$, $\lambda=0$, $uv+vu=\mu-(\alpha+\beta)$ is in $\mathfrak{F}$.

Two nonzero elements u and v of $\mathfrak{A}$ are said to be *J-orthogonal* if their Jordan product $uv+vu=0$. We call $u_1, \cdots, u_n$ *pairwise J-orthogonal* when u_i and u_j are J-orthogonal for every $i\neq j$. Then we have

LEMMA 2. *Let $u_1, \cdots, u_n$ be pairwise J-orthogonal nonscalar elements of a quadratic algebra $\mathfrak{A}$. Then* 1, $u_1, \cdots, u_n$ *are linearly independent in $\mathfrak{F}$.*

For if $t=\lambda_1u_1+\cdots+\lambda_nu_n+\lambda_0=0$ for $\lambda_0, \lambda_1, \cdots, \lambda_n$ in $\mathfrak{F}$ then $tu_i+u_it=2(\lambda_0u_i+\lambda_iu_i^2)=0$. If $\lambda_i\neq 0$ we would have $\nu(u_i)=0$ contrary to our hypothesis on $\mathfrak{A}$. Hence $\lambda_i=0$, $\lambda_0=0$ as desired.

LEMMA 3. *Let $u_1, \cdots, u_n$ be pairwise J-orthogonal square roots of $\mathfrak{A}$ and w be an element of $\mathfrak{A}$ not in $\mathfrak{B}=\mathfrak{F}+u_1\mathfrak{F}+\cdots+u_n\mathfrak{F}$. Then there exists an element b of $\mathfrak{B}$ such that $u=w-b$ is a square root of $\mathfrak{A}$ and $u_1, \cdots, u_n, u$ are pairwise J-orthogonal.*

---

[3] Cf. *Linear algebras with associativity not assumed*, Duke Math. J. vol. 1 (1935) pp. 113–125.

We have $w^2=2\beta w+\gamma$ for β and γ in $\mathfrak{F}$ and we put $v=w-\beta$ and see that $v^2=\gamma+\beta^2$, v is a square root. Since v is not in $\mathfrak{B}$ the quantities $1, u_i, v$ are linearly independent and we apply Lemma 1 to obtain $vu_i+u_iv=\delta_i$ in $\mathfrak{F}$. Define $b_0=2^{-1}\sum_{j=1}^{n}\alpha_j^{-1}\delta_ju_j$ where $u_j^2=\alpha_j$. Then $b_0u_i+u_ib_0=\alpha_i^{-1}\delta_iu_i^2=\delta_i$. But then $(v-b_0)u_i+u_i(v-b_0)=0$ and $u_1, \cdots, u_n, u$ are pairwise J-orthogonal if $u=v-b_0=w-b$ where $b=\beta+b_0$ is in $\mathfrak{B}$. Also $u^2=v^2+2^{-1}\sum_{j=1}^{n}\alpha_j^{-1}\delta_j(vu_j+u_jv)+4^{-1}\sum_{j=1}^{n}\alpha_j^{-2}\delta_j^2$ is in $\mathfrak{F}$ and our proof is complete.

We shall apply the simple results just obtained to prove the finite dimensionality of alternative *quadratic* algebras, that is, of quadratic algebras such that $x(xy)=(xx)y$, $(yx)x=y(xx)$ for every x and y of $\mathfrak{A}$. It is necessary to consider only the case where $\mathfrak{A}$ contains a subspace $\mathfrak{H}$ of dimension at least eight. Thus we may assume that $\mathfrak{A}$ contains a pair of J-orthogonal square roots u and v such that $u^2=\alpha\neq 0$ in $\mathfrak{F}$, $v^2=\beta\neq 0$ in $\mathfrak{F}$, $uv=-vu$. By the alternative law we have

$$u(uv) = \alpha v,\ (uv)u = -(vu)u = -v(uu) = -\alpha v \neq 0.$$

Then uv and u are J-orthogonal. Similarly uv and v are J-orthogonal. But then $uv\neq 0$ is a nonscalar and, by Lemma 2, $\mathfrak{A}$ contains the four-dimensional subspace $\mathfrak{Q}=\mathfrak{F}+u\mathfrak{F}+v\mathfrak{F}+uv\mathfrak{F}$ of $\mathfrak{A}$. Also $(uv)^2=\gamma+\delta uv$ for γ and δ in $\mathfrak{F}$, $u[(uv)^2]=[u(uv)](uv)=\alpha v(uv)=-\alpha v(vu)=-\alpha\beta u=\gamma u+\alpha\delta v$. It follows that $\alpha\delta=0$ and so $(uv)^2=\gamma=-\alpha\beta$. Evidently $\mathfrak{Q}$ is a linear algebra which is a subring of $\mathfrak{A}$ and it is easily verified, since all products of basal elements of $\mathfrak{Q}$ are now known, that $\mathfrak{Q}$ is an associative generalized quaternion algebra. We state the result as

LEMMA 4. *Let u and v be J-orthogonal square roots of an alternative quadratic algebra $\mathfrak{A}$. Then $\mathfrak{Q}=\mathfrak{F}+u\mathfrak{F}+v\mathfrak{F}+uv\mathfrak{F}$ is a four-dimensional subring of $\mathfrak{A}$ and is a generalized quaternion associative algebra with $u^2=\alpha$, $v^2=\beta$, $(uv)^2=-\alpha\beta$, $vu=-uv$ for $\alpha\neq 0$, $\beta\neq 0$ in $\mathfrak{F}$.*

Assume next that $\mathfrak{A}$ contains $\mathfrak{Q}$ as well as an element not in $\mathfrak{Q}$. Then Lemma 3 implies that there exists a square root which is J-orthogonal to u, v and uv. For every such square root we prove

LEMMA 5. *Let $u_1=u$, $u_2=v$, $u_3=uv$, $u_4=w$ be a set of pairwise J-orthogonal square roots of an alternative quadratic algebra $\mathfrak{A}$. Then $u_1, \cdots, u_4$, $u_5=uw$, $u_6=vw$, $u_7=(uv)w$ are pairwise J-orthogonal, $u_i^2=\alpha_i$ in $\mathfrak{F}$, $u_iu_j=\alpha_{ij}u_k$ for every $i\neq j$ where α_{ij} is in $\mathfrak{F}$ and k depends on i and j. Also $\mathfrak{B}=\mathfrak{F}+u_1\mathfrak{F}+\cdots+u_7\mathfrak{F}$ is an eight-dimensional subalgebra of $\mathfrak{A}$.*

For let $\alpha=\alpha_1$, $\beta=\alpha_2$, $\gamma=\alpha_4$. Then $(u+w)^2=\alpha+\gamma$ and by the alternative law $(\alpha+\gamma)v=(u+w)^2v=(u+w)(uv+wv)=u(uv)+w(uv)$

$+u(wv)+w(wv)=\alpha v+\gamma v-u(vw)-(uv)w$. Thus

$$u(vw) = -(uv)w. \tag{3}$$

Similarly $(\alpha+\gamma)v=v(u+w)^2=(vu+vw)(u+w)=(vu)u+(vw)w+(vu)w+(vw)u$, $(vw)u=-(vu)w$,

$$(vw)u = (uv)w = -u(vw). \tag{4}$$

We have proved that u and vw are J-orthogonal. By symmetry v is J-orthogonal to uw and

$$v(uw) = -(uw)v = (uv)w. \tag{5}$$

However we may replace u, v by uv, v to see that vw and uv are J-orthogonal. Also $\mathfrak{F}+v\mathfrak{F}+w\mathfrak{F}+vw\mathfrak{F}$ is a quaternion algebra and $(vw)^2=-\beta\gamma$, vw and v are J-orthogonal.

Formula (4) may be applied with u and v replaced by uv and u to yield

$$(uw)(uv) = [(uv)u]w = -\alpha vw. \tag{6}$$

Also $(uw)v = (vu)w = -(uv)w$. Then $(u+uw)^2(uv) = (\alpha-\alpha\gamma)uv = (u+uw)[(u+uw)(uv)]=(u+uw)(\alpha v-\alpha vw)=\alpha uv+\alpha(uw)v-\alpha u(vw)-\alpha(uw)(vw)$. But $\alpha\,(uw)v-\alpha u(vw)=-\alpha(uv)w+\alpha(uv)w=0$. Hence

$$(uw)(vw) = \gamma uv. \tag{7}$$

Interchange u and v to obtain $(vw)(uw)=\gamma vu=-(uw)(vw)$. It follows immediately that the symmetry among the elements u, v, uv implies that the elements $u_1, \cdots, u_7$ are pairwise J-orthogonal. They are square roots since a product such as uw is in a quaternion algebra $\mathfrak{F}+u\mathfrak{F}+w\mathfrak{F}+uw\mathfrak{F}$ and is a square root by Lemma 4. But (4) and (7) yield $u_iu_j=\alpha_{ij}u_k$.

We finally assume that $\mathfrak{A}$ contains $\mathfrak{B}=\mathfrak{F}+u_1\mathfrak{F}+\cdots+u_7\mathfrak{F}$ and an element not in $\mathfrak{B}$. Then $\mathfrak{A}$ contains an element z which is J-orthogonal to $u_1, \cdots, u_7$. We form the product $t=(uv)(wz)$. Now $z^2=\delta\neq0$ in $\mathfrak{F}$, $(uv)t=-\alpha\beta wz$, $[(uv)t]^2=\alpha\beta\gamma\delta\neq0$, $t\neq0$. However we use the quaternion algebra $\mathfrak{F}+uv\mathfrak{F}+w\mathfrak{F}+(uv)w\mathfrak{F}$ with z and apply (4) to obtain $t=-[(uv)w]z$. Then $t=[u(vw)]z$. Since z is orthogonal to u, vw and $u(vw)$ we now have $t=-u[(vw)z]=u[v(wz)]$. Apply Lemma 5 with $u_1=u$, $u_2=w$, $u_4=z$ to get wz orthogonal to u. The same values, except for $u_1=v$, yields wz orthogonal to v and similarly $u_1=uv$ implies that wz is orthogonal to uv. But then (4) yields $t=-(uv)(wz)=-t$. We have proved[4]

---

[4] Alternative quadratic algebras of finite dimension were determined by M. Zorn, Abh. Math. Sem. Hamburgischen Univ. vol. 9 (1933) pp. 395–402.

THEOREM 1. *Every alternative quadratic algebra is an algebra of order* $n=1, 2, 4, 8$ *over* $\mathfrak{F}$.

3. Absolute valued algebraic algebras. Let $\mathfrak{R}$ be the field of all real numbers and $\mathfrak{A}$ be an algebraic algebra over $\mathfrak{R}$. We assume the existence of a function $|a|$ on $\mathfrak{A}$ to $\mathfrak{R}$ such that $|0|=0$, $|a|>0$ if $a\neq 0$, $|ab|=|a|\,|b|$, $|a+b|\leqq |a|+|b|$ for every a and b of $\mathfrak{A}$. Assume also that if a is in $\mathfrak{A}$ and α is any real number then $|\alpha a|=|\alpha|\,|a|$ where $|\alpha|$ is the ordinary absolute value of α. Then we call $\mathfrak{A}$ an absolute-valued algebra.

We shall assume that $\mathfrak{A}$ has a unity quantity 1 so that $\mathfrak{R}$ may be taken to be a subalgebra of $\mathfrak{A}$ and the hypothesis $|\alpha a|=|\alpha|\,|a|$ now becomes simply the hypothesis that $|a|$ is the ordinary absolute value whenever a is a real number. The hypothesis that $\mathfrak{A}$ is an algebra over $\mathfrak{R}$ implies that $\mathfrak{R}[x]$ is an absolute-valued real algebra generated by x and containing 1. However all real, absolute-valued algebras with a unity quantity are known,[1] they all have degree one or two and hence $\mathfrak{R}[x]$ is isomorphic to either $\mathfrak{R}$ or the field $\mathfrak{C}$ of all complex numbers. It follows immediately that $\mathfrak{A}$ is a quadratic algebra over $\mathfrak{R}$.

Every element x of $\mathfrak{A}$ now has the form $x=\alpha+\beta u$ where α and β are real and $u^2=-1$. Then $\mathfrak{R}(u)$ is isomorphic to $\mathfrak{C}$, $(x-\alpha)^2+\beta^2=0$, the roots of $f(\xi, x)=\xi^2-2\alpha\xi+\alpha^2+\beta^2$ are ρ, $\bar{\rho}$ where $\rho=\alpha+\beta i$, $\bar{\rho}=\alpha-\beta i$, $\rho\bar{\rho}=\alpha^2+\beta^2$, $|\rho|=(\alpha^2+\beta^2)^{1/2}$. It is known[3] that the minimum function of x and R_x coincide with $f(\xi, x)$ and that $|x|=|\rho|=(\alpha^2+\beta^2)^{1/2}$.

Assume that $u_1, \cdots, u_n$ are pairwise J-orthogonal quantities such that $u_i^2=-1$. Then if $x=\alpha_0+\alpha_1u_1+\cdots+\alpha_nu_n$ we have $x^2-2\alpha_0x+\alpha_0^2+\cdots+\alpha_n^2=0$. It follows that $|x|^2=\alpha_0^2+\alpha_1^2+\cdots+\alpha_n^2$.

We are now able to compute the absolute values of quantities and shall assume that $\mathfrak{A}$ contains a pair of J-orthogonal quantities u and v such that $u^2=v^2=-1$. By Lemma 3 we may write $uv=\lambda_1+\lambda_2u+\lambda_3v+\lambda_4w$ where $\lambda_1, \cdots, \lambda_4$ are real, u, v, w are pairwise J-orthogonal and $w^2=-1$. Of course when uv is in $\mathfrak{F}+u\mathfrak{F}+v\mathfrak{F}$ this will be given by $\lambda_4=0$. Then $|uv|^2=|u|^2|v|^2=1=\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2$. But $|1+v|^2=2$, $|u(1+v)|^2=2=|u+uv|^2=|\lambda_1+(\lambda_2+1)u+\lambda_3v+\lambda_4w|^2=\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+1+2\lambda_2$. Then $\lambda_2=0$. A similar computation of $|(1+u)v|^2$ yields $\lambda_3=0$, $\lambda_1^2+\lambda_4^2=1$. Finally $|(1+u)(1+v)|^2=|(\lambda_1+1)+u+v+\lambda_4w|^2=4=\lambda_1^2+\lambda_4^2+3+2\lambda_1$, $\lambda_1=0$, $\lambda_4=\pm 1$, $uv=\pm w$ is J-orthogonal to u and v and $(uv)^2=w^2=-1$.

Apply the argument above to the J-orthogonal quantities u and uv to see that $[u(uv)]^2=-1$ and that $u(uv)$ is J-orthogonal to both u and uv. Then $u(uv)=\xi v+\eta z$ where ξ and η are real, z is J-orthogonal to

u, v and uv, $z^2=-1$ and of course the nonexistence of such a z implies that $\eta=0$. Since $|u(wv)|=|u|^2|v|=1$ we have $\xi^2+\eta^2=1$. We form the product $(-1+u)(v+uv)=-v+uv-uv+u(uv)=-v+u(uv)$ $=(\xi-1)v+\eta z$ and see that $4=|(\xi-1)v+\eta z|^2=\xi^2+\eta^2+1-2\xi$, -2ξ $=2$, $\xi=-1$, $\xi^2+\eta^2=1+\eta^2=1$, $\eta=0$. Hence $u(uv)=-v$.

The results above imply that $\mathfrak{A}$ is alternative. Indeed the relation $x(xy)=(xx)y$ is equivalent to $(\alpha+\beta u)[(\alpha+\beta u)y]=[(\alpha+\beta u)(\alpha+\beta u)]y$ for real α and β where $u^2=-1$. But $\alpha(\alpha y)+(\beta u)\alpha y+\alpha[(\beta u)y]$ $=\alpha^2 y+[(\alpha\beta)u]y+[(\beta u)\alpha]y$ and so we need only prove $u(uy)=(uu)y$. We may also write $y=\lambda+\mu u+\nu v$ where λ, μ, ν are real, $uv=-vu$ and $v^2=-1$, and the linearity of our relation in y implies that we need only prove the trivial $u(u^2)=u^2u=-u$ and the relation $u(uv)=(uu)v$ $=-v$ proved above. Similarly the relation $(yx)x=y(xx)$ is implied by $(vu)u=-(uv)u=u(uv)=-v=v(uu)$. We then apply Theorem 1 to obtain

THEOREM 2. *Let $\mathfrak{A}$ be an absolute-valued real algebraic algebra with a unity quantity. Then $\mathfrak{A}$ is a finite-dimensional algebra and is indeed the field $\mathfrak{R}$ or $\mathfrak{C}$, the quaternion algebra $\mathfrak{Q}$ or the Cayley algebra $\mathfrak{D}$.*

An algebra is called a *division algebra* if its nonzero elements form a quasigroup. For such algebras we have

THEOREM 3. *Let $\mathfrak{A}$ be an absolute valued real algebraic division algebra without a unity element. Then $\mathfrak{A}$ is an algebra of order four or eight.*

For the mappings $x\rightarrow xg=xR_g$ and $x\rightarrow gx=xL_g$ are one-to-one mappings of $\mathfrak{A}$ on $\mathfrak{A}$ for every $g\neq 0$ of $\mathfrak{A}$ and so R_g and L_g have inverses. Select g so that $|g|=1$ and take $P=R_g^{-1}$, $Q=L_g^{-1}$. Let $\mathfrak{A}_0$ be the same real vector space as $\mathfrak{A}$ and define multiplication in $\mathfrak{A}_0$ by $x\cdot y=(xP)(yQ)$. Then if $e=gg$ we have $eP=eQ=g$, $e\cdot y=g(yQ)$ $=yL_g^{-1}L_g=y$, $y\cdot e=(yR_g^{-1})g=y$ and so $\mathfrak{A}_0$ has e as its unity quantity. Since $\mathfrak{A}_0$ is the same vector space as $\mathfrak{A}$ the additive and scalar properties of $|x|$ all hold in $\mathfrak{A}_0$. But $|x\cdot y|=|xP|\cdot|yQ|=|x|\,|y|$ since $(xP)g=x$, $|x|=|xP|\,|g|=|xP|$, $g(yQ)=y$, $|y|=|g|\,|yQ|=|yQ|$. Thus $\mathfrak{A}_0$ is an absolute valued algebraic algebra with a unity quantity and is finite-dimensional by Theorem 2, $\mathfrak{A}$ is finite-dimensional. However the result on the dimension of $\mathfrak{A}$ of our Theorem is known[1] for algebras.

THE UNIVERSITY OF CHICAGO

INNER DERIVATIONS OF NON-ASSOCIATIVE ALGEBRAS

R. D. SCHAFER

In this note we propose a definition of inner derivation for non-associative algebras. This definition coincides with the usual one for Lie algebras, and for associative algebras with no absolute right (left) divisor of zero. It is well known that all derivations of semi-simple associative or Lie algebras over a field of characteristic zero are inner.

Recent correspondence with N. Jacobson has revealed that a number of the ideas in this note duplicate some of his current researches.[1] In particular, he has shown that every derivation of a semi-simple non-associative algebra (that is, direct sum of simple algebras) with a unity quantity over a field of characteristic zero is inner in this sense.

1. **Preliminaries.** A *derivation* of a non-associative algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ is a linear transformation D on $\mathfrak{A}$ satisfying

$$(xy)D = x(yD) + (xD)y \tag{1}$$

for all x, y in $\mathfrak{A}$. It is known [2][2] that the set $\mathfrak{D}$ of all derivations of $\mathfrak{A}$ is a Lie algebra over $\mathfrak{F}$ if multiplication in $\mathfrak{D}$ is defined by

$$[D_1, D_2] = D_1D_2 - D_2D_1 \tag{2}$$

where D_1D_2 is the ordinary (associative) multiplication of linear transformations. $\mathfrak{D}$ is called the *derivation algebra* of $\mathfrak{A}$.

If we write R_y for the *right multiplication*

$$x \to xy = xR_y \qquad \text{for all } x \text{ in } \mathfrak{A}$$

and L_x for the *left multiplication*

$$y \to xy = yL_x \qquad \text{for all } y \text{ in } \mathfrak{A},$$

the definition (1) is seen to be equivalent to either one of

$$[R_y, D] = R_{yD} \qquad \text{for all } y \text{ in } \mathfrak{A} \tag{3}$$

or

$$[L_x, D] = L_{xD} \qquad \text{for all } x \text{ in } \mathfrak{A}. \tag{4}$$

Presented to the Society, April 30, 1949; received by the editors June 20, 1948.

[1] N. Jacobson, *Derivation algebras and multiplication algebras of semi-simple Jordan algebras*, to appear in Ann. of Math.

[2] Numbers in brackets refer to the references cited at the end of the paper.

We denote by $R(\mathfrak{A})$ (or $L(\mathfrak{A})$) the set of all right (or left) multiplications of $\mathfrak{A}$. Then (3) and (4) imply

$$[R(\mathfrak{A}), \mathfrak{D}] \leqq R(\mathfrak{A}), \qquad [L(\mathfrak{A}), \mathfrak{D}] \leqq L(\mathfrak{A}), \tag{5}$$

where by $[\mathfrak{S}, \mathfrak{T}]$ we mean the linear set spanned by all $[S, T]$ for S in $\mathfrak{S}$, T in $\mathfrak{T}$. Anti-commutativity

$$[S, T] = -[T, S] \tag{6}$$

and the *Jacobi identity*

$$[[S, T], U] + [[T, U], S] + [[U, S], T] = 0 \tag{7}$$

for linear transformations on $\mathfrak{A}$ imply

$$[\mathfrak{S}, \mathfrak{T}] = [\mathfrak{T}, \mathfrak{S}], \tag{8}$$

$$[[\mathfrak{S}, \mathfrak{T}], \mathfrak{U}] \leqq [[\mathfrak{T}, \mathfrak{U}], \mathfrak{S}] + [[\mathfrak{U}, \mathfrak{S}], \mathfrak{T}] \tag{9}$$

for linear sets $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{U}$ of linear transformations on $\mathfrak{A}$.[3]

A non-associative algebra $\mathfrak{A}$ which is not the zero algebra of dimension one is called *simple* in case its only ideals are $\{0\}$ and $\mathfrak{A}$. A *semi-simple* non-associative algebra $\mathfrak{A}$ is the direct sum $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_s$ of simple components $\mathfrak{A}_i$.

The *center* $\mathfrak{Z}$ of a non-associative algebra $\mathfrak{A}$ consists of all elements c in $\mathfrak{A}$ such that

$$L_c = R_c, \qquad R_yR_c = R_cR_y = R_{cy} \tag{10}$$

for all y in $\mathfrak{A}$. The center of $\mathfrak{A}$ is *characteristic*; that is, $\mathfrak{Z}\mathfrak{D} \leqq \mathfrak{Z}$. For $L_{cD} = [L_c, D] = [R_c, D] = R_{cD}$, while $R_yR_{cD} = [R_yR_c, D] - [R_y, D]R_c = [R_cR_y, D] - R_c[R_y, D] = [R_c, D]R_y = R_{cD}R_y = R_{(cy)D} - R_{c(yD)} = R_{(cD)y}$.

Also the simple components of a semi-simple algebra are characteristic, for $\mathfrak{A}_i^2 = \mathfrak{A}_i$ implies that x in $\mathfrak{A}_i$ may be written in the form $x = \sum y_jz_j$ for y_j, z_j in $\mathfrak{A}_i$. Then $xD = \sum (y_jz_j)D = \sum y_j(z_jD) + \sum (y_jD)z_j$ is in $\mathfrak{A}_i$ since $\mathfrak{A}_i$ is an ideal of $\mathfrak{A}$.

The center $\mathfrak{Z}$ of a simple non-associative algebra $\mathfrak{A}$ is either $\{0\}$ or a field. If $\mathfrak{A}$ contains a unity element 1, then $\mathfrak{Z} \neq \{0\}$, and $\mathfrak{A}$ may be regarded as an algebra over $\mathfrak{Z}$. As such, it is central simple (that is, simple for all scalar extensions).

2. **Inner derivations.** Let $\mathfrak{M}$ be a linear set of transformations on $\mathfrak{A}$, and write

$$\mathfrak{M}_1 = \mathfrak{M}, \qquad \mathfrak{M}_i = [\mathfrak{M}_1, \mathfrak{M}_{i-1}], \qquad i = 2, 3, \cdots. \tag{11}$$

---

[3] Throughout this note we use the notation $\mathfrak{B} + \mathfrak{C}$ for the sum of the linear sets $\mathfrak{B}$, $\mathfrak{C}$; we do not mean necessarily that $\mathfrak{B} \cap \mathfrak{C} = \{0\}$.

Then the set $\mathfrak{L}=\mathfrak{M}_1+\cdots+\mathfrak{M}_i+\cdots$ is the smallest Lie algebra containing $\mathfrak{M}$. For, when we prove

$$[\mathfrak{M}_i, \mathfrak{M}_j] \leqq \mathfrak{M}_{i+j}, \qquad i, j = 1, 2, \cdots, \tag{12}$$

we have proved that $\mathfrak{L}$ is a Lie algebra. By virtue of (8) it is sufficient to prove (12) for $j \geqq i$. If $i=1$ we have (12) by definition. We assume therefore that $[\mathfrak{M}_k, \mathfrak{M}_l] \leqq \mathfrak{M}_{k+l}$ for any $l \geqq k$ and for $k<i$. Then $[\mathfrak{M}_i, \mathfrak{M}_j] = [[\mathfrak{M}_1, \mathfrak{M}_{i-1}], \mathfrak{M}_j] \leqq [[\mathfrak{M}_{i-1}, \mathfrak{M}_j], \mathfrak{M}_1] + [[\mathfrak{M}_j, \mathfrak{M}_1], \mathfrak{M}_{i-1}] \leqq [\mathfrak{M}_{i+j-1}, \mathfrak{M}_1] + [\mathfrak{M}_{j+1}, \mathfrak{M}_{i-1}] \leqq \mathfrak{M}_{i+j}$ by (9). Now any Lie algebra containing $\mathfrak{M}$ contains all $\mathfrak{M}_i$ and therefore $\mathfrak{L}$, so $\mathfrak{L}$ is the smallest Lie algebra containing $\mathfrak{M}$.

The role of $\mathfrak{L}$ is analogous to that of the *enveloping algebra* of $\mathfrak{M}$, which is the smallest associative algebra containing $\mathfrak{M}$. Clearly $\mathfrak{L}$ is contained in the enveloping algebra of $\mathfrak{M}$.

Let $\mathfrak{M}$ be the set $R(\mathfrak{A})+L(\mathfrak{A})$ spanned by the right and left multiplications of $\mathfrak{A}$. Then we call $\mathfrak{L}$ the *Lie transformation algebra* of $\mathfrak{A}$. $\mathfrak{L}$ is contained in the so-called *transformation algebra* $T(\mathfrak{A})$, the enveloping algebra of the right and left multiplications and the identity I.

DEFINITION. We call a derivation D of a non-associative algebra $\mathfrak{A}$ *inner* in case D is in the Lie transformation algebra $\mathfrak{L}$ of $\mathfrak{A}$.

We recall that an algebra $\mathfrak{A}$ is *associative* in case

$$(xy)z = x(yz) \qquad \text{for } x, y, z \text{ in } \mathfrak{A}. \tag{13}$$

It is easy to see that (13) is equivalent to any one of

$$L_{xy} = L_y L_x, \qquad [L_x, R_z] = 0, \qquad R_y R_z = R_{yz}. \tag{14}$$

Let

$$D = R_d - L_d \qquad \text{for } d \text{ in } \mathfrak{A}; \tag{15}$$

then D is a derivation of $\mathfrak{A}$ by (3) and (14) since $[R_y, R_d-L_d] = [R_y, R_d] = R_{yd-dy}$. Derivations of this form have always been called inner derivations of the associative algebra $\mathfrak{A}$. They are inner derivations by our definition.

THEOREM 1. *The Lie transformation algebra of an associative algebra* $\mathfrak{A}$ *is*

$$\mathfrak{L} = R(\mathfrak{A}) + L(\mathfrak{A}). \tag{16}$$

If $\mathfrak{A}$ *has no absolute right (left) divisor of zero, then a derivation* D *of* $\mathfrak{A}$ *is inner if and only if* D *has the form* (15).

Any element of $R(\mathfrak{A})+L(\mathfrak{A})$ has the form R_x+L_y. Then (14) implies that $[R_{x_1}+L_{y_1}, R_{x_2}+L_{y_2}]=[R_{x_1}, R_{x_2}]+[L_{y_1}, L_{y_2}]=R_{[x_1,x_2]}+L_{[y_2,y_1]}$ in $R(\mathfrak{A})+L(\mathfrak{A})$, so that $R(\mathfrak{A})+L(\mathfrak{A})$ is itself a Lie algebra, and (16) holds. Let R_d+L_f in $\mathfrak{L}$ be a derivation of $\mathfrak{A}$. Since (15) is a derivation, so is $(R_d+L_f)-(R_d-L_d)=L_{f+d}$. From (1) we have $(f+d)xy=x(f+d)y+(f+d)xy$, or $x(f+d)y=xR_{(f+d)y}=0$ for all x, y in $\mathfrak{A}$. Hence $R_{(f+d)y}=0$. Assuming there is no absolute right divisor of zero in $\mathfrak{A}$, we have $(f+d)y=yL_{f+d}=0$ for all y, or $L_{f+d}=0$, $R_d+L_f=R_d-L_d$. A similar argument may be made in case $\mathfrak{A}$ contains no absolute left divisor of zero.

The identities

$$(17) \qquad xy = -yx, \qquad (xy)z + (yz)x + (zx)y = 0$$

define a *Lie algebra*. These together are equivalent to

$$(18) \qquad L_x = -R_x, \qquad [R_y, R_z] = R_{yz}.$$

Then (3) and (18) imply that

$$(19) \qquad D = R_d \qquad \text{for any } d \text{ in } \mathfrak{A}$$

is a derivation of $\mathfrak{A}$; this has always been called an inner derivation of a Lie algebra. It is also inner by our definition.

THEOREM 2. *The Lie transformation algebra of a Lie algebra* $\mathfrak{A}$ *is*

$$(20) \qquad \mathfrak{L} = R(\mathfrak{A}).$$

A derivation D *of* $\mathfrak{A}$ *is inner if and only if* D *has the form* (19).

For $L(\mathfrak{A})=R(\mathfrak{A})$ and $[R(\mathfrak{A}), R(\mathfrak{A})]\leqq R(\mathfrak{A})$ by (18).

THEOREM 3. *The set* $\mathfrak{J}=\mathfrak{L}\cap\mathfrak{D}$ *of inner derivations of a non-associative algebra* $\mathfrak{A}$ *is an ideal in the derivation algebra* $\mathfrak{D}$.

If J is in $\mathfrak{J}$ and D in $\mathfrak{D}$, we have $[J, D]$ in $\mathfrak{D}$. Moreover, J may be written in the form $J=\sum M_i$, M_i in $\mathfrak{M}_i$. Now

$$(21) \qquad [\mathfrak{M}_i, \mathfrak{D}] \leqq \mathfrak{M}_i, \qquad i = 1, 2, \cdots .$$

The case $i=1$ of (21) is given by (5). We assume (21) in a proof by induction. Since M_{i+1} is a sum of elements of the form $[M_1, M_i]$, we have $[M_{i+1}, D]$ a sum of elements of the form $[[M_1, M_i], D] = -[[M_i, D], M_1]-[[D, M_1], M_i]$ in $[\mathfrak{M}_i, \mathfrak{M}_1]+[\mathfrak{M}_1, \mathfrak{M}_i]=\mathfrak{M}_{i+1}$. Then $[J, D]$ is in $\sum[\mathfrak{M}_i, \mathfrak{D}]\leqq\sum\mathfrak{M}_i=\mathfrak{L}$, and $[J, D]\in\mathfrak{J}=\mathfrak{L}\cap\mathfrak{D}$.

COROLLARY. *If* $\mathfrak{D}$ *is simple and* $\mathfrak{J}\neq\{0\}$, *then* $\mathfrak{J}=\mathfrak{D}$; *that is, all the derivations of* $\mathfrak{A}$ *are inner.*

THEOREM 4. *Let* $\mathfrak{A}=\mathfrak{A}_1\oplus \cdots \oplus\mathfrak{A}_t$ *be a semi-simple non-associa-*

tive algebra, $\mathfrak{A}_i$ *simple. All the derivations of* $\mathfrak{A}$ *are inner if and only if all the derivations of* $\mathfrak{A}_i$ *are inner* $(i=1, 2, \cdots, s)$.

Since $\mathfrak{A}_i$ is characteristic, any derivation D of $\mathfrak{A}$ induces a derivation $D^{(i)}$ on $\mathfrak{A}_i$, and

$$D = D_1 + \cdots + D_s, \qquad D_i \text{ derivations of } \mathfrak{A}, \tag{22}$$

where the restriction of D_i to $\mathfrak{A}_i$ is $D^{(i)}$ and $\mathfrak{A}_jD_i = \{0\}$ for $j \neq 1$. Conversely, if $D^{(i)}$ is a derivation of $\mathfrak{A}_i$, the transformation D_i on $\mathfrak{A}$, whose restriction to $\mathfrak{A}_i$ is $D^{(i)}$ and for which $\mathfrak{A}_jD_i = \{0\}$ if $j \neq i$, is a derivation of $\mathfrak{A}$; then so is the sum (22). Since the $\mathfrak{A}_i$ are pairwise orthogonal ideals, a similar decomposition of right and left multiplications is possible, and any element T of the Lie transformation algebra $\mathfrak{L}$ of $\mathfrak{A}$ may be written as

$$T = T_1 + \cdots + T_s \tag{23}$$

where $\mathfrak{A}_jT_i = \{0\}$ for $j \neq i$, and the restriction of T_i to $\mathfrak{A}_i$ is an element $T^{(i)}$ of the Lie transformation algebra of $\mathfrak{A}_i$. Conversely if $T^{(i)}$ is an element of the Lie transformation algebra of $\mathfrak{A}_i$, define the transformation T_i of $\mathfrak{A}$ whose restriction to $\mathfrak{A}_i$ is $T^{(i)}$ and for which $\mathfrak{A}_jT_i = \{0\}$ if $j \neq i$; then T_i is in $\mathfrak{L}$, and so is the sum (23). Then $D = T$ if and only if $D^{(i)} = T^{(i)}$ $(i=1, 2, \cdots s)$.

3. **Alternative algebras.** An *alternative algebra* $\mathfrak{A}$ is a non-associative algebra in which

$$x^2y = x(xy), \qquad yx^2 = (yx)x \qquad \text{for all } x, y \text{ in } \mathfrak{A}. \tag{24}$$

All associative algebras are alternative. Moreover, all simple alternative algebras are associative, except for algebras which are Cayley-Dickson algebras over their centers [4]. *Cayley-Dickson algebras* are algebras $\mathfrak{C}$ of dimension 8 formed from a (generalized) quaternion algebra $\mathfrak{Q}$ as follows: the elements of $\mathfrak{C} = \mathfrak{Q} + v\mathfrak{Q}$ are $q_1 + vq_2$, q_i in $\mathfrak{Q}$; multiplication in $\mathfrak{C}$ is defined by

$$(q_1 + vq_2)(q_3 + vq_4) = (q_1q_3 + \gamma q_4\bar{q}_2) + v(\bar{q}_1q_4 + q_3q_2) \tag{25}$$

where $\gamma \neq 0$ in $\mathfrak{F}$ and $\bar{q} = t(q)1 - q$ for q in $\mathfrak{Q}$ with $q^2 - t(q)q + n(q)1 = 0$.

In an alternative algebra $\mathfrak{A}$ the "associator" $[x_1, x_2, x_3] = (x_1x_2)x_3 - x_1(x_2x_3)$ "alternates"; that is, $[x_1, x_2, x_3] = \epsilon[x_{i_1}, x_{i_2}, x_{i_3}]$ for any permutation i_1, i_2, i_3 of 1, 2, 3, where ϵ is 1 in case the permutation is even, -1 in case it is odd. Equivalently,

$$\begin{aligned} R_{xz} - R_xR_z = R_zR_x - R_{zx} = L_{zx} - L_xL_z = L_zL_x - L_{xz} \\ = [L_x, R_z] = [R_x, L_z]. \end{aligned} \tag{26}$$

Now (26) implies that

$$[R_x, R_z] = R_{[x,z]} - 2[L_x, R_z] \tag{27}$$

and

$$[L_x, L_z] = -L_{[x,z]} - 2[L_x, R_z], \tag{28}$$

where by $[x, z]$ we mean $xz - zx$ in $\mathfrak{A}$. Also

$$D_1 = 3[L_x, L_z] + 2R_{[x,z]} + L_{[x,z]} \tag{29}$$

is a derivation of $\mathfrak{A}$ for any x, z in $\mathfrak{A}$, for it is easy to see from (26) that

$$[L_y, [L_x, L_z]] = L_{y[L_x,L_z]+y[x,z]}, \tag{30}$$

while (28) gives $[L_y, 2R_{[x,z]} + L_{[x,z]}] = -L_{[y,[x,z]]}$. With (27) and (28) it is easy to write several variants of (29); indeed, if the characteristic of $\mathfrak{F}$ is not two, we obtain the symmetrical expression

$$D = [L_x, L_z] + [L_x, R_z] + [R_x, R_z] \tag{31}$$

for $D = D_1/2$ in (29). Then the sum

$$D = \sum_i ([L_{x_i}, L_{z_i}] + [L_{x_i}, R_{z_i}] + [R_{x_i}, R_{z_i}]) \tag{32}$$

is also a derivation of $\mathfrak{A}$ for x_i, z_i in $\mathfrak{A}$.

THEOREM 5. *The Lie transformation algebra of an alternative algebra* $\mathfrak{A}$ *over* $\mathfrak{F}$ *of characteristic not two is*

$$\mathfrak{L} = R(\mathfrak{A}) + L(\mathfrak{A}) + [L(\mathfrak{A}), R(\mathfrak{A})]. \tag{33}$$

Let $\mathfrak{M}_1 = R(\mathfrak{A}) + L(\mathfrak{A})$ and $\mathfrak{L}_0 = \mathfrak{M}_1 + [L(\mathfrak{A}), R(\mathfrak{A})]$. Then $\mathfrak{M}_2 = [\mathfrak{M}_1, \mathfrak{M}_1] \leqq \mathfrak{L}_0$ by (27) and (28). Assume $\mathfrak{M}_{i-1} \leqq \mathfrak{L}_0$. Then $\mathfrak{M}_i = [\mathfrak{M}_1, \mathfrak{M}_{i-1}] \leqq [\mathfrak{M}_1, \mathfrak{M}_1 + [L(\mathfrak{A}), R(\mathfrak{A})]] = \mathfrak{M}_2 + [\mathfrak{M}_1, [L(\mathfrak{A}), R(\mathfrak{A})]]$. Since the characteristic of $\mathfrak{F}$ is not two, it follows from (28) and (30) that $[L_y, [L_x, R_z]]$ is in $\mathfrak{L}_0$. Similarly it may be shown that $[R_y, [L_x, R_z]]$ is in $\mathfrak{L}_0$. Hence $\mathfrak{M}_i \leqq \mathfrak{L}_0$ for all i, $\mathfrak{L} \leqq \mathfrak{L}_0$. But $\mathfrak{L}_0 \leqq \mathfrak{M}_1 + \mathfrak{M}_2$, so $\mathfrak{L} = \mathfrak{L}_0$. It is clear from (27) and (28) that we may also write $\mathfrak{L} = R(\mathfrak{A}) + L(\mathfrak{A}) + [L(\mathfrak{A}), L(\mathfrak{A})] = R(\mathfrak{A}) + L(\mathfrak{A}) + [R(\mathfrak{A}), R(\mathfrak{A})]$.

THEOREM 6. *All derivations of a Cayley-Dickson algebra* $\mathfrak{C}$ *over a field of characteristic zero are inner. They have the form* (32).[4]

---

[4] *Added in proof*: Let $\mathfrak{A}$ be an alternative algebra with unity element over $\mathfrak{F}$ of characteristic not two or three. Then (using $\mathfrak{L} = R(\mathfrak{A}) + L(\mathfrak{A}) + [L(\mathfrak{A}), L(\mathfrak{A})]$ and (29)) one may prove that a derivation D of $\mathfrak{A}$ is inner if and only if D is the sum of two derivations: one of the form (32), the other of the form (15) with d satisfying $[d, x, y] = 0$ for all x, y in $\mathfrak{A}$. For a Cayley-Dickson algebra $\mathfrak{C}$ the second derivation in the sum is zero.

The set of inner derivations of $\mathfrak{C}$ is not $\{0\}$, for it is easy to choose a pair of quaternions x, z in $\mathfrak{Q}$ such that $[z, x] \neq 0$. Then, applying D in (31) to v in (25), we obtain $vD = v[\bar{z}, \bar{x}] + v[z, \bar{x}] + v[z, x] = v[z, x]$ since $\bar{z} + z = t(z)1$. Then $D \neq 0$. But D is inner and, by the corollary to Theorem 3, all derivations of $\mathfrak{C}$ are inner, for it is known [**3**, p. 780] that the derivation algebra $\mathfrak{D}$ of $\mathfrak{C}$ is simple (the exceptional simple Lie algebra of dimension 14). To see that the set $\mathfrak{D}_0$ of derivations (32) is an ideal in $\mathfrak{D}$, it is sufficient to show that for D_1 in (29) we have $[D_1, D']$ in $\mathfrak{D}_0$ for any derivation D' of $\mathfrak{C}$. But this is an easy consequence of the Jacobi identity and the fact that $[x, z]D' = [x, zD'] + [xD', z]$. Since $\mathfrak{D}_0 \neq \{0\}$, $\mathfrak{D}_0 = \mathfrak{D}$.

THEOREM 7. *All derivations of a semi-simple alternative algebra* $\mathfrak{A}$ *over a field of characteristic zero are inner.*

Since $\mathfrak{A}$ is the direct sum of simple components which are either associative or Cayley-Dickson algebras over their centers, and since it is well known that derivations of the associative components are inner, this theorem is reduced by Theorem 4 to the case where $\mathfrak{A}$ is a Cayley-Dickson algebra $\mathfrak{C}$ over its center $\mathfrak{Z}$. But, since the characteristic is zero, the derivations of $\mathfrak{A}$ map the field $\mathfrak{Z}$ upon $\{0\}$; moreover, if $\mathfrak{D}$ is the derivation algebra of $\mathfrak{C}$, then $\mathfrak{D}_{\mathfrak{Z}}$ is the derivation algebra of $\mathfrak{A}$. Then by Theorem 6 all derivations of $\mathfrak{A}$ are inner.

4. Jordan algebras.

A *Jordan algebra* is a commutative algebra $\mathfrak{A}$ in which

$$x^2(xy) = x(x^2y) \qquad \text{for all } x, y \text{ in } \mathfrak{A}. \tag{34}$$

A. A. Albert has shown [**1**, equation (8), p. 550] that in a Jordan algebra $\mathfrak{A}$ over $\mathfrak{F}$ of characteristic not two the identity

$$[R_y, [R_x, R_z]] = R_{(xy)z - x(yz)} \tag{35}$$

holds. That is, $[R_x, R_z]$ is a derivation of $\mathfrak{A}$ for any x, z in $\mathfrak{A}$. So is the sum

$$D = \sum_i [R_{x_i}, R_{z_i}], \qquad \text{for } x_i, z_i \text{ in } \mathfrak{A}. \tag{36}$$

THEOREM 8. *The Lie transformation algebra of a Jordan algebra* $\mathfrak{A}$ *over a field of characteristic not two is*

$$\mathfrak{L} = R(\mathfrak{A}) + [R(\mathfrak{A}), R(\mathfrak{A})]. \tag{37}$$

If $\mathfrak{A}$ *has a unity element* 1, *then a derivation* D *of* $\mathfrak{A}$ *is inner if and only if* D *has the form* (36).

Since $\mathfrak{A}$ is commutative, $\mathfrak{M}_1=R(\mathfrak{A})$. Any element of $\mathfrak{M}_3=[R(\mathfrak{A}), [R(\mathfrak{A}), R(\mathfrak{A})]]$ is a sum of elements of the form (35), so $\mathfrak{M}_3\leqq\mathfrak{M}_1$. Then $[\mathfrak{M}_2, \mathfrak{M}_2]\leqq\mathfrak{M}_4=[\mathfrak{M}_1, \mathfrak{M}_3]\leqq[\mathfrak{M}_1, \mathfrak{M}_1]=\mathfrak{M}_2$ by (12). Hence $\mathfrak{M}_1+\mathfrak{M}_2$ is a Lie algebra, $\mathfrak{L}=\mathfrak{M}_1+\mathfrak{M}_2=R(\mathfrak{A})+[R(\mathfrak{A}), R(\mathfrak{A})]$. Let $D=R_y+\sum_i [R_{x_i}, R_{z_i}]$ be an inner derivation. Since $1D=0$, we have $y=0$.

We refer the reader to the proof of the following theorem by N. Jacobson in the paper mentioned in footnote 1.

THEOREM 9 (JACOBSON). *All derivations of a semi-simple Jordan algebra $\mathfrak{A}$ over a field of characteristic zero are inner; they have the form* (36).

REFERENCES

1. A. A. Albert, *A structure theory for Jordan algebras*, Ann. of Math. vol. 48 (1947) pp. 546–567.

2. N. Jacobson, *Abstract derivation and Lie algebras*, Trans. Amer. Math. Soc. vol. 42 (1937) pp. 206–224.

3. ———, *Cayley numbers and normal simple Lie algebras of type G*, Duke Math. J. vol. 5 (1939) pp. 775–783.

4. R. D. Schafer, *Alternative algebras over an arbitrary field*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 549–555.

THE INSTITUTE FOR ADVANCED STUDY

ON THE THEORY OF SUM-EQUATIONS

I. M. SHEFFER

1. **Introduction.** A system of linear equations is called a *sum-equation system* if it has the form

$$a_{n0}x_n + a_{n1}x_{n+1} + \cdots = c_n \qquad (n = 0, 1, \cdots), \tag{1.1}$$

with $a_{n0} \neq 0$. All quantities are assumed to be complex numbers. Let $\{A_n(t)\}$ be defined by

$$A_n(t) = \sum_{s=0}^{\infty} a_{ns}t^s \qquad (n = 0, 1, \cdots), \tag{1.2}$$

and define the *type* of a sequence $\{y_n\}$ as the number $((y_n))$ given by

$$((y_n)) \equiv \limsup_{n\to\infty} | y_n |^{1/n}. \tag{1.3}$$

Suppose the functions $\{A_n(t)\}$ are analytic in $|t| < q$, and that $((c_n)) = \delta < q$, so that the function

$$C(t) = \sum_{0}^{\infty} c_n t^n \tag{1.4}$$

has the radius of convergence $1/\delta$.

Let functions $\{H_{k,j}(t)\}$ $(j = 0, 1, \cdots, k-1;\ k = 0, 1, \cdots)$ be defined by

$$H_{k,j}(t) = c_j + c_{j+k}t^k + c_{j+2k}t^{2k} + \cdots, \tag{1.5}$$

and let

$$\Delta_k(t; H) = \begin{vmatrix} H_{k,0}(1/t) & (1/t)H_{k,1}(1/t) & \cdots & (1/t^{k-1})H_{k,k-1}(1/t) \\ A_0(\omega_k t) & \omega_k A_1(\omega_k t) & \cdots & \omega_k^{k-1} A_{k-1}(\omega_k t) \\ \cdots & \cdots & \cdots & \cdots \\ A_0(\omega_k^{k-1} t) & \omega_k^{k-1} A_1(\omega_k^{k-1} t) & \cdots & \omega_k^{(k-1)(k-1)} A_{k-1}(\omega_k^{k-1} t) \end{vmatrix}, \tag{1.6}$$

where

$$\omega_k = e^{2\pi i/k} \qquad [i = (-1)^{1/2}]. \tag{1.7}$$

The functions $H_{k,j}(t)$ are analytic in $|t| < 1/\delta$, the function $\Delta_k(t)$ given by

---

Presented to the Society, April 17, 1948; received by the editors June 28, 1948.

$$(1.8)\quad \Delta_k(t) = \begin{vmatrix} A_0(t) & A_1(t) & A_2(t) & \cdots & A_{k-1}(t) \\ A_0(\omega_k t) & \omega_k A_1(\omega_k t) & \omega_k^2 A_2(\omega_k t) & \cdots & \omega_k^{k-1} A_{k-1}(\omega_k t) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_0(\omega_k^{k-1} t) & \omega_k^{k-1} A_1(\omega_k^{k-1} t) & \omega_k^{2(k-1)} A_2(\omega_k^{k-1} t) & \cdots & \omega_k^{(k-1)(k-1)} A_{k-1}(\omega_k^{k-1} t) \end{vmatrix}$$

in $|t| < q$, and the function $\Delta_k(t; \bar{H})$ in the ring $\delta < |t| < q$.

The present article has two principal aims. One is to establish the following result on sum-equations:

THEOREM 1.1. *Let the functions* $\{A_n(t)\}$ *be analytic in* $|t| < q$, *with* $A_n(0) \neq 0$, *and let* $((c_n)) = \delta < q$. *If there exists an increasing sequence of integers* k_s $(s = 1, 2, \cdots)$ *such that*

$$(1.9)\qquad \lim_{s=\infty} k_s \Delta_{k_s}(t; \bar{H}) / \Delta_{k_s}(t) = \Delta^*(t),$$

the convergence being uniform in a ring R: $q_1 < |t| < q_2$, *with* $\delta < q_2 \leqq q$, *then*

$$(1.10)\qquad x_n = \frac{1}{2\pi i} \int_\gamma t^{n-1} \Delta^*(t)\, dt$$

is a solution of system (1.1), *where* γ *is a rectifiable simple closed curve surrounding* $t = 0$ *and lying in the ring* R. *Moreover,* $((x_n)) \leqq d$ *where* d *is the maximum distance from* γ *to the origin.*

This theorem will be obtained in §3 as the result of a limiting process based on the solution of associated k-periodic systems of equations. The second aim is to obtain solutions of homogeneous and nonhomogeneous k-periodic systems, *in closed form*, as contour integrals. This is done in §§2 and 4.

2. Nonhomogeneous k-periodic system.

Sum-systems that are k-periodic (with or without perturbation terms) were treated elsewhere.[1] There the existence theorems did not give solutions in closed form. We now obtain closed solutions. System

$$(2.1)\qquad a_{n0}x_n + a_{n1}x_{n+1} + \cdots = c_n \qquad (n = 0, 1, \cdots)$$

is *k-periodic* if

$$(2.2)\qquad A_{kn+p}(t) = A_p(t) \qquad (p = 0, 1, \cdots, k-1;\ n = 0, 1, \cdots),$$

where the functions $\{A_n(t)\}$ are given by (1.2). We suppose here that each $A_n(t)$ is analytic in $|t| \leqq q$ (so that q can be replaced by $q+\epsilon$ for sufficiently small positive ϵ), and that $((c_n)) = \delta \leqq q$.

---

[1] I. M. Sheffer, *On k-periodic systems of linear equations*, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 244–313. We shall refer to this article as Tr.

Let γ be the circle $|t| = q'$, where $\delta < q'$, and where q' is chosen close enough to δ so that the functions $\{A_n(t)\}$ are regular[2] in $|t| \leqq q'$ and also so that the function $\Delta_k(t)$ given by (1.8) has no zeros on γ. Define the sequence $z_n^{(j)}$ by

$$z_n^{(j)} = \frac{1}{2\pi i}\int_\gamma \frac{t^{n+\sigma}}{\Delta_k(t)}\cdot \Delta_{k;0,j}(t) H_{k,\nu}\left(\frac{1}{t}\right) dt, \tag{2.3}$$

where $H_{k,\nu}$ is given by (1.5) and $\Delta_{k;r,j}(t)$ *is the cofactor of the element in the rth row and jth column in* $\Delta_k(t)$. Moreover, σ and ν are (as yet undetermined) integers whose value will be made to depend on j.

Substitute the sequence $z_n^{(j)}$ into the $(ks+r)$th equation of (2.1). By uniform convergence, and the use of (2.2),

$$\begin{aligned} A_{ks+r}[\{z_n^{(j)}\}] &\equiv \sum_{n=0}^{\infty} a_{ks+r,n} z_{ks+r+n}^{(j)} \\ &= \frac{1}{2\pi i}\int_\gamma \frac{t^{ks+r+\sigma}}{\Delta_k(t)} A_r(t)\Delta_{k;0,j}(t) H_{k,\nu}\left(\frac{1}{t}\right) dt. \end{aligned} \tag{2.4}$$

Make the substitution $t = \omega_k^\lambda u$, where ω_k is given by (1.7) and λ is an integer. As t traverses γ, so does u, and on using the relations[3]

$$\Delta_k(\omega_k^\lambda t) = \Delta_k(t), \tag{2.5}$$

$$H_{k,\nu}(\omega_k^\lambda t) = H_{k,\nu}(t), \tag{2.6}$$

we obtain

$$\begin{aligned} A_{ks+r}[\{z_n^{(j)}\}] = \frac{1}{2\pi i}\int_\gamma &\frac{u^{ks+r+\sigma}}{\Delta_k(u)}\cdot H_{k,\nu}\left(\frac{1}{u}\right)\omega_k^{\lambda(r+\sigma+1)} \\ &\cdot A_r(\omega_k^\lambda u)\cdot \Delta_{k;0,j}(\omega_k^\lambda u) du. \end{aligned} \tag{2.7}$$

Now sum over $\lambda = 0, 1, \cdots, k-1$ and divide by k:

$$\begin{aligned} A_{ks+r}[\{z_n^{(j)}\}] = \frac{1}{2\pi k i}\int_\gamma &\frac{u^{ks+r+\sigma}}{\Delta_k(u)} \\ &\cdot H_{k,\nu}\left(\frac{1}{u}\right)\left\{\sum_{\lambda=0}^{k-1} \omega_k^{\lambda(r+\sigma+1)} A_r(\omega_k^\lambda u)\Delta_{k;0,j}(\omega_k^\lambda u)\right\} du. \end{aligned} \tag{2.8}$$

---

[2] This requirement is especially needed when $\delta = q$.

[3] (2.5) appears as Theorem 2.2 in Tr. (loc. cit.); and (2.6) is an immediate consequence of (1.5).

From Lemma 3.2 of Tr. (loc. cit) we find that

$$\Delta_{k;0,j}(\omega_k^{\lambda} t) = \omega_k^{\lambda j}\Delta_{k;\lambda,j}(t); \tag{2.9}$$

so

$$\begin{aligned} A_{ks+r}[\{z_n^{(j)}\}] = \frac{1}{2\pi k i}\int_\gamma & \frac{u^{ks+r+\sigma}}{\Delta_k(u)} \\ & \cdot H_{k,\nu}\left(\frac{1}{u}\right)\left\{\sum_{\lambda=0}^{k-1}\omega_k^{\lambda(j+\sigma+1)}\left[\omega_k^{\lambda r}A_r(\omega_k^{\lambda}u)\Delta_{k;\lambda,j}(u)\right]\right\}du. \end{aligned} \tag{2.10}$$

But from the definition of $\Delta_{k,\lambda,j}$ as a cofactor in the determinant $\Delta_k(t)$, we have

$$\sum_{\lambda=0}^{k-1}\omega_k^{\lambda r}A_r(\omega_k^{\lambda}u)\Delta_{k;\lambda,j}(u) = \begin{cases}\Delta_k(u) & \text{if } j = r; \\ 0 & \text{if } j \neq r.\end{cases} \tag{2.11}$$

Consequently, if we choose $\sigma = -j-1$, as we now do, then the brace in (2.10) has the value $\Delta_k(u)$ or 0 according as $j=r$ or $j\neq r$.

Thus, for $r, j = 0, 1, \cdots, k-1$; $s = 0, 1, \cdots,$

$$A_{ks+r}[\{z_n^{(j)}\}] = \begin{cases}\dfrac{1}{2\pi k i}\displaystyle\int_\gamma u^{ks-1}H_{k,\nu}\left(\frac{1}{u}\right)du, & j = r; \\ 0, & j \neq r.\end{cases} \tag{2.12}$$

Now choose $\nu = j$, and define $\{y_n\}$ by

$$y_n = \sum_{j=0}^{k-1} z_n^{(j)}, \tag{2.13}$$

so that

$$y_n = \frac{1}{2\pi i}\int_\gamma \frac{t^{n-1}}{\Delta_k(t)}\left\{\sum_{j=0}^{k-1} t^{-j}\Delta_{k;0,j}(t)H_{k,j}\left(\frac{1}{t}\right)\right\}dt. \tag{2.14}$$

Then from (2.12) we see that

$$\begin{aligned} A_{ks+r}[\{y_n\}] &= \frac{1}{2\pi k i}\int_\gamma u^{ks-1}H_{k,r}\left(\frac{1}{u}\right)du \\ &= \frac{1}{k}\left[\text{coefficient of } u^{-ks} \text{ in } H_{k,r}\left(\frac{1}{u}\right)\right] \\ &= \frac{1}{k}c_{ks+r}, \end{aligned} \tag{2.15}$$

so the following result has been established:

THEOREM 2.1. *Let system* (2.1) *be k-periodic, and let the functions* $\{A_n(t)\}$ *be analytic in* $|t| \leqq q$, *with* $A_n(0) \neq 0$. *Then for each sequence* $\{c_n\}$ *of type* $\delta \leqq q$, (2.1) *has the solution*

$$x_n = ky_n \tag{2.16}$$

where $\{y_n\}$ *is given by* (2.14), *the contour* γ *being the circle* $|t| = q'$, *where* q' *is larger than* δ *but smaller than the minimum radius of convergence of the sequence of functions* $\{A_n(t)\}$, *and where, moreover, there are no zeros of the function* $\Delta_k(t)$ *on* γ.

REMARKS. (i) Curve γ need not be a circle; it suffices that γ be a rectifiable simple closed curve surrounding $t=0$ and lying in the ring $\delta < |t| < R'$ where R' is the minimum radius of convergence of the power series $\{A_n(t)\}$ (defined by (1.2)).

(ii) Let R'' be chosen (not to exceed R') so that $\Delta_k(t)$ has no zeros in the ring $\delta < |t| < R''$. Then all contours γ, of Remark (i), that lie in this new ring will obviously yield the *same* solution (2.16). Inspection of (2.14) shows that the type of $\{y_n\}$ and hence of $\{x_n\}$ cannot exceed d, where d is the maximum distance from γ to the origin. Since γ can be chosen as close to the circle $|t| = \delta$ as we wish, it follows that for γ in the ring $\delta < |t| < R''$, the type of the solution (2.16) cannot exceed δ. If $((x_n)) < \delta$, it is easy to show that the sequence $A_n[\{x_n\}]$, that is, the sequence $\{c_n\}$, is also of type less than δ; which is a contradiction. Hence we conclude that *if contour* γ *lies in the ring* $\delta < |t| < R''$, *the solution* (2.16) *is of type equal to* δ.

COROLLARY. *The solution* (2.16) *can be written*

$$x_n = \frac{k}{2\pi i} \int_\gamma \frac{t^{n-1}}{\Delta_k(t)} \cdot \Delta_k(t;\, H)dt, \tag{2.17}$$

where $\Delta_k(t;\, H)$ *is given by* (1.6).

This follows at once from (1.6), (2.14) and the meaning of $\Delta_{k;0,j}(t)$

3. **Proof of the theorem.** We now establish Theorem 1.1. Choose the number Q to be less than q and also to lie in the range $\delta < Q \leqq q_2$. Consider an arbitrary positive integer k, and denote by $\mathcal{A}^{(k)}$ the k-periodic system obtained from (1.1) by using the first k equations of (1.1); that is, systems (1.1) and $\mathcal{A}^{(k)}$ agree in their first k equations. All the results of §2 apply to $\mathcal{A}^{(k)}$ if we use Q in place of q.

Since (1.9) holds uniformly (in the ring R), there exists an integer S such that if for an $s > S$ the denominator $\Delta_{ks}(t)$ has zeros in the ring, then these zeros are completely cancelled by corresponding zeros in the numerator $\Delta_{k_s}(t; H)$. It is therefore not necessary that the contour γ avoid zeros[4] of the denominator.

Choose γ as a circle whose radius r lies in the range $\delta < r < Q$. Given $\epsilon > 0$, there exists an $S = S_\epsilon$ such that for all $s > S$,

$$\left| \{ k_s \Delta_{k_s}(t; H)/\Delta_{k_s}(t) \} - \Delta^*(t) \right| < \epsilon, \tag{3.1}$$

uniformly on γ. Let $\{x_n^{(k_s)}\}$ be the solution given by (2.17) corresponding to the k_s-periodic system $\mathcal{A}^{(k_s)}$, and let $\{x_n\}$ be given by (1.10). Then

$$\begin{aligned} \left| A_p[\{x_n^{(k_s)}\}] - A_p[\{x_n\}] \right| &\equiv \left| \sum_{n=0}^{\infty} a_{pn} x_{n+p}^{(k_s)} - \sum_{n=0}^{\infty} a_{pn} x_{n+p} \right| \\ &= \left| \frac{1}{2\pi i} \int_\gamma t^{p-1} A_p(t) [\{ k_s \Delta_{k_s}(t; H)/\Delta_{k_s}(t) \} - \Delta^*(t)] dt \right| \\ &\leqq M_p \cdot r^p \cdot \epsilon, \end{aligned} \tag{3.2}$$

where $M_p = \max |A_p(t)|$ for t on γ.

Let p be arbitrary but fixed, and choose s (already subject to the condition $s > S$) so that $k_s > p$. Then

$$A_p[\{x_n^{(k_s)}\}] = c_p, \tag{3.3}$$

since $\{x_n^{(k_s)}\}$ satisfies system $\mathcal{A}^{(k_s)}$. Hence

$$\left| c_p - A_p[\{x_n\}] \right| < M_p \cdot r^p \cdot \epsilon. \tag{3.4}$$

Since p is fixed and ϵ is arbitrary, it follows that

$$A_p[\{x_n\}] = c_p.$$

That is, $\{x_n\}$ satisfies the pth equation of system (1.1); and from the arbitrariness of p, $\{x_n\}$ satisfies system (1.1). The portion of Theorem 1.1 referring to the type of solution $\{x_n\}$ follows at once from (1.10), so the proof of Theorem 1.1 is complete.

Remarks. (i) The hypothesis of Theorem 1.1 can be weakened in the following way without affecting the conclusion: Replace the condition that (1.9) *holds uniformly in the ring* R: $q_1 < |t| < q_2$ *with* $\delta < q_2 \leqq q$ by the condition that (1.9) *holds uniformly on* γ, *where* γ *is a rectifiable simple closed curve surrounding the origin and lying in the ring* $\delta < |t| < q$.

(ii) If $G(t)$ is analytic in and on γ, then $(1/2\pi i)\int_\gamma t^n G(t) dt = 0$,

---

[4] Since each $\Delta_k(t)$ has in $|t| \leqq Q$ only a finite number of zeros, it is possible to choose γ in the ring R so as to avoid all such zeros. For the purpose of a later Remark however, it is undesirable that γ be so restricted.

$n=0, 1, \cdots$. In the ratio $\Delta_k(t; H)/\Delta_k(t)$ that occurs in (1.9), one may therefore throw away all the *positive powers* of t (in the Laurent expansion for the ratio). Or one may state more generally: *If there exists a sequence of functions* $\{G_k(t)\}$, *analytic in and on* γ, *and a sequence* $\{k_s\}$, *such that*

$$\lim_{s=\infty} \left[\{k_s\Delta_{k_s}(t; H)/\Delta_{k_s}(t)\} + tG_{k_s}(t)\right] = \Delta^{**}(t),$$

the convergence being uniform on γ, *then* (1.1) *has the solution*

$$x_n^{**} = \frac{1}{2\pi i}\int_\gamma t^{n-1}\Delta^{**}(t)dt.$$

4. **k-periodic system, homogeneous case.** It will now be shown that the homogeneous k-periodic system also has solutions that can be exhibited in closed form. The homogeneous system is

$$a_{n0}x_n + a_{n1}x_{n+1} + \cdots = 0 \qquad (n = 0, 1, \cdots). \tag{4.1}$$

It is assumed as before that $A_n(t)$ is analytic in $|t| \leqq q$ and that $A_n(0) \neq 0$. Since the determinant $\Delta_k(t)$ of (1.8) has the property (2.5), it follows that the zeros (if any) of $\Delta_k(t)$ in $|t| \leqq q$ come in nests of k zeros each, such that if α is a zero, then the zeros in the same nest[5] are $\omega_k^s\alpha$, $s=0, 1, \cdots, k-1$. Moreover all zeros of a nest have the same multiplicity, so that the total number m of zeros of $\Delta_k(t)$ in $|t| \leqq q$ is a multiple of k: $m=lk$. We have termed l the *order*[6] of $\Delta_k(t)$ (in $|t| \leqq q$), and have shown[7] that system (4.1) has precisely l linearly independent solutions of type not exceeding q.

Consider a fixed nest of zeros $\{\omega_k^s\alpha\}$. Let C be a rectifiable simple closed curve[8] surrounding the origin and all the zeros of the nest, with all the other zeros of $\Delta_k(t)$ exterior to C; and define the sequences $X_{j,l}^{(r)}: \{x_{jl;n}^{(r)}\}$ by

$$x_{jl;n}^{(r)} = \frac{1}{2\pi i}\int_C \frac{t^{n+l}(t^k - \alpha^k)^r}{\Delta_k(t)}\cdot\Delta_{k;0,j}(t)dt, \tag{4.2}$$

where j, l are in the range $0, 1, \cdots, k-1$, and r is a non-negative integer. The value of this integral is the same for all curves C fulfilling the above conditions. It is therefore no restriction to suppose, as we now do, that C is chosen so as to be invariant under a rotation of angle $2\pi/k$ around the origin.

---

[5] Note their symmetric position about the origin.

[6] Tr., §2 loc. cit.

[7] Tr., Theorem 3.6, loc. cit.

[8] C is also to lie in the common circle of convergence of the functions $A_n(t)$, $n=0, 1, \cdots, k-1$.

Let $p=0, 1, \cdots, k-1$; $s=0, 1, \cdots$. Then, using (2.2),

$$\begin{aligned} A_{ks+p}[X_{j,l}^{(r)}] &\equiv \sum_{n=0}^{\infty} a_{pn} x_{jl;ks+p+n}^{(r)} \\ &= \frac{1}{2\pi i}\int_C \frac{t^{ks+p+l}(t^k-\alpha^k)^r}{\Delta_k(t)} \cdot A_p(t)\Delta_{k;0,j}(t)dt. \end{aligned} \tag{4.3}$$

Since the only singularities of the integrand interior to C are at the zeros $\{\omega_k^s\alpha\}$, we can replace C by k contours $C_0, \cdots, C_{k-1}$, where C_λ surrounds only the zero $\omega_k^\lambda\alpha$. In fact, C_λ may be chosen as a small circle centered at this zero, and if the radii of these circles are equal, then a rotation of angle $2\pi/k$ around the origin will carry each circle into the next one.

This replacement of $\int_C$ by $\sum_{\lambda=0}^{k-1}\int_{C_\lambda}$ being accomplished, make the change of variable $t=\omega_k^\lambda u$ in the integral $\int_{C_\lambda}$. As t traces C_λ the variable u traces C_0. Consequently,

$$\begin{aligned} A_{ks+p}[X_{j,l}^{(r)}] = \frac{1}{2\pi i}\int_{C_0} & \frac{(u^k-\alpha^k)^r}{\Delta_k(u)} \\ & \cdot u^{ks+p+l}\left\{\sum_{\lambda=0}^{k-1}\omega_k^{\lambda(p+l+1)}A_p(\omega_k^\lambda u)\Delta_{k;0,j}(\omega_k^\lambda u)\right\}du; \end{aligned} \tag{4.4}$$

and on using (2.9),

$$A_{ks+p}[X_{j,l}^{(r)}] = \frac{1}{2\pi i}\int_{C_0}\frac{(u^k-\alpha^k)^r}{\Delta_k(u)}\cdot u^{ks+p+l}\cdot\Phi_{p,l,j}(u)du, \tag{4.5}$$

where

$$\Phi_{p,l,j}(u) = \sum_{\lambda=0}^{k-1}\omega_k^{\lambda(p+l+j+1)}A_p(\omega_k^\lambda u)\Delta_{k;\lambda,j}(u)du. \tag{4.6}$$

For each j, choose $l=k-j-1$. Then from (2.11) we obtain the relation

$$\Phi_{p,k-j-1,j}(u) = \begin{cases}\Delta_k(u) & \text{if } p=j, \\ 0 & \text{if } p\neq j.\end{cases} \tag{4.7}$$

In both cases (4.5) becomes

$$A_{ks+p}[X_{j,k-j-1}^{(r)}] = 0. \tag{4.8}$$

To sum up:

THEOREM 4.1. *Let the functions* $\{A_n(t)\}$ *be analytic in* $|t|\leq q$, *with*

$A_n(0) \neq 0$. If $\{\omega_k^s \alpha\}$, $s=0, 1, \cdots, k-1$, is a nest of zeros of $\Delta_k(t)$ with $|\alpha| \leqq q$, then for each $j=0, 1, \cdots, k-1$ and for each $r=0, 1, \cdots$, the sequence (4.2) *with $l=k-j-1$ is a solution of the k-periodic homogeneous system* (4.1).

In Theorem 4.1 it is conceivable that one or more of the solutions (4.2), with $l=k-j-1$, is identically zero. (This is certainly true for r sufficiently large.) We therefore examine the problem of linear independence. On eliminating l in (4.2) we obtain sequences to be denoted by $X_j^{(r)}$:

$$X_j^{(r)}: \quad x_{j,n}^{(r)} = \frac{1}{2\pi i}\int_C \frac{t^{n+k-j-1}(t^k - \alpha^k)^r}{\Delta_k(t)} \cdot \Delta_{k;0,j}(t)dt; \tag{4.9}$$

and these are the sequences that have been shown to satisfy[9] (4.1).

If we replace C by its equivalent, namely the sum of the contours $C_0, \cdots, C_{k-1}$, and on C_p set $t=\omega_k^p u$, then (4.9) is seen to become

$$x_{j,n}^{(r)} = \frac{1}{2\pi i}\int_{C_0} \frac{(u^k - \alpha^k)^r}{\Delta_k(u)} \cdot u^{n+k-j-1} \cdot \Delta_{k;j,n}(u)du, \tag{4.10}$$

where

$$\Delta_{k;j,n}(u) = \sum_{p=0}^{k-1} \omega_k^{pn} \Delta_{k;p,j}(u). \tag{4.11}$$

It is clear that

$$\Delta_{k;j,n+sk}(u) = \Delta_{k;j,n}(u). \tag{4.12}$$

Also, by use of (2.9), one obtains the relation $\Delta_{k;p,j}(\omega_k u) = \omega_k^j \Delta_{k;p+1,j}(u)$. Applying this to (4.11) one readily shows that

$$\Delta_{k;j,n}(\omega_k u) = \omega_k^{j-n} \Delta_{k;j,n}(u). \tag{4.13}$$

From (4.13) it follows that

$$(\omega_k u)^{n+k-j-1} \Delta_{k;j,n}(\omega_k u) = \frac{1}{\omega_k} \left[u^{n+k-j-1} \Delta_{k;j,n}(u)\right], \tag{4.14}$$

so that the bracket possesses a power series of the following type:

$$u^{n+k-j-1} \Delta_{k;j,n}(u) = \sum_{s=1}^{\infty} b_{j,n;s-1} u^{sk-1}; \tag{4.15}$$

---

[9] They are also of type not exceeding q.

and from (4.12),

$$u^{n+pk-j-1}\Lambda_{k;j,n+pk}(u) = u^{pk}\sum_{s=1}^{\infty} b_{j,n;s-1}u^{sk-1}. \tag{4.16}$$

Suppose α is a zero of multiplicity h for $\Delta_k(t)$. Then this is also the multiplicity of each of the zeros in the nest $\{\omega_k^s\alpha\}$, so that

$$\Delta_k(t) = (t^k - \alpha^k)^h \cdot Q_k(t) \tag{4.17}$$

where $Q_k(t)$ is analytic in $|t| \leqq q+\epsilon$ ($\epsilon > 0$ sufficiently small) and has the properties $Q_k(\omega_k^s\alpha) \neq 0$ and $Q_k(\omega_k t) = Q_k(t)$.

Let the circles $C_0, \cdots, C_{k-1}$ be chosen so small that as u traces C_0, then $z = u^k$ traces exactly once a rectifiable simple closed curve T_0 that surrounds $z = \alpha^k$ and has $z = 0$ in its exterior. We have

$$\Delta_k(u) = (z - \alpha^k)^h \cdot q_k(z)$$

where $q_k(z)$ is analytic in and on T_0 and nowhere zero there. Consequently

$$x_{j,n}^{(r)} = \frac{1}{2k\pi i}\int_{T_0} (z - \alpha^k)^{r-h} \cdot D_{k;j,n}(z)dz, \tag{4.18}$$

where

$$D_{k;j,n}(z) = \frac{1}{q_k(z)} \cdot \sum_{s=0}^{\infty} b_{j,n;s}z^s = \sum_{s=0}^{\infty} d_{j,n;s}(z - \alpha^k)^s. \tag{4.19}$$

If a j and a p exist in the range $0, 1, \cdots, k-1$ such that

$$D_{k;j,p}(\alpha^k) \neq 0, \tag{4.20}$$

then the h sequences (4.9), for $r = 0, 1, \cdots, h-1$, constitute a linearly independent set[10] of solutions of (4.1). To see this, we first observe that from (4.15) and (4.16) follows the relation $D_{k;j,m+sk}(z) = z^s D_{k;j,m}(z)$, so that

$$x_{j,m+sk}^{(r)} = \frac{1}{2k\pi i}\int_{T_0} (z - \alpha^k)^{r-h} \cdot z^s D_{k;j,m}(z)dz. \tag{4.21}$$

Hence if $X_j^{(0)}, \cdots, X_j^{(h-1)}$ are linearly dependent, then constants λ_r exist, not all zero, such that

$$\{0\} = \sum_{r=0}^{h-1} \lambda_r X_j^{(r)};$$

---

[10] If $r \geqq h$, (4.18) shows that $\{x_{j,n}^{(r)}\} = \{0\}$.

so, setting $n=m+sk$,

$$(4.22)\qquad 0=\sum_{r=0}^{k-1}\lambda_r x_{j,n}^{(r)}=\frac{1}{2k\pi i}\int_{T_0} z^s D_{k;j,m}(z)\left\{\sum_{r=0}^{k-1}\lambda_r(z-\alpha^k)^{r-k}\right\}dz.$$

This is an identity for $m=0, 1, \cdots, k-1$ and for all $s=0, 1, \cdots$. Choose $s=0, 1, \cdots, h-1$ and multiply the sth relation by μ_s where μ_s is chosen so that

$$\sum_{s=0}^{k-1}\mu_s z^s=(z-\alpha^k)^{k-1}.$$

Also take $m=p$. Then

$$0=\lambda_0\int_{T_0} D_{k;j,p}(z)(z-\alpha^k)^{-1}dz=\lambda_0 D_{k;j,p}(\alpha^k);$$

and by assumption (4.20), we must therefore have $\lambda_0=0$. In the same way it is shown that $\lambda_1=\cdots=\lambda_{k-1}=0$. This contradiction establishes the linear independence of $X_j^{(0)}, \cdots, X_j^{(k-1)}$, as was asserted.

From (4.15) and (4.19) we see that condition (4.20) is equivalent to this: p and j exist in $0, 1, \cdots, k-1$ such that[11]

$$(4.23)\qquad \Lambda_{k;j,p}(\omega_k^s\alpha)\neq 0,\qquad s=0, 1, \cdots, k-1.$$

Now suppose that (4.20), and hence (4.23), does not hold. Then for *every* j, p, s in $0, 1, \cdots, k-1$ we shall have

$$\text{(a)}\qquad \Lambda_{k;j,p}(\omega_k^s\alpha)=0.$$

For each fixed j and s, take $u=\omega_k^s\alpha$ in (4.11). We obtain a system of linear homogeneous equations with a nonvanishing Vandermond determinant, so the "unknowns" must all be zero. Thus,

$$\text{(b)}\qquad \Delta_{k;p,j}(\omega_k^s\alpha)=0\quad (p, j, s=0, 1, \cdots, k-1).$$

Conversely, (a) follows from (b).

If we agree to say that system (4.1) is *nonsingular at the nest of zeros* $\{\omega_k^s\alpha\}$ if there exists a triple of integers (p, j, s) in the range $0, 1, \cdots, k-1$ such that

$$(4.24)\qquad \Delta_{k;p,j}(\omega_k^s\alpha)\neq 0,$$

then we have established the following result:

---

[11] If (4.23) holds for *one* value of s it will then hold for all s, as we see from (4.12).

THEOREM 4.2. *In the k-periodic homogeneous system* (4.1) *let the functions* $A_n(t)$ *be analytic in* $|t| \leqq q$, *with* $A_n(0) \neq 0$. *If* (4.1) *is nonsingular at the nest of zeros* $\{\omega_k^s \alpha\}$, $s = 0, 1, \cdots, k-1$, *then the h sequences* $X_j^{(0)}, \cdots, X_j^{(h-1)}$ *of* (4.9) *are linearly independent solutions of system* (4.1) *and are of type not exceeding q.*

REMARKS. (i) Let the order of $\Delta_k(t)$ in $|t| \leqq q$ be l. If system (4.1) is nonsingular at every nest of zeros, then Theorem 4.2 provides us with sets of solutions, the total number of such solutions being exactly l, and the solutions in each set are linearly independent. It can be shown (although we shall not do so here) that the totality of these l solutions are linearly independent, and that they therefore constitute a complete set of solutions of type not exceeding q, of (4.1).

(ii) If system (4.1) is not nonsingular at some nest of zeros $\{\omega_k^s \alpha\}$, then (4.9) will not yield h linearly independent solutions, but may yield some. There *are* h linearly independent solutions corresponding to this nest, but their representation is not so simple as (4.9), and we do not discuss them here.

(iii) One can proceed from (4.9) by a limiting process, and arrive at a result analogous to Theorem 1.1 for the homogeneous system corresponding to (1.1). However, one cannot be certain that the solution so obtained is not actually $x_n \equiv 0$.

THE PENNSYLVANIA STATE COLLEGE

APPROXIMATION BY CURVES OF A UNISOLVENT FAMILY

TH. MOTZKIN

The aim of this paper is to extend theorems on the best approximation of a given function by a polynomial of a given degree, or more generally by a linear combination of given functions, first to the case where the approximating function is to be taken from a more general family of functions, which satisfies the requirements of continuity, solvence and unisolvence, to be specified below, and secondly to a class of related geometrical problems, for which that of approximating a curve, or six given points, by an ellipse may serve as an example. The best approximation is said to be furnished by that curve for which the maximum distance between corresponding points of the approximating and the approximated curves is as small as possible.[1]

Let (S) be a family of curves S represented by one-valued functions $y=S(x)$, $-1\leqq x\leqq 1$. The family (S) is assumed to be n-parametric, solvent, unisolvent, and continuous; more explicitly we assume:

1. *Solvence*: for any n values $x_1, \cdots, x_n$ with $-1\leqq x_1< \cdots <x_n\leqq 1$ and arbitrary real numbers $y_1, \cdots, y_n$ there exists a function S of (S) with $S(x_i)=y_i$, $i=1, \cdots, n$;

2. *Unisolvence*: only one such function exists, in the extended sense that not only, for any two different functions S_0 and S_1 of (S), S_0-S_1 has less than n roots (zeros), but also that this is true if any root x with $|x|<1$ for which S_0-S_1 does not change sign between $x-\epsilon$ and $x+\epsilon$ is counted as two roots;

3. *Continuity*: $S(x)=S(x; x_1, \cdots, x_n; y_1, \cdots, y_n)$ is a continuous function of $x, y_1, \cdots, y_n$.

It follows that there cannot exist $n+1$ values $-1\leqq x_0< \cdots <x_n \leqq 1$ for which S_0-S_1 has "alternating signs," that is, is alternatingly non-negative and non-positive.

For any curve S, $\sigma=\sigma(S)$ shall denote the supremum of the values

---

Received by the editors May 21, 1948.

[1] For the approximation by linear systems of functions see S. Bernstein, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle*, Paris, 1926, Chap. 1. The approximation of systems of points in space and to some extent also of functions of several variables by linear functions and other polynomials is considered by P. Kirchberger, *Ueber Tchebychefsche Annaeherungsmethoden*, Math. Ann. vol. 57 (1903) pp. 509–540. It would be desirable to extend the method of the present note to the case of several independent variables.

According to Kirchberger loc. cit. p. 510, the approximation with least maximal deviation was first considered by Poncelet, and more systematically by Chebyshev.

$|S(x)|$; because of the continuity of S in the closed interval $-1 \leqq x \leqq 1$ this supremum is actually attained. We define furthermore the *oscillation number* $v = v(S) \geqq 0$ as one less than the maximum number of values $x_0 < \cdots < x_v$ such that $|S(x_0)| = \sigma$, $S(x_i) = -S(x_{i-1})$, $i = 1, \cdots, v$. If the number of such values is not bounded then $v(S) = \infty$.

A *nearest* curve S_0 of a family (S) to $y = 0$ will be a curve for which $\sigma_0 = \sigma(S_0)$ is as small as possible. The existence, uniqueness and properties of S_0 are given in the following three theorems.

THEOREM I. *Under the above assumptions on* (S) *there exists a curve* $y = S_0(x)$, *and* $v_0 = v(S_0) \geqq n$.

THEOREM II. *For every other curve* S_1 *of* (S) *there are no values* $-1 \leqq x_0 < \cdots < x_n \leqq 1$ *with* $|S_1(x_i)| \geqq \sigma_0$, $i = 0, \cdots, n$, *and alternating signs of* $S_1(x_i)$.[3] *In particular* $v(S_1) < n$, *so that* S_0 *is unique and characterized by* $v(S_0) \geqq n$.

THEOREM III. *Also when* x *is restricted to* $n+1$ *values* $-1 \leqq x_0 < \cdots < x_n \leqq 1$ *with* $|S_0(x_i)| = \sigma_0$, $i = 0, \cdots, n$, *and alternating signs of* $S_0(x_i)$, S_0 *is the nearest curve to* $y = 0$. *Moreover* S_0 *is the only curve of* (S) *with equal and alternating ordinates at the abscissae* x_i.

Theorems II and III follow immediately by remarking that if a curve $S_1 \neq S_0$ of (S) were situated in contradiction to either of them, $S_1 - S_0$ would have alternating signs for $x_0, \cdots, x_n$.

To prove Theorem I we observe first that since $\sigma(S)$ is a non-negative and continuous function $\sigma(y_1, \cdots, y_n)$ of the values of S for any n fixed numbers $-1 \leqq x_1 < \cdots < x_n \leqq 1$, and since the functions S with $\sigma \leqq N$ are bounded and form a closed family, the infimum $\sigma_0 \geqq 0$ of σ is in fact attained for a certain curve S_0.

Now if $v_0 < n$ (and thus $\sigma_0 > 0$), then let Ξ be the closed set of all roots of $|S_0(x)| = \sigma_0$, and choose $n-1$ values ξ_i, $i = 1, \cdots, n-1$, such that:

1. Each of the v_0 open intervals between consecutive numbers of Ξ that give alternating signs to S_0 contains one, or an odd number, of the values ξ_i;

2. Each of the other open intervals between consecutive numbers of Ξ (there may be 0 or a finite or infinite number of such intervals) contains none, or an even number, of the values ξ_i;

3. A half-open interval between -1 (inclusive) and the smallest number of Ξ, or 1 (inclusive) and the greatest number of Ξ, contains none, or any number, of the values ξ_i;

---

[3] This contains as a special case a result of de la Vallée-Poussin and its generalization by Bernstein loc. cit. p. 6.

4. Finally $\xi_{n-1}=1$, if

(*) -1 and 1 belong to Ξ and $n-1-v_0$ is odd.

Such a choice is obviously possible.

We now consider the curves S' of (S) for which $S'(\xi_i)=S_0(\xi_i)$, $i=1, \cdots, n-1$; because of the unisolvence assumed they meet only for $x=\xi_i$, and cross each other (that is, their difference changes sign) there. Due to the solvence of (S), S' may be chosen so that $S'(\xi)=S_0(\xi)(1-\epsilon)$ for a given value ξ of Ξ (different from 1 if (*) holds). For small $\epsilon>0$, S' will be near S_0 everywhere, because of the continuity of (S), and owing to the choice of the ξ_i we have $|S'|<\sigma_0$ for every value of Ξ (different from 1 if (*) holds) and therefore everywhere. Hence $\sigma(S')<\sigma_0$, against the definition of σ_0 (in case (*) holds we obtain $|S'(-1)|<\sigma_0$ and apply the same procedure to S' instead of S_0).[3] Thus the oscillation number v_0 must be at least n. Q.E.D.

The approximation of a given function $f(x)$, continuous for $-1\leqq x\leqq 1$, by a function of the family (S) has analogous properties; to see this replace every function S by $S-f$ (which does not affect the conditions imposed on the family) and apply Theorems I, II and III.

More generally the strip $|x|\leqq 1$, $|y|<\infty$ may undergo a topological mapping, the distance between the images of (x, y) and (x, y') being defined as $|y'-y|$.

If all functions S have the period 2, then the lines $x=-1$ and $x=1$ may be identified, and we obtain an approximation of a closed curve on a cylinder or within an annulus.

The extension of the theorems to approximations on the whole axis of real numbers, or in an open interval, requires additional assumptions.

The proof of Theorems I–III remains the same if (every $S(x)$ being still defined for $-1\leqq x\leqq 1$) $\sigma(S)$ denotes the supremum of $|S(x)|$ for values x belonging to a given closed partial set X of $-1\leqq x\leqq 1$ that contains at least $n+1$ numbers.

In the case of a finite set $X=(x_0, \cdots, x_m)$ the nearest curve can be found by determining, for every $n+1$ numbers of X, the one curve whose ordinates at these abscissae are equal but alternating in sign, and choosing the best one from among the finite number of curves obtained.

The related geometrical problem of finding, from a unisolvent

---

[3] In this case also, a single application of the procedure is sufficient: fix only $\xi_1, \cdots, \xi_{n-2}$ and put $S'(\xi)=S_0(\xi)(1-\epsilon)$ for $\xi=-1$ and $\xi=1$. There cannot be $n-1$ common points of S_0 and S' within $-1\leqq x\leqq 1$, since $n-1-v_0$ is odd.

n-parametric family (S) of plane curves, the nearest curve S_0 to a finite number of given points $P_0, \cdots, P_m$, $m \geqq n$, can be treated similarly. The family (S) is supposed to contain, within a certain domain of the real plane, through every n points one and only one curve, depending continuously on the situation of the n points. The nearest curve S_0 (that one for which the greatest distance $\sigma(S)$ from it of one of the points P_i is as small as possible), is again to be chosen from those curves for which $n+1$ from among the points P_i have equal distances from the curve. Indeed, if the number of approximated points P_k with maximal distance from S_0 were less than $n+1$, a small change of the curve could be effected (because of the solvence and continuity of the family) diminishing the maximal distance.

Owing to the unisolvence we can also prove by the same method of fixing $n-1$ points of the approximating curve as in the proof of Theorem I that $(n+1)/2(\pm 1/2)$ of the points P_k are on either side of S_0, with alternating nearest points Q_k on S_0, provided that there are exactly $n+1$ points P_k and that to every P_k there exists only one nearest point Q_k. In the general case it is impossible to divide S_0 by $n-1$ points into closed arcs, alternatingly called positive and negative, so that every point P_k on one side of S_0 has one of its nearest points Q_k on a positive arc, and every point P_k on the other side of S_0 has one of its nearest points Q_k on a negative arc. The value of $2\sigma(S_0)$ may be called the *breadth* of the given point set (P_i) with regard to the unisolvent family (S).[4]

As examples of unisolvent families we mention the family of all straight lines, the family of all conics, the family of all circles or more generally of all curves (positively) homothetic to a given, closed or infinite, convex curve that contains no straight segment. Each family has to be closed by including its limiting curves: points, straight lines and pairs of straight lines. Evidently a point cannot be a nearest curve, but a pair of straight lines can, and this possibility must be separately taken care of when determining the nearest conic to a point set (P_i). A straight line as S_0 behaves like a general member of the family.

It is easily seen that for a family of closed and bounded curves n will be odd.

The family of all parabolas is neither solvent nor unisolvent. Still since, in a bounded domain, parabolas that are sufficiently near to each other intersect only in $3=n-1$ points of the domain ("local unisolvence"), and since four points through which a parabola exists do not lose this property after a small change of position ("local solvence") the former results remain valid. However not only pairs of parallel straight lines but also single straight lines must be separately considered, because the local solvence fails for groups of four collinear points.

---

[4] For other properties of unisolvent families cf. Th. Motzkin, *Sur les arcs dont les courbes osculatrices ne se coupent pas*, C. R. Acad. Sci. Paris vol. 206 (1938) pp. 1700–1701.

There are, of course, families without local solvence or unisolvence, for example the family of all cubics, or the family of all circles of given radius. The nearest circle of given radius to the vertices of an equilateral triangle has the same centre as the triangle, and the approximated points are thus all on the same side of the circle. The nearest circle of given radius to the end points and mid point of a large segment has also the same centre as the segment, and there are only two farthest points though $n=2$.

In the geometrical problems, Theorems II and III, and in particular the uniqueness of the nearest curve, do not subsist. Thus the vertices of an equilateral m-gon and its centre have $2m/(3+(-1)^m)$ nearest straight lines, and m nearest circles. It seems however that in the vicinity of $m+1$ given points there are always $m+1$ points with a unique nearest curve.

When approximating an infinite (closed and bounded) set some of the $n+1$ extremal points P_k may coincide. Thus the nearest straight line to an ellipse is its major axis, with 2, instead of 3, farthest points. There may also be an infinite number of nearest curves: to a circle, every diameter is a nearest straight line, and similarly for any curve of constant breadth; to a circle and its centre there are even doubly infinitely many nearest circles, most of them with 2, instead of 4, farthest points.

Similar considerations apply to unisolvent families of curves on a surface. The generalization to different definitions of the distance of a point from a curve appears to be more difficult.

THE HEBREW UNIVERSITY

APPROXIMATION IN LIP (α, p)

PAUL CIVIN

Let L_p, $1<p<\infty$, denote the class of measurable functions of period 2π for which $(\int_{-\pi}^{\pi}|f(x)|^p dx)^{1/p}=M_p(f)<\infty$, and let Lip (α, p), $0<\alpha<\infty$, represent that subclass of L_p for which $(\int_{-\pi}^{\pi}|f(x+h)-f(x)|^p dx)^{1/p}=O(h^{-\alpha})$ as $h\to 0$. The object of the present note is to demonstrate the following theorem.

THEOREM. *If $f(x)\in$ Lip (α, p) and $\{P_n(x)\}$ is a sequence of trigonometric polynomials of order n such that*

$$M_p(f-P_n)\leqq Kn^{-\alpha}, \tag{1}$$

then

$$\left(\int_{-\pi}^{\pi}|P_n'(x)|^p dx\right)^{1/p}\leqq\begin{cases}A(1-\alpha)^{-1}n^{1-\alpha}, & 0<\alpha<1,\\ A\log n, & \alpha=1,\\ A(\alpha-1)^{-1}, & 1<\alpha<\infty\end{cases} \tag{2}$$

where in each case A depends only on α and the sequence $P_n(x)$ but not on n.

The method is that of M. Zamansky[1] [2] who obtained the corresponding results for functions in Lip α, $0<\alpha\leqq 1$.

An application of the inequality of Zygmund [3] concerning the pth mean of the derivative of a trigonometric polynomial together with the Minkowski inequality shows that if (1) and (2) are satisfied by a sequence $\{P_{n_j}\}$ with $(n_{j+1}/n_j)=O(1)$ and if $\{\lambda_n\}$ is any sequence of trigonometric polynomials of order n such that $M_p(\lambda_n)=O(n^{-\alpha})$, then the sequence $\{P_{n_j}+\lambda_n\}$ $(n=n_j, n_j+1, \cdots, n_{j+1}-1; j=1, 2, \cdots)$ also satisfies (1) and (2). A further application of the same inequalities shows that if $\{P_n\}$ satisfies (1) and (2) and if $\{Q_n\}$ satisfies (1), then $\{Q_n\}$ also satisfies (2). The proof of the theorem is thus reduced to the exhibition of a sequence $\{P_{n_j}\}$ of trigonometric polynomials of order n_j with $(n_{j+1}/n_j)=O(1)$ such that (1) and (2) hold for $\{P_{n_j}\}$.

Let r be the smallest integer greater than $(1+\alpha)/2$ and $q=p/(p-1)$. If $f(x)\in L_p$ and

$$u(r)=\int_{-\infty}^{\infty}(\sin t/t)^{2r}dt$$

Presented to the Society, November 27, 1948; received by the editors June 21, 1948.

[1] Numbers in brackets refer to the references at the end of the note.

then

$$F_j(x) = (u(r))^{-1}\int_{-\infty}^{\infty} f(x + 2^{1-j}t)(\sin t/t)^{2r}dt$$

is a trigonometric polynomial[2] [1] of order less than $r2^j$. Let $P_{n_j}(x) = F_j(x) + n_j^{-\alpha}\cos n_jx$ for $n_j = r2^j$, so that $P_{n_j}(x)$ is a trigonometric polynomial of order[3] n_j and $(n_{j+1}/n_j) = O(1)$.

In view of the definition of r, it is possible to select β so that $\beta q > 1$ and $p(2r-\beta-\alpha) > 1$. Therefore

$$\begin{aligned}
M_p(f - P_{n_j}) \\
&\leqq A_1\left(\int_{-\pi}^{\pi}\left(\int_{-\infty}^{\infty} | f(x) - f(x + 2^{1-j}t) | \, | \sin t/t |^{2r}dt\right)^p dx\right)^{1/p} \\
&\quad + Bn_j^{-\alpha} \\
&\leqq A_1\left(\int_{-\pi}^{\pi}\left(\int_{-\infty}^{\infty} | \sin t/t |^{pq}dt\right)^{p/q}\right. \\
&\qquad \left.\cdot \int_{-\infty}^{\infty} | f(x) - f(x + 2^{1-j}t) |^p | \sin t/t |^{(2r-\beta)p}dtdx\right)^{1/p} \\
&\quad + Bn_j^{-\alpha} \\
&\leqq A_2\left(\int_{-\infty}^{\infty} | \sin t/t |^{(2r-\beta)p}\int_{-\pi}^{\pi} | f(x) - f(x + 2^{1-j}t) |^p dxdt\right)^{1/p} \\
&\quad + Bn_j^{-\alpha} \\
&\leqq A_3\left(\int_{-\infty}^{\infty} | \sin t/t |^{(2r-\beta)p} | 2^{1-j}t |^{\alpha p}dt\right)^{1/p} + Bn_j^{-\alpha} \\
&\leqq A_4 n_j^{-\alpha},
\end{aligned}$$

with the various A_k and B independent of the n_j.

By a technique identical with that in the preceding paragraph, it is easily seen that

$$M_p(P_{n_j} - P_{n_{j-1}}) \leqq An_j^{-\alpha}.$$

Another application of the Zygmund inequality shows that

---

[2] The proof given in [1] is for continuous $f(x)$. However, the proof is identical for $f(x) \in L_p$.

[3] A term of order n_j is also required for the completeness of the Zamansky proof [2] for $f(x) \in \text{Lip}\ \alpha$. As that proof was given, the difference in the actual and apparent order of the approximating polynomials causes the proof to fail in the case of functions $f(x)$ with "large" gaps in their Fourier series.

$$M_p(P'_{n_j} - P'_{n_{j-1}}) \leqq A n_j^{1-\alpha}.$$

The application of the Minkowski inequality to the sum

$$P'_{n_k}(x) = \sum_{j=1}^{k-1} (P'_{n_{j+1}}(x) - P'_{n_j}(x)) + P'_{n_1}(x),$$

followed by summation over j, completes the proof of the theorem.

REFERENCES

1. C. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, Paris, 1919.

2. M. Zamansky, *Sur l'approximation des fonctions continues*, C. R. Acad. Sci. Paris vol. 224 (1947) pp. 704–706.

3. A. Zygmund, *A remark on conjugate series*, Proc. London Math. Soc. vol. 3 (1932) pp. 392–400.

UNIVERSITY OF OREGON

ON DESCARTES-HARRIOT'S RULE[1]

PAUL TURÁN

The classical rule of Descartes-Harriot asserts that if

$$f(x) = a_0 + a_1x + \cdots + a_nx^n \tag{1}$$

is a polynomial with real coefficients, then the number P of its positive zeros cannot exceed the number V_0 of the sign changes of the sequence

$$a_0, a_1, \cdots, a_n \tag{2}$$

(if some of the coefficients vanish they can be omitted). This rule has been extended and refined in various ways. V_0 yields obviously an upper bound for the number P^* of zeros lying in $0<x<1$; then Laguerre[2] observed the P^* is majorised also by the number V_1 of sign changes of the sequence

$$a_0, (a_0 + a_1), \cdots, (a_0 + a_1 + \cdots + a_n), (a_0 + \cdots + a_n), \cdots \tag{3}$$

which is not greater than V_0. More generally P^* is majorised by V_k, where V_k denotes the number of sign changes of that sequence which arises from (2), completing it with zeros to an infinite sequence and forming the sequence of k times iterated partial sums; these V_k's form a nonincreasing sequence. One trend of the investigations is the study of the V_k for large k; this was done mainly by M. Fekete and G. Pólya.[3] Another trend is to obtain similar rules representing $f(x)$ in various other forms (Runge,[4] Sylvester,[5] Obreschkoff,[6] I. J. Schoenberg[7]) or even generally for the linear combinations

$$h(x) = a_0\phi_0(x) + \cdots + a_n\phi_n(x)$$

Received by the editors June 22, 1948.

[1] To the memory of my late friend, Ervin Feldheim.

[2] E. Laguerre, *Mémoire sur la théorie des équations numériques*, J. Math. Pures Appl. (3) vol. 9 (1883) pp. 99–146.

[3] See in particular their joint paper in Rend. Circ. Mat. Palermo vol. 34 (1912) pp. 89–120 entitled *Über ein Problem von Laguerre.*

[4] See the paper of G. Pólya, *Über einige Verallgemeinerungen der Descartesschen Zeichenregel*, Archiv der Mathematik und Physik vol. 23 (1915) pp. 22–32.

[5] J. J. Sylvester, *Mathematical papers*, vol. 2, pp. 360 and 401.

[6] N. Obreschkoff, *Über die Wurzeln algebraischer Gleichungen*, J. Deutschen Math. Verein vol. 33 (1924) pp. 52–64.

[7] I. J. Schoenberg, *Zur Abzählung der reellen Wurzeln algebraischen Gleichungen*, Math. Zeit. vol. 38 (1934) pp. 546–564.

of a prescribed system $\phi_0(x), \phi_1(x), \cdots$ (Pólya-Szegö,[8] S. Bernstein[9]). A further trend is the comparison of the different rules (F. Klein,[10] I. J. Schoenberg[7]), extensions permitting complex coefficients in (1) and substituting the segments of the positive axis by more general domains (Obreschkoff,[11] I. J. Schoenberg[12]), and finally the improving the rule by suitable Tschirnhausen-transformation (S. Lipka[13]). But as far as we know no similar rule has been given to a *lower* estimation of the number P of the positive roots. The aim of this note is to give such a simple rule. This is based on the Laguerre-polynomial-representation of $f(x)$:

$$f(x) = \sum_{\nu=0}^{n} b_\nu L_\nu(x), \qquad b_\nu \text{ real,} \tag{4}$$

where the Laguerre-polynomials $L_\nu(x)$ are defined as is well known by

$$e^{-x} L_\nu(x) = \frac{1}{\nu!} \frac{d^\nu}{dx^\nu} (e^{-x} x^\nu). \tag{5}$$

The rule we are going to prove will assert that *the number P of positive roots of $f(x)$ is not less than the number of sign changes in the sequence of successive differences of the coefficients b_ν, that is, in the sequence*

$$\begin{gathered} b_0, \ [b_0 - b_1], \ [b_0 - 2b_1 + b_2], \cdots, \\ [b_0 - C_{n,1} b_1 + C_{n,2} b_2 - \cdots + (-1)^n C_{n,n} b_n]. \end{gathered} \tag{6}$$

It is perhaps not uninteresting to notice that if η_n denotes the greatest positive root of $L_n(x)$, then[6] the number of sign changes of the coefficient-sequence

$$b_0, b_1, \cdots, b_n$$

itself furnishes an *upper* bound for the number of the roots of $f(x)$ in the interval $\eta_n \leqq x < +\infty$.

---

[8] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, chap. 5, probleme 87–90.

[9] S. Bernstein, *Leçons sur les propriétés extremales et la meilleure approximation des fonctions analytiques d'une variable reelle*, Paris, 1926.

[10] F. Klein, *Geometrisches zur Abzählung der reellen Wurzeln algebraischer Gleichungen*, Gesammelte Mathematische Abhandlungen, pp. 198–208.

[11] N. Obreschkoff, *Sur un problème de Laguerre*, C. R. Acad. Sci. Paris vol. 177 (1923) pp. 102–104.

[12] I. J. Schoenberg, *Extensions of theorems of Descartes and Laguerre to the complex domain*, Duke Math. J. vol. 2 (1936) pp. 84–94.

[13] S. Lipka, *Über die Descartessche Zeichenregel*, Acta Univ. Szeged. vol. 7 (1934–1935) pp. 177–185.

The proof of the rule (6) is extremely simple and actually gives more since it turns out, as a matter of fact, that even the number of the positive sign changing places of $f(x)$ is not less than the number of sign changes of the sequence (6). It is based on the following theorem of Fejér.[14]

The number of the sign variations of a real function $g(x)$ in the interval $0 < x < a$ ($a \leqq +\infty$) is not less than the number of sign changes of the sequence of moments:

$$M_0 = \int_0^a g(t)dt, \cdots, M_n = \int_0^a g(t)t^n dt. \tag{7}$$

In order to prove our rule we form instead of the moments of $f(x)$ those of $f(x)e^{-x}$

$$M_k = \int_0^\infty f(x)e^{-x}x^k dx, \qquad k = 0, 1, \cdots, n. \tag{8}$$

Owing to the expansion[15]

$$x^k = k! \sum_{\nu=0}^{k} (-1)^\nu C_{k,\nu} L_\nu(x) \tag{9}$$

and the well known orthogonality property[16]

$$\int_0^\infty e^{-x} L_\mu(x) L_\nu(x) dx = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 1 & \text{if } \mu = \nu, \end{cases}$$

we obtain from (9)

$$M_k = k! \sum_{\nu=0}^{k} (-1)^\nu C_{k,\nu} b_\nu, \qquad k = 0, 1, \cdots, n. \tag{10}$$

---

[14] L. Fejér, *Nombre des changement de signe d'une fonction dans un intervalle et ses moments*, C. R. Acad. Sci. Paris vol. 158 (1914) pp. 1328–1331. If some of the moments are 0, then we can of course simply drop them; we can get an improved lower estimation in the following way. Let, for example, $M_\mu \neq 0$. Let $c_\mu = M_\mu$; further for $\nu < \mu$, $c_\nu = M_\nu$ if $M_\nu \neq 0$ and $c_\nu = -\text{sg}\, c_{\nu+1}$ if $M_\nu = 0$; finally for $\nu > \mu$, $c_\nu = M_\nu$ if $M_\nu \neq 0$ and $c_\nu = -\text{sg}\, c_{\nu-1}$ if $M_\nu = 0$. Then the number of sign changes of $g(x)$ in $0 < x < a$ is not less than even the number of sign changes in the sequence $c_0, c_1, \cdots, c_n$. See Pólya-Szegö, loc. cit. footnote 8, vol. 2, p. 50.

[15] I don't know to whom this identity is due. I have seen it stated and proved in the paper of E. Feldheim, *Contributions à la théorie des polynômes de Jacobi* (Hungarian with French summary), Matematikai és Physikai Lapok vol. 48 (1941) pp. 453–504.

[16] See, for example, G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939, p. 96.

This together with the quoted theorem of Fejér shows that the number of positive sign changing places of $f(x)e^{-x}$ (or of $f(x)$) is not less than the number of sign changes in the sequence (10) or, what amounts to the same, in the sequence (6).

Of course a similar reasoning yields similar rules for other polynomial expansions but the rule given for Laguerre expansion seems to be particularly simple and to a certain extent dual to the rule of Laguerre (3). The question of constructing such simple rules leads to interesting questions in the theory of orthogonal polynomials; to these I shall return elsewhere.

INSTITUTE FOR ADVANCED STUDY

A BOUND FOR THE MEAN VALUE OF A FUNCTION

J. ERNEST WILKINS, JR.

Let $f(t)$ be a bounded measurable function defined when $0 \leqq t \leqq \pi$. The Fourier sine series associated with $f(t)$ is

$$\sum_{n=1}^{\infty} b_n \sin nt, \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(t) \sin nt \, dt.$$

We shall be interested in this paper in establishing a bound for the mean value[1]

$$a = \frac{1}{\pi}\int_0^{\pi} f(t) dt$$

when $f(t)$ is such that one of the coefficients b_n vanishes.

We can suppose without essential loss of generality that $|f(t)| \leqq 1$. Since $b_{2n} = 0$ whenever $f(t)$ is constant, it is clear that the only conclusion on a that can be drawn from the inequality $|f(t)| \leqq 1$ and the equality $b_{2n} = 0$ is that $|a| \leqq 1$, and this conclusion is valid whether b_{2n} vanishes or not. Hence we shall restrict attention to b_{2n+1}. For the same reason we shall not discuss the vanishing of the coefficient

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t) \cos nt \, dt$$

of the Fourier cosine series of $f(t)$.

Suppose that $b_{2n+1} = 0$. Define a positive number y by the equation $y = \sin [\sec^{-1}(2n+2)]$ where the $\sec^{-1}$ lies between 0 and $\pi/2$. Let E be the sum of the intervals

$$\frac{2p\pi + \sin^{-1} y}{2n+1} \leqq t \leqq \frac{(2p+1)\pi - \sin^{-1} y}{2n+1} \qquad (p = 0, 1, \cdots, n),$$

where the $\sin^{-1}$ lies between 0 and $\pi/2$. Then it is clear that

$$(1) \qquad \begin{aligned} &\sin (2n+1)t \geqq y && \text{if } t \text{ is in } E, \\ &\sin (2n+1)t < y && \text{if } t \text{ is not in } E. \end{aligned}$$

Received by the editors February 11, 1948, and, in revised form, June 14, 1948.

[1] The importance of the concept of mean value in the study of Fourier series can be seen by consulting Bohr [1, pp. 7–29]. Numbers in brackets refer to the bibliography at the end of the paper.

Now let $f_0(t)=-1$ if t is in E and $f_0(t)=+1$ if t is not in E. It follows from the definitions of y and E that

$$\int_0^\pi f_0(t)\sin(2n+1)t\,dt=0,$$

and that the mean value of $f_0(t)$ is

$$\begin{aligned} a_0 &= 1-(2/\pi)\text{ meas } E \\ &= [4(n+1)/(2n+1)\pi]\sec^{-1}(2n+2)-1/(2n+1). \end{aligned}$$

We shall now prove that if $f(t)$ is an arbitrary real-valued measurable function such that $|f(t)|\leqq 1$ and $b_{2n+1}=0$, then $|a|\leqq a_0$. Let $g(t)=f(t)-f_0(t)$. Then $0\leqq g(t)$ on E and $0\geqq g(t)$ on the complement cE of E. By virtue of relations (1) we conclude that

$$\int_E g(t)\sin(2n+1)t\,dt \geqq y\int_E g(t)dt,$$

$$\int_{cE} g(t)\sin(2n+1)t\,dt \geqq y\int_{cE} g(t)dt.$$

Adding and remembering that $b_{2n+1}=0$ for both f and f_0 we see that

$$0\geqq y\int_0^\pi g(t)dt,$$

with equality if and only if $g(t)=0$ almost everywhere. Since $y>0$, we have that

$$\int_0^\pi f(t)dt \leqq \int_0^\pi f_0(t)dt, \tag{2}$$

with equality if and only if $f(t)=f_0(t)$ almost everywhere.

Now let $h(t)=f(t)+f_0(t)$. Then $0\geqq h(t)$ on E, $0\leqq h(t)$ on cE, and so

$$\int_E h(t)\sin(2n+1)t\,dt \leqq y\int_E h(t)dt,$$

$$\int_{cE} h(t)\sin(2n+1)t\,dt \leqq y\int_{cE} h(t)dt,$$

$$0\leqq y\int_0^\pi h(t)dt,$$

$$-\int_0^\pi f(t)dt \leqq \int_0^\pi f_0(t)dt, \tag{3}$$

with equality if and only if $f(t)=-f_0(t)$ almost everywhere. Combining the inequalities (2) and (3) we conclude that *when* $f(t)$ *is a real-valued measurable function such that* $|f(t)|\leqq 1$ *and* $b_{2n+1}=0$, *then*

$$(4)\qquad |a|=\left|\frac{1}{\pi}\int_0^\pi f(t)dt\right|\leqq\frac{4(n+1)}{(2n+1)\pi}\sec^{-1}(2n+2)-\frac{1}{2n+1},$$

with equality if and only if $f(t)=\pm f_0(t)$ *almost everywhere.*

In particular, if $b_1=0$, then $|a|\leqq 1/3=.3333$, while if $b_3=0$, then $|a|\leqq .7855$. The right-hand side of the inequality (4) approaches unity as n approaches infinity.

This conclusion may be extended to complex functions $f(t)$ as follows. Let $f(t)=f_1(t)+if_2(t)$, where $f_1(t)$ and $f_2(t)$ are real. There exist real numbers x and y such that

$$x^2+y^2=1,\qquad x\int_0^\pi f_2(t)dt+y\int_0^\pi f_1(t)dt=0.$$

Hence it is true that the mean value of $f(t)$ has the same absolute value as the mean value of the real function $xf_1(t)-yf_2(t)$. This real function has a Fourier coefficient b_{2n+1} equal to zero since this is true for both $f_1(t)$ and $f_2(t)$ and is bounded by one since $f(t)$ is and $x^2+y^2=1$. Since the inequality (4) is valid for xf_1-yf_2, it is therefore true for $f(t)$. Moreover since equality for xf_1-yf_2 implies that $xf_1-yf_2=\pm f_0(t)$, equality for $f(t)$ implies that $f(t)=cf_0(t)$ where c is a constant of absolute value unity.

BIBLIOGRAPHY

1. H. Bohr, *Almost periodic functions*, New York, Chelsea, 1947.

AMERICAN OPTICAL COMPANY

A GENERALIZED CONVOLUTION FOR FINITE FOURIER TRANSFORMATIONS[1]

A. W. JACOBSON

1. **Introduction.** The finite sine transformation and the finite cosine transformation of $F(x)$ with respect to x are defined as follows:

$$S\{F(x)\} = \int_0^\pi F(x) \sin nx dx = f_s(n) \qquad (n = 1, 2, \cdots),$$

$$C\{F(x)\} = \int_0^\pi F(x) \cos nx dx = f_c(n) \qquad (n = 0, 1, 2, \cdots)$$

respectively. For example, the sine transforms of the first and second derivatives of $F(x)$ are $S\{F'(x)\} = -nC\{F(x)\}$ and $S\{F''(x)\} = -n^2S\{F(x)\} + n[F(0) - (-1)^nF(\pi)]$.

If $F(x)$ in $(-2\pi, 2\pi)$ and $G(x)$ in $(-\pi, \pi)$ are bounded and integrable, then the function

$$F(x)*G(x) = \int_{-\pi}^{\pi} F(x - y)G(y)dy \tag{1}$$

is called the convolution of F and G on the interval $(-\pi, \pi)$.

If $F(x)$ and $G(x)$ are bounded and integrable on the interval $0 \leqq x \leqq \pi$, and if $F_1(x)$ is an odd periodic extension of $F(x)$ and $G_1(x)$ an odd extension of $G(x)$, then the product of the sine transforms of $F(x)$ and $G(x)$ can be written in terms of the transform of the convolution as follows:

$$S\{F_1(x)\}S\{G_1(x)\} = - 2^{-1}C\{F_1(x)*G_1(x)\}. \tag{2}$$

See [1, p. 274][2] and [2, p. 270].

2. **The Fourier transformation of a generalized convolution.** The purpose of this paper is to generalize the above results and to illustrate the use of the generalized convolution. This generalization consists, primarily, of extending the concept of the convolution to any

---

Presented to the Society, April 16, 1948; received by the editors June 25, 1948.

[1] The author wishes to thank Professor R. V. Churchill for his advice in the preparation of this paper. The content of this paper is part of a dissertation submitted in partial fulfillment of the requirements for the degree of doctor of philosophy in the University of Michigan.

[2] The numbers in brackets refer to the bibliography.

integrable function of two variables. The aim here is to obtain a generalization similar to the one made by Bartels and Churchill [3] regarding the Laplace transformation. It will be seen in the next section that with the aid of this generalized convolution, very general steady state boundary value problems can be resolved into simpler problems.

Let $F(x, y)$ be a bounded and integrable function of x and y over the square $0 \leqq x \leqq \pi$, $0 \leqq y \leqq \pi$. A generalized Fourier convolution $F^*(x)$ of $F(x, y)$ corresponding to the iterated finite sine transformation

$$S\{S\{F(x, y)\}\} = \int_0^\pi \int_0^\pi F(x, y) \sin nx \sin n'y dx dy = \bar{f}(n, n') \qquad (n, n' = 1, 2, \cdots) \tag{3}$$

is defined by

$$F^*(x) = -\int_{-\pi}^{\pi} F_1(x - y, y) dy, \tag{4}$$

where $F_1(x, y)$ is an odd periodic extension of $F(x, y)$ with respect to x and an odd extension with respect to y.

In case $F(x, y) = F_1(x)$, $G_1(y)$, the function $F^*(x)$ is the convolution (1) of F_1 and G_1.

THEOREM. *If $F(x, y)$ is a bounded and integrable function of x and y over the square $0 \leqq x \leqq \pi$, $0 \leqq y \leqq \pi$ and if $n' = n$, then*

$$S\{S\{F(x, y)\}\} = 2^{-1}C\{F^*(x)\}.$$

PROOF. Over the square $0 \leqq \lambda \leqq \pi$, $0 \leqq \mu \leqq \pi$ the iterated integral (3) can be written as

$$2^{-1}\int_0^\pi \int_0^\pi F(\lambda, \mu) \cos n(\lambda - \mu) d\lambda d\mu - 2^{-1}\int_0^\pi \int_0^\pi F(\lambda, \mu) \cos n(\lambda + \mu) d\lambda d\mu. \tag{5}$$

In the first integral set $\lambda - \mu = x$ and $\mu = y$. Then the lines $\lambda = 0$ and $\lambda = \pi$ in the $\lambda\mu$-plane become the lines $x + y = 0$ and $x + y = \pi$ in the xy-plane. Let A denote the parallelogram bounded by the last two lines and the lines $\mu = 0$, $\mu = \pi$. Since the Jacobian for this change of variables is unity, the first of the integrals (5) becomes

$$\iint_A F(x + y, y) \cos nx dx dy,$$

or

$$\int_{-\pi}^{0}\cos nx dx\int_{-x}^{\pi}F(x+y,\,y)dy \tag{6}$$
$$+\int_{0}^{\pi}\cos nx dx\int_{0}^{\pi-x}F(x+y,\,y)dy.$$

If in the first integral in (6) x is replaced by $-x$, then the first integral in (5) can be written

$$2^{-1}\int_{0}^{\pi}\cos nx dx\left[\int_{0}^{\pi-x}F(x+y,\,y)dy+\int_{x}^{\pi}F(y-x,\,y)dy\right].$$

By setting $\lambda+\mu=x$ and $\mu=y$ in the second integral in (5), it can be shown to take the form

$$2^{-1}\int_{0}^{\pi}\cos nx dx\left[\int_{0}^{x}F(x-y,\,y)dy+\int_{\pi-x}^{\pi}F(2\pi-x-y,\,y)dy\right].$$

Hence for the iterated sine transformation (3) there results the equation

$$S\{S\{F(x,\,y)\}\} = 2^{-1}\int_{0}^{\pi}\cos nx dx\left[-\int_{0}^{x}F(x-y,\,y)dy\right.$$
$$+\int_{x}^{\pi}F(y-x,\,y)dy+\int_{0}^{\pi-x}F(x+y,\,y)dy \tag{7}$$
$$\left.-\int_{\pi-x}^{\pi}F(2\pi-x-y,\,y)dy\right].$$

If $F_1(x, y)$ is an odd periodic extension of $F(x, y)$ with respect to x with period 2π and an odd extension with respect to y, then the quantity in brackets reduces to

$$-\int_{-\pi}^{\pi}F_1(x-y,\,y)dy,$$

which according to definition (4) is the generalized convolution $F^*(x)$ of $F(x, y)$. So

$$S\{S\{F(x,\,y)\}\} = 2^{-1}C\{F^*(x)\}.$$

If, in (7), I_1, I_2, I_3, and I_4 denote the four integrals in brackets, it can be shown under the same conditions as in the above theorem that

$$S\{C\{F(x,\,y)\}\} = 2^{-1}S\{F^*(x)\},$$

where $F^*(x)=I_1-I_2-I_3+I_4$; and $C\{C\{F(x, y)\}\}=2^{-1}C\{F^*(x)\}$, where $F^*(x)=I_1+I_2+I_3+I_4$; and finally

$$C\{S\{F(x, y)\}\} = 2^{-1}S\{F^*(x)\},$$

where

$$F^*(x) = I_1 + I_2 - I_3 - I_4.$$

If $F(x, y)$ is also periodic with respect to y with period 2π, then

$$S\{C\{F(x, y)\}\} = C\{S\{F(x, y)\}\}.$$

3. **Resolution of steady state temperature problem.** With the aid of the generalized convolution a very general steady temperature problem can be resolved into a problem with simpler boundary conditions and source function. Furthermore, this method makes it possible to extend the Duhamel integral formula from time to space coordinates.

Consider the function $V(x, y_1, y_2)$. Let Λ and λ denote the differential operators defined by

$$\Lambda\{V\} = C_0V + \sum_{i=1}^{2} C_i \frac{\partial}{\partial y_i}\left(K_i \frac{\partial V}{\partial y_i}\right)$$

and

$$\lambda\{V\} = c_0V + \sum_{i=1}^{2} c_i \frac{\partial V}{\partial y_i},$$

where the coefficients of V and its derivatives are functions of y_1 and y_2 only. Let the region R be bounded by a cylinder whose elements are parallel to the x-axis and whose bases are $x=0$ and $x=\pi$. Let Q denote an arbitrary point on the lateral surface, and P_0 and P_1 arbitrary points on the bases. If the coefficients of the second partial derivatives of V with respect to y_1 and y_2 are positive, then the following boundary value problem represents a very general steady state temperature problem:

$$\text{(A)}\qquad \begin{aligned} &\frac{\partial^2 V}{\partial x^2}(x, y_1, y_2) + \Lambda\{V\} = F(x, y_1, y_2), && x > 0,\\ &\lambda\{V(x, Q)\} = G(x, Q), && x > 0,\\ &V(+0, y_1, y_2) = H_0(P_0), \qquad V(\pi - 0, y_1, y_2) = H_1(P_1), \end{aligned}$$

where F, G, H_0, H_1 are known functions. In the second of these equations it is understood that $\lambda\{V(x, Q)\}$ represents the limit of $\lambda\{V(x, P)\}$ as the interior point P approaches the point Q on the lateral surface in a prescribed manner.

Let the corresponding small letter denote the sine transform of the function with respect to x. Applying the sine transformation to problem (A) yields:

$$\begin{aligned} -n^2v(n, y_1, y_2) + n[H_0(P_0) - (-1)^nH_1(P_1)]& \\ + \Lambda\{v(n, y_1, y_2)\} &= f(n, y_1, y_2), \\ \lambda\{v(n, Q)\} &= g(n, Q). \end{aligned} \tag{A'}$$

Let the function $U(x, x', y_1, y_2)$, depending on the parameter x' independent of x, y_1, y_2, be the solution in the region R of the following problem:

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2}(x, x', y_1 y_2) + \Lambda\{U\} &= \frac{\pi - x}{\pi}F(x', y_1, y_2), \\ \lambda\{U(x, x', Q)\} &= \frac{\pi - x}{\pi}G(x', Q), \\ U(+0, x', y_1, y_2) &= \frac{\pi - x'}{\pi}H_0(P_0), \\ U(\pi - 0, x', y_1, y_2) &= \frac{\pi - x'}{\pi}H_1(P_1). \end{aligned} \tag{B}$$

The sine transformation of problem (B) with respect to x gives:

$$\begin{aligned} -n^2u(n, x', y_1, y_2) + n\frac{\pi - x'}{\pi}[H_0(P_0) - (-1)^nH_1(P_1)]& \\ + \Lambda\{u(n, x', y_1, y_2)\} &= n^{-1}F(x', y_1, y_2), \\ \lambda\{u(n, x', Q)\} &= n^{-1}G(x', Q). \end{aligned} \tag{B'}$$

Let $\bar{u}(n, n', y_1, y_2)$ be the sine transform of $u(n, x', y_1, y_2)$ with respect to x'. When $n' = n$, problem (B') becomes

$$\begin{aligned} -n^2\bar{u}(n, y_1, y_2) + [H_0(P_0) - (-1)^nH_1(P_1)]& \\ + \Lambda\{\bar{u}(n, y_1, y_2)\} &= n^{-1}f(n, y_1, y_2), \\ \lambda\{\bar{u}(n, Q)\} &= n^{-1}g(n, Q). \end{aligned} \tag{B''}$$

If the equations in problem (B'') are multiplied by the parameter n, it becomes evident that problems (A') and (B'') are equivalent and that $n\bar{u}(n, y_1, y_2)$ is also a solution of (A'). Suppose that the solution of (A′) is unique, then

$$v(n, y_1, y_2) = n\bar{u}(n, y_1, y_2). \tag{8}$$

Since $\bar{u}(n, y_1, y_2)$ is the iterated sine transform of $U(x, x', y_1, y_2)$, then according to our theorem of §2,

$$\begin{aligned} u(n, y_1, y_2) &= S\{S\{U(x, x', y_1, y_2)\}\} \\ &= 2^{-1}C\{U^*(x, y_1, y_2)\}. \end{aligned}$$

Hence in view of (8)

$$\begin{aligned} v(n, y_1, y_2) &= 2^{-1}nC\{U^*(x, y_1, y_2)\} \\ &= -2^{-1}S\left\{\frac{\partial}{\partial x} U^*(x, y_1, y_2)\right\}. \end{aligned}$$

Since $v(n, y_1, y_2) = S\{V(x, y_1, y_2)\}$, then

$$V(x, y_1, y_2) = -\frac{1}{2}\frac{\partial}{\partial x} U^*(x, y_1, y_2).$$

According to the definition (4) of the generalized convolution, this can be written as

$$V(x, y_1, y_2) = \frac{1}{2}\frac{\partial}{\partial x}\int_{-\pi}^{\pi} U_1(x - x', x', y_1, y_2)dx', \tag{9}$$

where U_1 is an odd periodic extension of U with respect to x an odd extension with respect to x'.

Formula (9) is then an extension of the Duhamel integral formula from time to space coordinates.

It can be shown that the above problem (B) can further be resolved into simpler problems, the solutions of which can be expressed in terms of some key functions. See [4].

Bibliography

1. R. V. Churchill, *Modern operational mathematics in engineering*, McGraw-Hill, 1944.

2. Hans Kniess, *Lösung von Randwertproblemen bei System gewöhnlicher Differentialgleichungen vermittels der endlichen Fourier-Transformation*, Math. Zeit. vol. 44 (1938).

3. R. C. F. Bartels and R. V. Churchill, *Resolution of boundary value problems by the use of a generalized convolution*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 276–282.

4. A. W. Jacobson, *A generalized convolution for the finite Fourier transformation*, Thesis, University of Michigan, 1948.

University of Michigan

A THEOREM ON MONOTONE INTERIOR TRANSFORMATIONS

EDWIN E. MOISE

B. Knaster[1] has raised the question whether there is a compact metric continuum M, irreducible between two of its points, and a monotone interior transformation T, throwing M into the unit interval, such that for each x of $T(M)$, $T^{-1}(x)$ is an arc. In the present note, we shall answer this question in the negative.

Suppose that such a continuum exists, and let $T(M)=I=[0, 1]$. Let K be a subcontinuum of M which contains points of $T^{-1}(x)$ and $T^{-1}(y)$, where $x, y \in I$ and $x \neq y$. Since M is an irreducible continuum, K contains $T^{-1}(z)$ for each z between x and y; and since T is interior, K contains $T^{-1}(x)$ and $T^{-1}(y)$. It follows that each subcontinuum of M either is an arc or contains an open subset of M, but not both. In the first case, K lies in the inverse image of a point of I, and in the second case, K is the inverse image of a subinterval of I. In either case, K is decomposable.

Now let C_1 be a simple chain of open subsets of M, with links c_1, $c_2, \cdots, c_k$, covering $T^{-1}(0)$, such that each link of C_1 contains a point of $T^{-1}(0)$ which does not lie in the closure of the sum of the other links of C_1. There is a subcontinuum K_1 of M, lying in $\sum c_i$ and containing $T^{-1}(0)$, such that for each link c of C_1, each component of $K_1 - c \cdot K_1$ is a boundary subset of M; each such component is therefore an arc. Let K be $T^{-1}(I')$, $I' \subset I$. For each x of I', $\bar{c}_k$ is the sum of two mutually exclusive closed point-sets H and H', containing $\bar{c}_k T^{-1}(0)$ and $\bar{c}_k T^{-1}(x)$ respectively. In fact, for each $j<k$, the closure of $c_2+c_3+\cdots+c_j$ has a separation into closed sets which induces such a separation of $\bar{c}_k$. But the closure of[2] C_1^* obviously has no such separation; whence it follows that there is a component L of $K_1 - K_1 \cdot \bar{c}_k$ which has a limit-point in H and a limit-point in H'. L must contain a point not in the closure of $c_2+c_3+\cdots+c_{k-1}$; and being a boundary set, L is an arc. L is therefore a subset of the inverse image of a point y of I. Let A_1 be $T^{-1}(0)$, and let A_2 be $T^{-1}(y)$.

By repeated application of the above procedure, we obtain a sequence $A_1, A_2, \cdots$ of arcs lying in M, and a sequence $C_1, C_2, \cdots$ of simple chains of open subsets of M, such that (1) C_i covers A_i, (2)

Presented to the Society, April 17, 1948; received by the editors June 10, 1948.

[1] B. Knaster, *Un continu irréductible à décomposition continue en tranches*, Fund. Math. vol. 25 (1935) p. 577.

[2] If C is a collection of sets, then C^* denotes the sum of the elements of C.

the closure of C^*_{i+1} lies in C_i^*, (3) the maximum diameter of the links of C_i is less than $1/i$, and (4) C_{i+1} contains two mutually exclusive chains, each of which has a link in each link of C_i.

Let N be the common part of the closures of the sets C_i^*. Suppose that N is the sum of two mutually exclusive closed point-sets H and H'. Condition (3) now implies that for some i, C_i is the sum of two mutually exclusive open sets each of which is a sum of links of C_i; since C_i is a simple chain, this is impossible. Therefore N is a continuum.

It is not difficult to show that N is indecomposable. The proof indicated below is very similar to a proof given in another connection by Knaster.[3] We wish to show that if N' is a proper subcontinuum of N, then every point of N' is a limit-point of $N-N'$. For each i, let C_i' be the set of all links of C_i that contain a point of N'. There is a k such that for $i>k$, C_i-C_i' contains two intersecting links of C_i. It follows that for $i>k$, C_{i+1} contains two mutually exclusive chains, one of which covers N' and both of which have a link in each link of C_{i+1}. Therefore N is the closure of $N-N'$. Since M, by hypothesis, contains no indecomposable continuum, the proof of the following theorem is now complete:

THEOREM. *For no compact, metric, irreducible continuum M is there a monotone interior transformation throwing M into an arc A such that the inverse image of each point of A is an arc.*

UNIVERSITY OF MICHIGAN

---

[3] B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fund. Math. vol. 3 (1922) p. 279.

A CONVEX METRIC FOR A LOCALLY CONNECTED CONTINUUM

R. H. BING

A topological space is metrizable if there is a distance function $D(x, y)$ such that if x, y, z are points, then

$$(1) \qquad D(x, y) \geqq 0, \text{ the equality holding only if } x = y,$$

$$(2) \qquad D(x, y) = D(y, x) \text{ (symmetry)},$$

$$(3) \qquad D(x, y) + D(y, z) \geqq D(x, z) \text{ (triangle condition)},$$

$$(4) \qquad D(x, y) \text{ preserves limit points.}$$

By (4) we mean that x is a limit point of the set T if and only if for each positive number ϵ there is a point of T at a positive distance from x of less than ϵ. We say that the metric $D(x, y)$ is convex if for each pair of points x, y there is a point u such that

$$(5) \qquad D(x, u) = D(u, y) = D(x, y)/2.$$

A subset M of a topological space S is said to have a convex metric (even though S may have no metric) if the subspace M of S has a convex metric.

It is known [5][1] that a compact continuum is locally connected if it has a convex metric. The question has been raised [5] as to whether or not a compact locally connected continuum M can be assigned a convex metric. Menger showed [5] that M is convexifiable if it possesses a metric D such that for each point p of M and each positive number ϵ there is an open subset R of M containing p such that each point of R can be joined in M to p by a rectifiable arc of length (under D) less than ϵ. Kuratowski and Whyburn proved [4] that M has a convex metric if each of its cyclic elements does. Beer considered [1] the case where M is one-dimensional. Harrold found [3] M to be convexifiable if it has the additional property of being a plane continuum with only a finite number of complementary domains.

We shall show that if M_1 and M_2 are two intersecting compact continua with convex metrics D_1 and D_2 respectively, then there is a convex metric D_3 on M_1+M_2 that preserves D_1 on M_1 (Theorem 1). Using this result, we show that any compact n-dimensional locally connected continuum has a convex metric (Theorem 6). We do not

---

Presented to the Society, February 28, 1948; received by the editors June 21, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

answer the question: *Does each compact locally connected continuum have a convex metric?*[3]

In my paper *Extending a metric* it is shown [2, Theorem 5] that if K is a closed subset of the metrizable space S and D_1 is a metric on K, then there is a metric D_2 on S that preserves D_1 on K. The following result is a modification of that result.

THEOREM 1. *If M_1 and M_2 are two intersecting compact continua with convex metrics D_1 and D_2 respectively, there is a convex metric D_3 on M_1+M_2 that preserves D_1 on M_1.*

PROOF. Let $G(x)$ be the least upper bound of $D_1(p, q)$ for all points p, q of $M_1 \cdot M_2$ such that $D_2(p, q) \leqq x$. Then

$$G[D_2(p, q)] \geqq D_1(p, q) \qquad (p,\ q \text{ elements of } M_1 \cdot M_2). \tag{6}$$

Since $G(x)$ is a monotone nondecreasing function of x that approaches zero as x approaches zero, there is a function $F(x)$ $(x>0)$ such that $F(x)$ approaches zero as x approaches zero, $F(x) \geqq G(x)$, and the derivative of $F(x)$ with respect to x is a continuous monotone nonincreasing function greater than one.

If C is an arc which lies except for possibly its end points in $M_2 - M_1 \cdot M_2$ and $\int_C F'[D_2(p, M_1)]ds$ exists, where $F'(x)$ represents the derivative of $F(x)$ with respect to x, p is a variable point of C, s is the length along C under D_2, and $D_2(p, M_1)$ is the greatest lower bound of $D_2(p, q)$ for all points q of M_1, we define the length $L_0(C)$ of C under D_0 to be $L_0(C) = \int_C F'[D_2(p, M_1)]ds$. We note that $L_0(C)$ is not defined for all arcs C. However, if r and q are two points which lie on an arc C with such a length $L_0(C)$, then we call $D_0(r, q)$ the greatest lower bound of $L_0(C)$ for all such arcs C from r to q. Since $F'(x) > 1$, $D_0(r, q) \geqq D_2(p, q)$.

If r and q are two points of M_1 and $D_0(r, q)$ is defined, then

$$D_0(r, q) \geqq D_1(r, q) \tag{7}$$

because for each positive number ϵ there is a curve C from r to q whose interior lies in $M_1 - M_1 \cdot M_2$ such that

$$\begin{aligned} D_0(r, q) + \epsilon > \int_C F'[D_2(p, M_1)]ds &\geqq \int_C F'(s)ds \\ &= F(\text{length } C \text{ under } D_2) \geqq G[D_2(r, q)] \geqq D_1(r, q). \end{aligned} \tag{8}$$

The first "$\geqq$" relationship in (8) follows from the facts that $F'(x)$ is a nonincreasing function and $s \geqq D_2(p, M_1)$ since the end points of C lie on M_1; the second "$\geqq$" results from the facts that $F(x) \geqq G(x)$, length C under $D_2 \geqq D_2(r, q)$, and $G(x)$ is a monotone nondecreasing function; the third "$\geqq$" is a consequence of (6).

---

[3] Since this paper was submitted, both E. E. Moise [8] and the author [9], working independently, have answered this question in the affirmative.

If q is a point of $M_2 - M_1 \cdot M_2$ and C is a shortest arc from q to M_1 under D_2, then

$$L_0(C) = \int_C F'[D_2(p, M_1)]ds = \int_C F'(s)ds = F[D_2(q, M_1)].$$

Hence, if $p_1, p_2, \cdots$ is a sequence of points of $M_2 - M_1 \cdot M_2$ such that $D_2(p_i, M_1)$ approaches zero as i increases without limit, then $D_0(p_i; M_1)$ approaches zero as i increases without limit.

If p and q are two points of M_1, we define $D_3(p, q)$ to be $D_1(p, q)$; if p is a point of M_1 and q is a point of $M_2 - M_1 \cdot M_2$, we define $D_3(p, q)$ to be the greatest lower bound of $D_1(p, a) + D_0(a, q)$ for all points a of M_1; if both p and q are points of $M_2 - M_1 \cdot M_2$, we define $D_3(p, q)$ to be the minimum of $D_0(p, q)$ and the greatest lower bound of $D_0(p, a) + D_1(a, b) + D_0(b, q)$, where a is a point of M_1 and so is b. If $p = q$, we define $D_3(p, q)$ to be equal to zero.

The above definition of $D_3(p, q)$ is equivalent to defining $D_3(p, q)$ to be the greatest lower bound of the lengths of all arcs C from p to q where length in M_1 is measured under D_1 and length in $M_2 - M_1 \cdot M_2$ is measured under D_0. It follows from (7) that we need only consider those arcs C which intersect M_1 in a connected piece if at all.

Now the function $D_3(p, q)$ may be shown to satisfy conditions (1), (2), (3), (4), and (5). Hence, it is a convex metric for $M_1 + M_2$ that preserves D_1 on M_1.

THEOREM 2. *If D_1 and D_2 are convex metrics for the intersecting compact continua M_1 and M_2 respectively and $D_2 \geqq D_1$ on $M_1 \cdot M_2$, then for each positive number ϵ there is a convex metric D_3 for $M_1 + M_2$ such that $D_3 = D_1$ on M_1, $D_3 \leqq D_2$ on M_2, and the diameter under D_3 of each component of $M_2 - M_1 \cdot M_2$ is less than ϵ plus twice the diameter under D_1 of the boundary with respect to $M_1 + M_2$ of this component.*

PROOF. Define $E(p, q)$ to be the greatest lower bound of all sums of the type $f(p, p_1) + f(p_1, p_2) + \cdots + f(p_n, q)$ where adjacent points of $p, p_1, \cdots, p_n, q$ belong to the same one of the continua M_1, M_2 and $f(p_i, p_j)$ is $D_1(p_i, p_j)$ or $D_2(p_i, p_j)$ according as $p_i + p_j$ is or is not a subset of M_1. Since $D_2 \geqq D_1$ on $M_1 \cdot M_2$, n need be no larger than 2 for $E(p, q)$ to attain this greatest lower bound. The convex metric E on $M_1 + M_2$ preserves D_1 on M_1 and $E \leqq D_2$ on M_2.

Let X be the set of all points p of $M_2 - M_1 \cdot M_2$ such that the distance from p to M_1 under E is greater than one-half the diameter under E of the boundary with respect to M_1+M_2 of the component of $M_2-M_1\cdot M_2$ containing p.

Let n be a positive number so small that ϵ/n is greater than twice the diameter of M_2 under E. If C is a rectifiable (under E) arc in M_1+M_2 from p to q, define $L(C)$ to be the greatest lower bound of all sums of the type $f(p, p_1)+f(p_2, p_3)+\cdots+f(p_k, q)$ where $p_1, p_2, \cdots, p_k$ are points of C and $f(p_i, p_j)$ is either n times or 1 times the length under E of the subarc of C from p_i to p_j according as this subarc is or is not a subset of X. If $D_3(p, q)$ is defined to be the greatest lower bound of all such values $L(C)$, D_3 satisfies the conditions of Theorem 2.

THEOREM 3. *Suppose M_2 is a compact continuum with a convex metric, M_2 lies in a complete locally connected space S with a metric D, each component of $S-M_2$ is of diameter under D of less than θ and M_1 is a subcontinuum of M_2 with a convex metric D_1 such that*

$$(9)\qquad D_1(p, q) \leqq D(p, q) \qquad \textit{if } D_1(p, q) > \theta \ (p, q \textit{ elements of } M_1).$$

For each positive number ϵ there is a continuum M_3 containing M_2 and a convex metric D_3 for M_3 such that D_3 preserves D_1 on M_1 and the boundary of each component of $S-M_2$ is of diameter less than $\theta+\epsilon$ under D_3.

PROOF. By Theorem 1, there is a convex metric D_2 for M_2 that preserves D_1 on M_1. Let n be an integer so large that

$$(10)\qquad nD_2(p, q) > D(p, q) \qquad \text{if } D(p, q) > \epsilon/8 \ (p, q \text{ elements of } M_2).$$

Let X denote the collection of all pairs of points (x, y) such that both x and y are points of the boundary of the same component of $S-M_2$ and

$$(11)\qquad nD_2(x, y) > \theta + \epsilon.$$

There is a finite subcollection X' of X such that for each element (x, y) of X there is an element (x', y') of X' such that both x' and y' are accessible from the same component of $S-M_2$ and

$$(12)\qquad nD_2(x, x') + nD_2(y', y) < \epsilon/2.$$

Let $C_1, C_2, \cdots, C_j$ be a finite collection of components of $S-M_2$ irreducible with respect to the property that for each element of X' there is an integer i less than or equal to j such that both points of this element of X' are accessible from C_i.

There is a dendron T_i $(i=1, 2, \cdots, j)$ such that T_i lies except for its ends in C_i and the sum of the ends of T_i is a finite subset Y_i of the boundary of C_i such that Y_i contains all points of the sum of the elements of X' that are accessible from C_i and for each point q of the boundary of C_i there is a point r of Y_i such that $nD_2(r, q) < \epsilon/2$. Let $D(T_i)$ be a convex metric for T_i such that if p and q are two end points of T_i, then

$$\theta + \epsilon/4 < D(T_i; p, q) < \theta + \epsilon/2. \tag{13}$$

Let E be a metric for $M_2 + \sum T_i = M_3$ such that the distance between two points of M_3 under E is the greatest lower bound of the lengths of arcs containing them where length is measured by nD_2 in M_2 and by $D(T_i)$ in T_i. If (x, y) is an element of X, it follows from (12) and (13) that

$$E(x, y) < \theta + \epsilon. \tag{14}$$

The diameter of the boundary of each component of $S - M_3$ is less than $\theta + \epsilon$ under E, for suppose p and q are two points of this boundary; if $p + q$ is a subset of M_2, $E(p, q) < \theta + \epsilon$ by (11) and (14); if neither p nor q is a point of M_2, both belong to some T_i whose diameter is less than $\theta + \epsilon/2$; if p is an interior point of a T_i which does not contain q, there is an end point r of T_i such that $nD_2(r, q) < \epsilon/2$ and then $E(p, q) \leqq D(T_i; p, r) + nD_2(r, q) < \theta + \epsilon$.

We shall show that if r and s are two points of M_1, then $D_1(r, s) \leqq E(r, s)$. Suppose this is not the case and that rs is an arc from r to s in M_3 whose length is less than $D_1(r, s)$ under E. If $D_1(r, s) \leqq \theta$, then rs is a subset of M_2 alone and

$$E(r, s) = nD_2(r, s) \geqq D_2(r, s) = D_1(r, s). \tag{15}$$

Suppose rs is not a subset of M_2 and $p_1p_2, p_3p_4, \cdots, p_{2j-1}p_{2j}$ are the subarcs of rs which lie except for their end points in $M_3 - M_2$ where $p_{2i-1}p_{2i}$ precedes $p_{2i+1}p_{2i+2}$ on rs in the order from r to s. Let $Z(t)$ be 0 or t according as t is less than $\epsilon/8$ or not. If $D_1(r, s) > \theta$, it follows from (9), (3), and (10) that

$$\begin{aligned}
D_1(r, s) &\leqq D(r, s) \leqq D(r, p_1) + D(p_1, p_2) + \cdots + D(p_{2j}, s) \\
&< D(r, p_1) + \theta + D(p_2, p_3) + \cdots + \theta + D(p_{2j}, s) \\
&< Z[D(r, p_1)] + Z[D(p_2, p_3)] + \cdots + Z[D(p_{2j}, s)] \\
&\quad + j\theta + (j + 1)\epsilon/8 \\
&\leqq nD_2(r, p_1) + nD_2(p_2, p_3) + \cdots + nD_2(p_{2j}, s) \\
&\quad + j(\theta + \epsilon/4) \\
&\leqq \text{length } rs \text{ under } E.
\end{aligned} \tag{16}$$

It follows from (15) and (16) that $E(r, s) \geqq D_1(r, s)$.

It follows from Theorem 2 that there is a convex metric D_3 for M_3 such that $D_3 = D_1$ on M_1 and $D_3 \leqq E$. The boundary of each component of $S - M_3$ is of diameter less than $\theta + \epsilon$ under D_3 because it is under E.

THEOREM 4. *Suppose M is a compact locally connected continuum such that if p is a point contained in an open subset R_1 of M, there is an open subset R_2 of R_1 containing p such that the boundary of R_2 with respect to M is a subset of a subcontinuum of M with a convex metric. Then M has a convex metric.*

PROOF. Let F be a metric for M. We shall show that there is a collection of continua $M_1, M_2, \cdots$ in M and a collection of metrics $D_1, D_2, \cdots$ such that:

(a) M_{i+1} contains M_i.

(b) D_i is a convex metric for M_i.

(c) D_{i+1} preserves D_i on M_i.

(d) Under F, each component of $M - M_i$ is of diameter less than $1/4^i$.

(e) Under D_i, the boundary (with respect to M) of each component of $M - M_i$ is of diameter less than $1/4^i$.

(f) Under D_{i+1}, the common part of the M_{i+1} and each component of $M - M_i$ is of diameter less than $3/4^i$.

First, we show that if ϵ is a positive number, there is a subcontinuum W of M with a convex metric such that each component of $M - W$ is of diameter less than ϵ under F. By the Heine-Borel Theorem, we find that there is a finite collection G of subcontinua of M such that each element of G has a convex metric, for each point p of M there is an element g of G such that p belongs either to g or to a component of $M - g$ of diameter under F of less than ϵ, and the sum W of the elements of G is a continuum. Each component of $M - W$ is of diameter less than ϵ under F and it follows from Theorem 1 that W has a convex metric.

Let M_1 be a subcontinuum of M with a convex metric E such that each component of $M - M_1$ is of diameter less than $1/4$ under F. A suitable multiple of E gives a metric D_1 for M_1 which will satisfy conditions (b) and (e).

Let n be an integer so large that

$$nF(p, q) > D_1(p, q) \quad \text{if} \quad D_1(p, q) > 1/17 \quad (p, q \text{ elements of } M_1).$$

There is a continuum W in M containing M_1 such that W has a convex metric and the diameter of each component of $M - W$ is less than

$1/17$ under nF. By Theorem 3, there is a continuum M_2 with a convex metric E such that M_2 contains W, E preserves D_1 on M_1, and the diameter under E of the boundary with respect to M of each component of $M-M_1$ is less than $1/4^2$. Applying Theorem 2 we find that M_2 has a convex metric D_2 such that M_2 and D_2 satisfy conditions (a), (b), (c), (d), (e), and (f).

Similarly, there exist continua $M_3, M_4, \cdots$ and convex metrics $D_3, D_4, \cdots$ satisfying conditions (a), (b), (c), (d), (e), and (f). Let D be a function of the pairs of points of M such that $D(p, q)$ is the lower limit of $D_1(p_1, q_1), D_2(p_2, q_2), \cdots$ where $p_1, p_2, \cdots$ and $q_1, q_2, \cdots$ are sequences of points converging to p and q respectively and p_i+q_i is a subset of M_i. We shall show that D preserves limit points in M.

Let ϵ be a positive number and n an integer so large that $3/4^n+3/4^{n+1}+\cdots<\epsilon/4$. If p and q are points of the same component of $M-M_n$, then $D(p, q)<\epsilon/2$ by (f). If p is a point of M_n, let R be the set of all points r of M_n such that $D_n(p, r)<\epsilon/2$. The sum V of R and all components of $M-M_n$ that have a point of R on their boundaries is an open subset of M containing p and if q is a point of V, then $D(p, q)<\epsilon$. Hence, the set of points q such that $D(p, q)>\epsilon$ is not a limit point of p.

If V is an open subset of M containing p, we shall show that there is a positive number ϵ such that $D(p, M-V)\geqq\epsilon$. Let R be an open subset of V containing p and n an integer such that $F(R, M-V)>3/4^n$. There is a positive number ϵ such that $D_n(r, s)>\epsilon$ if $F(r, s)>1/4^n$. Since each component of $M-M_n$ is of diameter less than $1/4^n$ under F, each arc in M from R to $M-V$ contains points r and s of M_n such that $F(r, s)>1/4^n$. Hence, if k is an integer bigger than n, $D_k(R\cdot M_k, [M-V]\cdot M_k)>\epsilon$. Hence, $D(p, M-V)\geqq\epsilon$.

We have shown that D satisfies conditions (1), (2), and (4). As each D_i satisfies conditions (3) and (5) and D is the limit of $D_1, D_2, \cdots$, then D satisfies these conditions. Hence, it is a convex metric for M.

THEOREM 5. *If M is an n-dimensional locally connected compact continuum and ϵ is a positive number, there is a locally connected continuum W in M such that each component of $M-W$ is of diameter less than ϵ, W is $(n-1)$-dimensional if $n>1$, and W is a dendron (acyclic continuous curve) if $n=1$.*

PROOF. An application of the Heine-Borel Theorem gives that there is an $(n-1)$-dimensional closed subset H of M such that each component of $M-H$ is of diameter less than ϵ. If $n-1>0$, H is contained in an $(n-1)$-dimensional locally connected subcontinuum of M [6, Theorem 1]. If $n-1=0$, a dendron in M contains H [7, Theorem 1].

THEOREM 6. *Each n-dimensional compact locally connected continuum has a convex metric.*

PROOF. If $n=1$, Theorem 6 follows from Theorems 4 and 5 and the fact that a dendron has a convex metric. If $n>1$, Theorem 6 follows from Theorems 4 and 5 and induction on n.

DEFINITION. A set S is said to be finite-dimensional if for each point p of S and each open subset R of S containing p there is an integer n and an open subset R' of R containing p such that the boundary of R' with respect to S is n-dimensional.

The following result may be established by using Theorems 4, 5, and 6.

THEOREM 7. *Each finite-dimensional compact locally connected continuum has a convex metric.*

REFERENCES

1. Gustav Beer, *Beweis des Satzes, dass jede im kleinen zusammenhangende Kurve konvex metrisiert werden kann*, Fund. Math. vol. 31 (1938) pp. 281–320.
2. R. H. Bing, *Extending a metric*, Duke Math. J. vol. 14 (1947) pp. 511–519.
3. O. G. Harrold, Jr., *Concerning the convexification of continuous curves*, Amer. J. Math. vol. 61 (1939) pp. 210–216.
4. C. Kuratowski and G. T. Whyburn, *Sur les éléments cycliques et leurs applications*, Fund. Math. vol. 16 (1930) pp. 305–331.
5. K. Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. vol. 100 (1928) pp. 75–163.
6. R. L. Wilder, *On the imbedding of subsets of a metric space in Jordan continua*, Fund. Math. vol. 19 (1932) pp. 45–64.
7. Leo Zippin, *A study of continuous curves and their relation to the Janiszewski-Mullikin Theorem*, Trans. Amer. Math. Soc. vol. 31 (1929) pp. 744–770.
8. E. E. Moise, *Grille decomposition and convexification theorems for compact locally connected continua*, to appear in Bull. Amer. Math. Soc.
9. R. H. Bing, *Partitioning a set*, to appear in Bull. Amer. Math. Soc.

UNIVERSITY OF WISCONSIN

CYCLIC INVARIANCE UNDER MULTI-VALUED MAPS[1]

A. D. WALLACE

In what follows it is always assumed that X, Y are compact (=bicompact) connected Hausdorff spaces each containing more than one point.

Let f denote a function which assigns to each x in X a *subset* $f(x)$ of Y. We suppose that the sets $\{f(x)\}$ cover Y. By definition

$$f^{-1}(y) = \{x \mid x \in f(y)\}.$$

It is assumed that the sets $\{f^{-1}(y)\}$ cover X. The functions f and f^{-1} play dual roles inasmuch as $f=(f^{-1})^{-1}$. If f is single-valued, then f^{-1} is the inverse of f in the usual meaning of the term. For $A \subset X$, $B \subset Y$ we define

$$f(A) = \bigcup\{f(x) \mid x \in A\}, \qquad f^{-1}(B) = \bigcup \{f^{-1}(y) \mid y \in B\}.$$

When f is single-valued we know that continuity is equivalent to the assertion that A, B closed imply $f(A)$, $f^{-1}(B)$ closed. When f is multi-valued we take this as a *definition of continuity*. It does not follow, as in the single-valued case, that $f^{-1}(B)$ is open if B is open. These definitions include both a single-valued map (=continuous function) and its inverse.

In this note we show that certain theorems of analytic topology carry over to multi-valued maps (=continuous multi-valued functions as defined above). Some of our results are new even for single-valued maps. Except for fixed-point theorems there seem to be no results in the literature for multi-valued maps.

We say that f is *anarthric* if it is continuous and if for $y \in Y$ no $x \in X - f^{-1}(y)$ separates $f^{-1}(y)$ in X. If f is single-valued and non-alternating, then f is anarthric. See Wallace [2],[2] [3], and [4] and Whyburn [5] and [6]. It is clear that if f is the inverse of a single-valued map, then f is anarthric.

For simplicity we write $P \mid Q$ to mean that the sets P and Q are mutually separated. Also if p, $q \in X$, then $p \sim q$ means that no point separates p and q in X.

THEOREM 1. *In order that the multi-valued map f be anarthric each of the following conditions is both necessary and sufficient:*

Received by the editors June 28, 1948.

[1] This work was done under Contract N7-onr-434, Task Order 3, Navy Department, The Office of Naval Research.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

(i) *If* $X=M\cup N$, *where* M *and* N *are continua meeting in a cutpoint* x, *and* K *is any continuum meeting* M, *then* $f(M\cap K)=f(M)\cap f(K)$.

(ii) *If* H *is a subcontinuum of* Y *and* K *is a subcontinuum of* X *and* $K\cap f^{-1}(H)=P\cup Q$, $P|Q$, *then there exist points* $p\in P$, $q\in Q$ *such that* $p\sim q$.

Proof. We show that (i) holds if f is anarthric. Now the conclusion follows at once if K is disjoint with either $M-x$ or $N-x$. We assume therefore that K meets both of these sets and that $y\in f(K)\cap f(M)-f(K\cap M)$, the inclusion $f(M\cap K)\subset f(K)\cap f(M)$ clearly holding. Then $f^{-1}(y)$ meets both K and M but not $K\cap M$. Hence x is not in $f^{-1}(y)$ since $x\in K\cap M$. But $f^{-1}(y)$ meets both $M-x$ and $N-x$, a contradiction.

Next, (i) implies (ii). For let K_0 be a continuum contained in K and irreducible between the disjoint closed sets P and Q. Let $p\in P\cap K_0$, $q\in Q\cap K_0$ and suppose that x separates p and q in X. Then $p\cup q\cup(K_0-(P\cup Q))$ is connected and so contains x. Thus x is not in $f^{-1}(H)$ and so $f(x)$ does not intersect H. We have a decomposition $X=M\cup N$ with M and N closed, $M\cap N=x$ and $p\in M$, $q\in N$. Now $f(x)=f(M)\cap f(N)$ as we see by taking the K of (i) to be the present N. But $H=(H\cap f(M))\cup(H\cap f(N))$ and so $H\cap f(M)\cap f(N)$ is not void since H is a continuum contrary to the fact that this intersection is $H\cap f(x)$.

Finally (ii) implies that f is anarthric by taking $H=y$ and $K=X$.

We remark that it is *sufficient* to take $K=N$ in (i) and in (ii) to take $K=X$.

We recall briefly some definitions and results mostly contained in Wallace [2]. These reduce to the well known cyclic element theory if X, Y are metric and locally connected. See Whyburn [5].

By an A-set we mean a closed set H such that if $z\in X-H$, then $X=M\cup N$ with $M\cap N=x$, $(M-x)|(N-x)$, $H\subset M$, $z\in N-x$. It is easily seen that an A-set is a chain (Wallace [2]) and hence a continuum, that the intersection of any collection of A-sets is again an A-set and that the union of two intersecting A-sets is also an A-set.

A prime-chain is a chain which is either an end point, a cutpoint or a nondegenerate minimal chain. One can replace "chain" by "A-set" in the last sentence. It is readily seen that if a chain A is met by a prime-chain E in two points or in a non-cutpoint, then $E\subset A$.

A nodal set is a closed set which meets the closure of its complement in a single point. It is readily seen that an A-set is the intersection of all the nodal sets containing it and (since each nodal set is an A-set) that any intersection of nodal sets is an A-set.

THEOREM 2. *In order that the multi-valued map f be anarthric it is necessary and sufficient that for any A-set H in X and any subcontinuum K of X meeting H we have* $f(H\cap K)=f(H)\cap f(K)$.

PROOF. Suppose that f is anarthric. It is enough to show that, for any y in Y, $f^{-1}(y)\cap H$, $f^{-1}(y)\cap K$ nonvoid imply $f^{-1}(y)\cap H\cap K$ nonvoid. Now $H\cup K$ is a continuum and so by (ii) there exist points $p\in H\cap f^{-1}(y)$, $q\in K\cap f^{-1}(y)$ with $p\sim q$, assuming of course that our implication is not valid. Let E be the prime-chain containing $p\cup q$ (Wallace [2]). Then $E\cap H$ nonvoid implies that $E\cup H$ is an A-set. Thus, since K is a continuum, we know that $K\cap(E\cup K)=(K\cap E)\cup(K\cap H)$ is connected and so $E\cap H\cap K$ is not void. Hence E must contain two distinct points of H since $p\in f^{-1}(y)$ and this latter set does not meet $H\cap K$ by assumption. From this it follows that $E\subset H$ and so $q\in H$, a contradiction.

The sufficiency is readily inferred from the remarks following Theorem 1. The result fails unless it is required that H and K meet. For if X is the union of the unit circle and the segment from (1, 0) to (2, 0) and Y is the unit circle and f is the map X onto Y carrying the segment into (1, 0), then taking $Y=H$ and $K=(2, 0)$ we see that the conclusion fails.

THEOREM 3. *In order that the multi-valued map f be anarthric it is necessary and sufficient that if $\{A\}$ is any collection of A-sets with the finite intersection property, then* $f(\bigcap A)=\bigcap f(A)$.

PROOF. If f is anarthric it is sufficient to prove that, if $y\in Y$, the proposition "$f^{-1}(y)$ meets every set in $\{A\}$" implies "$f^{-1}(y)$ meets $\bigcap A$." To this end show that $\{f^{-1}(y)\cap A\}$ has the finite intersection property. Or, for any $A_1, A_2, \cdots, A_n$ in $\{A\}$ we have $f^{-1}(y)\cap A_1\cap\cdots\cap A_n$ nonvoid. Now by Theorem 2 we see that $f(A_1\cap\cdots\cap A_n)=f(A_1)\cap\cdots\cap f(A_n)$. Thus if $f^{-1}(y)$ intersects every A_i, then $f^{-1}(y)$ also intersects $A_1\cap\cdots\cap A_n$.

The sufficiency follows from the fact that, in Theorem 2, it is enough to take K an A-set.

THEOREM 4. *If f is anarthric and the image of each cutpoint is a point, then the image of a nodal set is a nodal set.*

This follows without difficulty from (i) of Theorem 1. The result is false if the condition, that the image of a cutpoint be a point, is deleted. In the (u, v) plane let Y be the circle $u^2+v^2=4$ and X the union of the circles $(u+1)^2+v^2=1$, $(u-1)^2+v^2=1$. Define $g(u, v)=(u, (2u-u^2)^{1/2})$ if v is non-negative and $g(u, v)=(u, -(2u-u^2)^{1/2})$ if v is nonpositive. Let $f=g^{-1}$; then f is anarthric but the left-hand circle is mapped by f into the left-hand semicircle of Y, which is not a nodal set.

THEOREM 5. *If f is anarthric and the image of a cutpoint is a point, then the image of an A-set is an A-set or a point.*

PROOF. If H is an A-set, then H is the intersection of all the nodal sets $\{N\}$ which contain it. By Theorems 3 and 4 we have $f(H)=\bigcap f(N)$. But each $f(N)$ is a nodal set and thus an A-set. Then $f(H)$ is an A-set since it is an intersection of A-sets.

In the case in which f is non-alternating and X is a Peano space this result is due to G. E. Schweigert.

Let us denote, for any non-null set A, the intersection of all the A-sets which contain A by $C(A)$. It then follows from Theorem 5 (see the proof of (3.14) in Wallace [2]) that we have the following corollaries.

COROLLARY. *For any nonempty set $A\subset X$, $C(f(A))\subset f(C(A))$.*

COROLLARY. *Let f be a single-valued map of X onto Y such that $f^{-1}(y)$ is a point for each cutpoint y in Y. Then the inverse of an A-set is an A-set or a point.*

According to Kelley [1] a central set is an intersection of a finite number of nodal sets. From Theorems 3 and 4 we have the following theorem.

THEOREM 6. *If f is anarthric and the image of each cutpoint is a point, then the image of a central set is a central set.*

THEOREM 7. *In order that a multi-valued map be anarthric it is necessary and sufficient that no A-set separate the inverse of a point.*

PROOF. The condition is clearly sufficient since each cutpoint is an A-set. Suppose that some A-set A separates the inverse of a point so that we have $X-A=U\cup V$, $U|V$, with $f^{-1}(y)$ meeting both U and V but not A. Now $A\cup U$ and $A\cup V$ are A-sets, say H and K. Then $f^{-1}(y)$ intersects both H and K but not $H\cap K$ contrary to the fact $f(H\cap K)=f(H)\cap f(K)$.

Our next result generalizes a noteworthy theorem of G. T. Whyburn [5].

THEOREM 8. *Let f be an anarthric map such that the image of a cutpoint is a point and let E be a prime-chain in Y. Then there is a prime-chain F in X such that $E\subset f(F)$. If F' is any other prime-chain in X, then $f(F')$ meets E in at most one point.*

PROOF. We may suppose that E is nondegenerate. Using the Hausdorff maximality principal (Zorn's lemma) we see that there exists a collection $\{A\}$ of A-sets maximal relative to the properties that (i) $E \subset f(A)$ and (ii) no finite collection of A's has a void intersection. Let F be the intersection of these A's so that, since $f(F) = \bigcap f(A)$, we know that E is a subset of $f(F)$. Now if F is a point it is either an end point or a cutpoint and so a prime-chain. Suppose that F contains more than one point and is not a prime-chain. Then $X = M \cup N$ with $M \cap N = x$, $(M-x) \mid (N-x)$ with F meeting each of these separands. We conclude that $Y = f(M) \cup f(N)$, $f(x) = y = f(M) \cap f(N)$. Now if E is contained in $f(M)$, then also $E \subset f(M) \cap f(F) = f(M \cap F)$. Since M and F are A-sets so also is $M \cap F$ and this latter is a proper subset of F contrary to the maximality of $\{A\}$. Accordingly we conclude that E meets both $f(M) - y$ and $f(N) - y$, an impossibility since E is a prime-chain.

Let F' be a prime-chain distinct from F such that $f(F')$ contains two points of E. Then $X = M \cup M'$ where M, M' are closed, intersect in a cutpoint x and $F \subset M$, $F' \subset M'$. Then $f(x) = y \in Y$. As before $Y = f(M) \cup f(M')$. If $f(M') \subset f(M)$, then $f(M') = y$ contrary to the fact that $f(F') \subset f(M')$ and contains two points. It then follows that, as above, the point y cuts E in X, a contradiction.

BIBLIOGRAPHY

1. J. L. Kelley, *Simple links and fixed sets under continuous mappings*, Amer. J. Math. vol. 69 (1947) p. 349.

2. A. D. Wallace, *Monotone transformations*, Duke. Math. J. vol. 9 (1942) pp. 487–506.

3. ———, *On non-alternating transformations*, Bull. Amer. Math. Soc. vol. 49 (1943) p. 62.

4. ———, *A structural property of transformations*, Bull. Amer. Math. Soc. vol. 49 (1943) p. 699.

5. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, 1942.

6. ———, *Uniqueness of the inverse of a transformation*, Duke Math. J. vol. 12 (1945) pp. 317–323.

THE TULANE UNIVERSITY

BOUNDARY VALUE PROBLEMS IN AERODYNAMICS OF LIFTING SURFACES IN NON-UNIFORM MOTION

ERIC REISSNER

1. **Introduction.** In the present paper we propose to discuss certain aspects of the theory of lifting surfaces in non-uniform motion. Briefly, lifting surface theory is concerned with the motion of an impenetrable, deformable surface through an incompressible or compressible non-viscous fluid. In general the impenetrable surface is intended to represent approximately an airplane wing, a tail surface, or a propeller. The adjective lifting indicates the nature of the interaction desired between the impenetrable surface and the surrounding fluid.

The mathematical nature of the problems arising in this theory is that of boundary value problems of partial differential equations. Our principal object here is formulation of these boundary value problems and presentation of some of the methods, exact or approximate, which have been used in the solution of some of these problems. As may be seen from the list at the end of this paper the amount of work done in this field is considerable and the following account is restricted to those aspects of the theory which have been of particular interest to the writer.

Lifting surface theory as developed may be designated as a perturbation theory in the following sense. Because of the assumption of no viscosity there are evidently types of motion of an impenetrable surface which proceed without disturbing the surrounding fluid at all. One now asks for such motions which proceed *nearly* without producing any disturbances and one uses the assumption of small disturbances to simplify the differential equations and boundary conditions of the theory. In general this simplification leads to a linearized theory and it is this linearized theory which will here be discussed. The main reason for the considerable literature on the subject is the fact that the range of applicability of the linearized theory has been found adequate for many problems arising in engineering, and in particular in aeronautical engineering.

Evidently one may, if one wishes, consider separately problems of uniform and non-uniform motion in lifting surface theory. Histori-

---

An address delivered before the New York Meeting of the Society on February 28, 1948, by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors November 12, 1948.

cally, uniform-motion theory, as initiated by Prandtl, precedes non-uniform motion theory by about ten years. Solution of problems of non-uniform motion theory has turned out to be of considerably greater mathematical complexity than solution of uniform-motion problems.

In what follows we shall formulate the problems of non-uniform motion in somewhat greater generality than has heretofore been done. After this we shall discuss in some detail various aspects of the theory of nearly plane lifting surfaces in incompressible flow, and in particular the step from two-dimensional to three-dimensional theory.

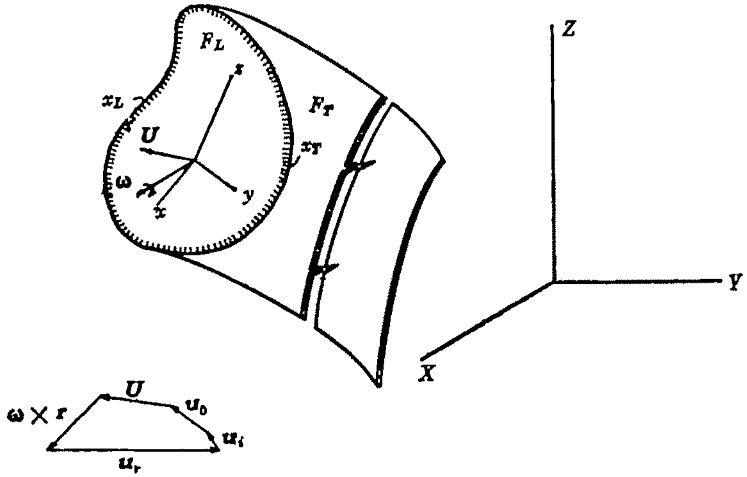

Fig. 1

2. **The general problem.** The actual problem of linearized lifting surface theory will be considered as an approximation to the following nonlinear problem. An impenetrable, deformable surface of given extent moves in a prescribed manner through a compressible perfect fluid. From part of the edge of the impenetrable surface emanates a surface of velocity discontinuity in such a manner that the fluid velocity remains finite along this part of the edge,[1] henceforth called the *trailing* edge. Along the remainder of the edge, called *leading*

---

[1] This condition of finite trailing edge velocity, first introduced in two-dimensional airfoil theory by Kutta and Joukovsky in order to obtain a definite lifting action, was subsequently found to represent rather well the effect of viscosity of actual fluids in a perfect-fluid theory of airfoils. The fact that in a three-dimensional theory enforcement of this condition necessitated introduction of a trailing surface of discontinuity was first observed by Prandtl.

edge, the fluid velocity will on account of the sharpness of the edge in general assume infinite values for an incompressible fluid. For a compressible fluid the assumption of a sharp leading edge will in general make impossible a continuous solution in the region exterior to the surfaces of discontinuity. We need not for the present purposes consider this difficulty as it disappears in the linearized form of the problem.

Let X, Y, Z be the axes of a fixed frame of reference and let x, y, z be the axes of a frame of reference moving with the impenetrable surface (Fig. 1). Let $\boldsymbol{U}(t)$ be the velocity of the origin of the moving system with reference to the fixed system and let $\boldsymbol{\omega}(t)$ be the angular velocity of the moving system with reference to the fixed system. Let $\boldsymbol{u}_r$ be the velocity of a fluid particle relative to the moving system and let $\boldsymbol{u}$ be the velocity of the same particle relative to the fixed system. The velocity vector $\boldsymbol{u}$ may be written in the form $\boldsymbol{u}_0+\boldsymbol{u}_i$, where $\boldsymbol{u}_0$ exists without being caused by the presence of the impenetrable surface and where $\boldsymbol{u}_i$ is induced by the motion of the impenetrable surface. Correspondingly we have a pressure $p=p_0+p_i$ and a density $\rho=\rho_0+\rho_i$.

We then have the following kinematical relations involving velocity $\boldsymbol{u}$ and acceleration $\boldsymbol{a}$

$$\boldsymbol{u} = \boldsymbol{u}_r + \boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{r}, \tag{1}$$

$$\boldsymbol{a} = (\partial \boldsymbol{u}/\partial t) + \boldsymbol{u}_r\cdot\nabla \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u}. \tag{2}$$

The differential equations of the problem are of the following form

$$\rho \boldsymbol{a} + \nabla p = 0, \tag{3}$$

$$(\partial\rho/\partial t) + \nabla\cdot(\rho \boldsymbol{u}_r) = 0, \tag{4}$$

$$p = f(\rho). \tag{5}$$

Equations (3) to (5) are to be solved in the space exterior to two surfaces $F_L=0$ and $F_T=0$, where F_L represents the given surface of pressure and velocity discontinuity and where F_T represents a surface of velocity discontinuity, determination of which is part of the problem. On F_L we have the condition of no relative normal flow. On F_T we have the two conditions that the normal velocities of points of the surface are given by the corresponding velocities of the surrounding fluid and that the pressure is continuous across this surface. Thus

$$F_L(x, y, z, t) = 0; \qquad \partial F_L/\partial t + \boldsymbol{u}_r\cdot\nabla F_L = 0, \tag{6}$$

$$F_T(x, y, z, t) = 0; \qquad \partial F_T/\partial t + \boldsymbol{u}_{r\pm}\cdot\nabla F_T = 0, \quad p_+ = p_-, \tag{7}$$

where the subscripts + and − distinguish the two sides of F_T.

The surfaces F_L and F_T are connected along a line C_T which is part of the edge of $F_L=0$ and which is sufficiently described for the present purposes by the designation "trailing" edge. Along C_T we have the additional condition that $\mathbf{u}$ remains finite.

In addition to the boundary condition (6) and (7) and the trailing edge conditions there are needed conditions at infinity. The form of these conditions will evidently depend on the form of the motion and on whether the fluid is compressible or incompressible.

For incompressible flow these conditions are roughly vanishing of all disturbances at an infinite distance from lifting surface and trailing surface. For compressible flow no such general statement can be made. In some cases all that is required is to superimpose on the conditions for imcompressible flow a condition stating that radiation energy is not created or reflected at infinity. In other cases the disturbances caused by the motion of the lifting surface cannot be required to vanish at infinity. General determination of these conditions is outside the scope of this report.

When the velocity distribution $\mathbf{u}_0$ which exists without the presence of the surface F_L is such that $\nabla\times\mathbf{u}_0=0$, equations (3) to (5) may be reduced to one scalar equation by means of the introduction of a velocity potential ϕ which, for incompressible flow, satisfies the Laplace equation but which for compressible flow is of a more general type.

We postpone here introduction of the velocity potential and first linearize the problem.

3. **The linearized form of the general problem.** Basic assumption for a linearized theory is that the lifting surface moves through the fluid *nearly* without disturbing the fluid such that powers and products of the quantities $\mathbf{u}_i$, p_i and ρ_i and their derivatives may be neglected.

The equation of motion (3) becomes then

$$\partial\mathbf{u}_i/\partial t-(\mathbf{U}+\boldsymbol{\omega}\times\mathbf{r})\cdot\nabla\mathbf{u}_i+\boldsymbol{\omega}\times\mathbf{u}_i+\mathbf{u}_i\cdot\nabla\mathbf{u}_0+\mathbf{u}_0\cdot\nabla\mathbf{u}_i = -\frac{\nabla p_i}{\rho_0}+\frac{\nabla p_0}{\rho_0}\frac{\rho_i}{\rho_0}. \tag{8}$$

The equation of continuity (4) becomes

$$(\partial\rho_i/\partial t)+\nabla(\rho_0\mathbf{u}_i)+\nabla[(\mathbf{u}_0-\mathbf{U}-\boldsymbol{\omega}\times\mathbf{r})\rho_i]=0 \tag{9}$$

and the equation of change of state becomes

$$p_i=f'(\rho_0)\rho_i. \tag{10}$$

We shall in some of what follows write as abbreviations

$$f'(\rho_0) = a_0^2, \qquad U + \omega \times r = V. \tag{11}$$

The quantity a_0 is the velocity of sound at a point of the undisturbed medium.

Turning now to the boundary conditions (6) and (7) we begin by establishing the condition for the motion of a surface $F_{LP} = 0$ without any disturbance of the surrounding fluid. Equation (6) indicates that this condition is as follows

$$\partial F_{LP}/\partial t + (u_0 - V)\cdot\nabla F_{LP} = 0. \tag{12}$$

We have chosen the subscript P to indicate this surface because we wish to refer to it henceforth as the projection of the lifting surface. As the actual lifting surface must deviate only slightly from this projection in order to move nearly without causing disturbances we may write

$$F_L = F_{LP} + f_L = 0 \tag{13}$$

where f_L is small in the same sense as u_i, ρ_i, and p_i are small. Then, considering (12), and except for quantities small of higher order, the boundary condition at the lifting surface becomes

$$F_{LP} = 0; \qquad \partial f_L/\partial t + (u_0 - V)\cdot\nabla f_L + u_i\cdot\nabla F_{LP} = 0. \tag{14}$$

Note that in satisfying the boundary condition at the projection of the lifting surface rather than at the lifting surface itself we again depend on the perturbation properties of the solution to be obtained.

The next step is the determination of the form of the trailing surface of discontinuity F_T. As the equation of this surface is one of the unknowns of the problem we must, in order to have a linear problem, omit the term u_i in (7) and the shape of F_T is then such that the equation

$$\partial F_T/\partial t + (u_0 - V)\cdot\nabla F_T = 0 \tag{15}$$

is satisfied. In addition to this we have the condition that at the trailing edge C_T the surface F_T is connected to the surface F_{LP}. The meaning of (15) is that within the linearized theory the shape of the trailing surface of discontinuity is independent of the velocity distribution u_i induced by the motion of the lifting surface.

Having the equation of F_T we then obtain from (17) the two conditions of continuous normal velocity and pressure across the surface in the form

$$F_T = 0; \qquad (u_{i+} - u_{i-})\cdot\nabla F_T = 0, \; p_+ - p_- = 0. \tag{16}$$

To the formulation of the problem as contained in equations (8) to (16) we must add the condition of finite u_i along C_T and appropriate conditions at infinity.

We remark that previous formulations of this problem of nonsteady motion in their most general form are based on the assumptions $u_0=0$, $\omega=0$, $U=Ui$. Under this assumption, and under the additional assumptions that $dU/dt=0$ and $\partial F_{LP}/\partial t=0$, Kuessner [44][2] has obtained an integral equation for the pressure distribution at the lifting surface.

4. **Velocity potential formulation of the linearized problem.** In what follows we shall assume that the fluid is at rest except for the motion induced by the lifting surface, that is, we put

$$u_0 = 0, \qquad \nabla p_0 = 0. \tag{17}$$

With these assumptions we have the existence of a velocity potential ϕ in terms of which

$$u_i = \nabla\phi. \tag{18}$$

Combination of (18), (17), and (8) gives for the pressure p_i the following expression

$$p_i/\rho_0 = (\mathbf{U} + \boldsymbol{\omega} \times \mathbf{r})\cdot\nabla\phi - \partial\phi/\partial t. \tag{19}$$

Combination of (19), (11), (10), and (9) gives the following differential equation for ϕ,

$$\nabla^2\phi - (1/a_0^2)[\partial/\partial t - (\mathbf{U} + \boldsymbol{\omega} \times \mathbf{r})\cdot\nabla]^2\,\phi = 0. \tag{20}$$

The boundary condition (14) becomes

$$F_{LP} = 0;\quad \frac{\partial\phi}{\partial n}\left|\nabla F_{LP}\right| = (\mathbf{U} + \boldsymbol{\omega} \times \mathbf{r})\cdot\nabla f_L - \frac{\partial f_L}{\partial t}. \tag{21}$$

The transition conditions (16) become

$$F_T = 0\begin{cases} \left(\dfrac{\partial\phi}{\partial n}\right)_+ = \left(\dfrac{\partial\phi}{\partial n}\right)_-, \\ \left[\dfrac{\partial\phi}{\partial t} - (\mathbf{U} + \boldsymbol{\omega} \times \mathbf{r})\cdot\nabla\phi\right]_+ \\ \qquad\qquad = \left[\dfrac{\partial\phi}{\partial t} - (\mathbf{U} + \boldsymbol{\omega} \times \mathbf{r})\cdot\nabla\phi\right]_- \end{cases} \tag{22}$$

---

[2] Numbers in brackets refer to the references cited at the end of the paper.

and the trailing edge condition is that along C_T we have $\nabla\phi$ finite.

The problem from here on is the solution of the mixed boundary value problem (20) to (22), with appropriate conditions at infinity. The object of such solutions is primarily the determination of the pressures p_i on both sides of the lifting surface.

Solutions obtained so far are all for nearly plane lifting surfaces such that $F_{LP} \equiv z = 0$. Predominant among these are solutions for the two-dimensional problem, which may be characterised by the requirement that $\partial\phi/\partial y \equiv 0$.

For two-dimensional incompressible flow essential contributions are due to H. Wagner [78], W. Birnbaum [3], H. Glauert [22], T. Theodorsen [73], H. G. Kuessner [42], P. Cicala [7], and G. Ellenberger [14].

For two-dimensional subsonic compressible flow which is less completely solved than the problem of incompressible flow, one must mention the work of C. Possio [53], R. Timman [75], D. Haskind [24], and that given in [60]. Various approximate methods for the solution of Possio's integral equation of the problem are described in [39].

The corresponding problem in the supersonic range has been dealt with by Possio [52], S. von Borbely [6], H. A. John and G. Temple [72], I. E. Garrick and I. Rubinov [19], and I. A. Panichkin [51].

The perturbation theory of non-steady two-dimensional transonic flow has recently been considered by C. C. Lin, H. S. Tsien and the writer [47].

In the three-dinensional theory one has the solution of Schade and Krienes [41, 62] for the lifting surface of circular plan-form in incompressible flow and, also for incompressible flow, a number of developments for an approximate analysis of the three-dimensionality of the flow for surfaces whose span is appreciably greater than their chord. We shall in what follows describe a particular approach to this problem based on earlier publications on this subject [56, 57]. A discussion of various other methods of analysis for this problem by Cicala [8, 9], W. R. Sears [69], R. T. Jones [32, 33], and Kuessner [44] can be found in [56].

Finally, we mention work by Garrick and Rubinov [20] and by E. A. Krasilschikova [40] on three-dimensional supersonic flow and a forthcoming publication on three-dimensional subsonic flow [59]. In both problems it appears that further work is required before all difficulties inherent in the problem are overcome.

5. **Motion of nearly plane lifting surface in incompressible flow.** Further discussion will be carried out for this subclass of the general

problem. Our object is to indicate the particular nature of the boundary value problem in question and to outline one of the possible methods of solution.

We assume that the projection of the lifting surface lies in the x, y-plane and that the motion of the lifting surface is in the direction of negative x. We further assume incompressible flow. We have then

$$U = -U\mathbf{i}, \qquad \omega = 0, \qquad a_0 = \infty \tag{23}$$

and, in accordance with (12) and (15),

$$F_{LP} = z = 0, \qquad F_T = z = 0. \tag{24}$$

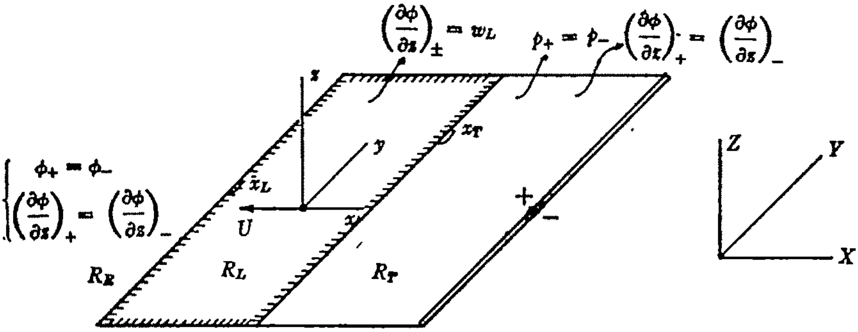
$\left(\frac{\partial\phi}{\partial z}\right)_\pm = w_L$
$p_+ = p_-$
$\left(\frac{\partial\phi}{\partial z}\right)_+ = \left(\frac{\partial\phi}{\partial z}\right)_-$
z
x_T
y
Z
Y
$\phi_+ = \phi_-$
$\left(\frac{\partial\phi}{\partial z}\right)_+ = \left(\frac{\partial\phi}{\partial z}\right)_-$
x_L
U
x
+
−
X
R_E
R_L
R_T

Fig. 2

We shall designate the region occupied by the projection of the lifting surface by R_L and the region occupied by the trailing surface of discontinuity by R_T (Fig. 2).

From (20) follows that when $a_0 = \infty$ the differential equation is, for steady as well as for nonsteady motion,

$$\nabla^2\phi = 0. \tag{25}$$

From (19) follows, for the pressure induced by the motion of the lifting surface,

$$\frac{p_i}{\rho_0} = -\left(\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x}\right). \tag{26}$$

If the instantaneous distance of a point of the lifting surface from the x, y-plane is designated by $Z(x, y, t)$ we have $F_L = F_L + f_L = z - Z(x, y, t) = 0$. Consequently the boundary condition (21) becomes

$$x,\ y \text{ inside } R_L\colon\ \frac{\partial\phi}{\partial z} = \frac{\partial Z}{\partial t} + U\frac{\partial Z}{\partial x} \equiv w_L. \tag{27}$$

The transition conditions (22) become

$$(28)\qquad x,\ y\text{ inside }R_T\quad\begin{cases}\left(\dfrac{\partial\phi}{\partial z}\right)_+=\left(\dfrac{\partial\phi}{\partial z}\right)_-,\\[2ex]\left(\dfrac{\partial\phi}{\partial t}+U\dfrac{\partial\phi}{\partial x}\right)_+=\left(\dfrac{\partial\phi}{\partial t}+U\dfrac{\partial\phi}{\partial x}\right)_-.\end{cases}$$

To equations (27) and (28) is to be added the condition of finite trailing edge velocities

$$(29)\qquad z=0,\ x=x_T(y);\qquad\nabla\phi\text{ finite}$$

and conditions at infinity which for incompressible flow may be taken in the form

$$(30)\qquad\begin{matrix}x=-\infty\\ z=\pm\infty\end{matrix}\left\{\nabla\phi=0,\qquad\frac{\partial\phi}{\partial t}+U\frac{\partial\phi}{\partial x}=0.\right.$$

The above problem is to be understood as a boundary value problem for the exterior of an infinitely thin semi-infinite tube surrounding the regions R_L and R_T, in the sense that (27) holds for $z=+0$ and for $z=-0$. It may be recalled that the main object is the determination of p_{i+} and p_{i-} in R_L, with $p_{i-}-p_{i+}$ being the lift intensity produced by the motion of the impenetrable surface F_L.

The form of the boundary conditions (27) to (30) is such that the problem for the exterior of the semi-infinite tube may be transformed by a symmetry consideration into a mixed boundary-value problem for one of the half spaces $z>0$ or $z<0$. This is done by observing that equations (27) and (30) are compatible with the assumption that ϕ is an odd function of z. If we define a region R_R as the x, y-plane minus the regions R_L and R_T and take into account that ϕ is continuous except across R_L and R_T we may replace the boundary conditions (27) and (28) by the following system of conditions at $z=0$:

$$(31)\qquad\begin{cases}x,\ y\text{ inside }R_L:\ \dfrac{\partial\phi}{\partial z}=w_L,\\[2ex]x,\ y\text{ inside }R_T:\ \dfrac{\partial\phi}{\partial t}+U\dfrac{\partial\phi}{\partial x}=0,\\[2ex]x,\ y\text{ inside }R_R:\ \phi=0.\end{cases}$$

We then must determine ϕ in one of the half-spaces, say $z>0$, with the conditions (29) and (31) at the boundary $z=0$ and with equations (30) giving the conditions at infinity.

Let us remark that explicit solutions of the problem thus formulated are possible in the two-dimensional case by the use of elliptic cylinder coordinates, while for the circular plan form wing the use of spheroidal coordinates is appropriate [41, 62]. The problem would be of a standard nature if the conditions in R_T were the same as in R_B. The main difficulty of obtaining an explicit solution comes from the particular form of the boundary condition holding in R_T inasmuch as all that can be said on the basis of (31) about the values of ϕ in R_T is

$$\phi(x, y, 0, t) = \Phi\left(x - \int U dt, y\right) \tag{32}$$

where Φ is an arbitrary function of its two arguments. Compensating for this arbitrariness is, as will be seen, the finiteness condition (29).

6. **Integral equation formulation of the problem.** As one is interested primarily in the values of p_i in R_L it suggests itself to derive an integral equation for this quantity. This procedure, adopted by Birnbaum [3], Possio [53], Kuessner [42], and others at the suggestion of Prandtl, and known under the name acceleration potential method, has the advantage that it can be developed without explicit introduction of the trailing surface of discontinuity. It does however have the disadvantage of leading to an integral equation with a distinctly more complicated kernel than the corresponding integral equation for the values of $\partial\phi/\partial x$ in R_L which we propose to discuss here. The main advantage of the latter formulation is that it permits immediate recognition of the explicit solvability of the problem of non-steady motion in terms of the solution of the corresponding problem of steady motion.

Setting as an abbreviation

$$\gamma(x, y, t) = \partial\phi(x, y, 0, t)/\partial x \tag{33}$$

we have the following representation for $\partial\phi/\partial x$ in terms of the boundary values γ,

$$\frac{\partial\phi}{\partial x} = \frac{-1}{2\pi}\int\int \gamma(\xi, \eta, t)\frac{\partial}{\partial z}\left(\frac{1}{r}\right)d\xi d\eta \tag{34}$$

where

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + z^2. \tag{35}$$

Equation (34) may be converted into an expression for $\partial\phi/\partial z$, by appropriate differentiation and integration, of the following form

$$\frac{\partial\phi}{\partial z} = \frac{-1}{2\pi}\int\int \gamma(\xi, \eta, t)\left[\int_{-\infty}^{x} \frac{\partial^2}{\partial z^2}\left(\frac{1}{r'}\right)dx'\right]d\xi d\eta \tag{36}$$

$$
\begin{aligned}
&= \frac{-1}{2\pi}\iint \gamma(\xi, \eta, t)\left\{\frac{x-\xi}{r^3}\right.\\
&\qquad\left. + \frac{\partial}{\partial y}\left[\frac{y-\eta}{(y-\eta)^2+z^2}\left(\frac{x-\xi}{r}+1\right)\right]\right\} d\xi d\eta.
\end{aligned} \tag{36}
$$

Note that when $\partial\gamma/\partial\eta = 0$ which corresponds to the assumption of two-dimensional flow the η-integration in (36) can be carried out with the result that

$$
\frac{\partial\phi(x, z, t)}{\partial z} = -\frac{1}{\pi}\int \frac{\gamma(\xi, t)(x-\xi)}{(x-\xi)^2+z^2} d\xi \tag{36'}
$$

which equation can of course be obtained directly in a simpler manner.

To separate two-dimensional from three-dimensional effects the following transformation is useful. We write (omitting for brevity the t in γ)

$$
\gamma(\xi, \eta) = \gamma(\xi, y) + [\gamma(\xi, \eta) - \gamma(\xi, y)]. \tag{37}
$$

Appropriate integration by parts then gives the following relation

$$
\begin{aligned}
\frac{\partial\phi}{\partial z} &= -\frac{1}{\pi}\int \frac{\gamma(\xi, y)(x-\xi)}{(x-\xi)^2+z^2} d\xi\\
&\quad - \frac{1}{2\pi}\iint \frac{\partial\gamma}{\partial\eta}\left\{\frac{y-\eta}{(y-\eta)^2+z^2}\left(\frac{x-\xi}{r}+1\right)\right.\\
&\quad \left. + \frac{(x-\xi)(y-\eta)}{(x-\xi)^2+z^2}\left(\frac{1}{r}-\frac{1}{|y-\eta|}\right)\right\} d\xi d\eta.
\end{aligned} \tag{38}
$$

We must now in equation (38) let z tend to zero and substitute the boundary conditions (31). It is advantageous that (38) is in such a form that, as can be proved, the two limiting processes of integration and of letting z tend to zero can be interchanged, provided the integrals are defined where appropriate as Cauchy principal values. Thus from (31) and (38)

$$
\begin{aligned}
w_L(x, y, t) &= -\frac{1}{\pi}\int_{x_L(y)}^{x_T(y)} \frac{\gamma(\xi, y, t)}{x-\xi} d\xi - \frac{1}{\pi}\int_{x_T(y)}^{\infty} \frac{\gamma_T(\xi, y, t)}{x-\xi} d\xi\\
&\quad - \frac{1}{2\pi}\iint_{B_L} \frac{\partial\gamma}{\partial\eta}\left\{\frac{1}{y-\eta} + K(x-\xi, y-\eta)\right\} d\xi d\eta\\
&\quad - \frac{1}{2\pi}\iint_{B_T} \frac{\partial\gamma_T}{\partial\eta}\left\{\frac{1}{y-\eta} + K(x-\xi, y-\eta)\right\} d\xi d\eta.
\end{aligned} \tag{39}
$$

In (39) x_L and x_T indicate the coordinates of the leading and trailing edge, γ_T is still to be determined as far as possible from the boundary conditions and K is of the following form

$$(40)\qquad K = \frac{((x-\xi)^2+(y-\eta)^2)^{1/2}-|y-\eta|}{(y-\eta)(x-\xi)}.$$

We shall from now on assume for simplicity's sake that the region R_L is the rectangle $|x| \leqq b$, $|y| \leqq sb$ and that the velocity U which occurs in (31) is constant. We further introduce dimensionless variables,

$$(41)\qquad x' = x/b, \qquad y' = y/b, \qquad t' = \omega t$$

and a dimensionless parameter k of the form

$$(42)\qquad k = \frac{\omega b}{U}.$$

For the case of simple harmonic motion k is referred to as the "reduced frequency" of the motion. We may again for simplicity's sake omit in what follows the primes designating the dimensionless variables.

We then have from (32) that

$$(43)\qquad \gamma_T(\xi, y, t) = \frac{\partial\Phi(\xi - t/k, y)}{\partial(\xi - t/k)}.$$

Furthermore on account of the finiteness condition (29)

$$(44)\qquad \Phi(1 - t/k, y) = \int_{-1}^{1} \gamma(x, y, t)dx \equiv \Gamma(t, y).$$

Then

$$(45)\qquad \Phi(\xi - t/k, y) = \Gamma(t - k(\xi - 1), y)$$

and with

$$(46)\qquad t - k(\xi - 1) = \tau, \qquad d(k\xi - t) = -d\tau$$

we have from (43)

$$(47)\qquad \gamma_T(\xi, y, t) = -k\frac{\partial\Gamma(\tau, y)}{\partial\tau}.$$

We introduce (46) and (47) into (39) and obtain the following form of the integral equation of the problem

$$
\begin{aligned}
w_L(x, y, t) = &-\frac{1}{\pi}\int_{-1}^{1}\frac{\gamma(\xi, y, t)}{x-\xi}\,d\xi \\
&+\frac{1}{\pi}\int_{-\infty}^{t}\frac{\partial\Gamma}{\partial\tau}\,\frac{d\tau}{(x-1)-(t-\tau)/k} \\
&-\frac{1}{2\pi}\int_{-s}^{s}\int_{-1}^{1}\frac{\partial\gamma}{\partial\eta}\left\{\frac{1}{y-\eta}+K(x-\xi, y-\eta)\right\}d\xi d\eta \\
&+\frac{1}{2\pi}\int_{-s}^{s}\int_{-\infty}^{t}\frac{\partial^2\Gamma}{\partial\eta\partial\tau}\left\{\frac{1}{y-\eta}\right. \\
&\left.+K\left(x-1-\frac{t-\tau}{k},\ y-\eta\right)\right\}d\xi d\tau.
\end{aligned}
\tag{48}
$$

For uniform motion we have $\partial\Gamma/\partial\tau = 0$ and the second and fourth integrals in (48) are absent. The form of equation (48) indicates clearly the manner in which for non-uniform motion the values of the solution $\gamma(x, y, t)$ depend on the past history of the motion through the cumulative effect of successive changes of

$$
\Gamma(\tau, y) = \int_{-1}^{1}\gamma(x, y, \tau)dx. \tag{49}
$$

The quantity Γ is one-half of what is usually referred to as the circulation intensity at a station $y = \text{const.}$ of the lifting surface.

Equation (48) is now to be solved for γ, in terms of w_L and Γ. Once this is done Γ is found by integration in terms of w_L and therewith γ is expressed in terms of w_L only. The pressure p_l at the lifting surface is then, according to (26), of the form

$$
\frac{p_l}{\rho_0 U} = -\left[k\frac{\partial}{\partial t}\left(\int_{-1}^{x}\gamma(\xi, y, t)d\xi\right)+\gamma(x, y, t)\right]. \tag{50}
$$

The advantage of (48) compared with the corresponding equation for the values of p_l lies in the form of the kernel $1/(x-\xi)$ in the first term. This permits solution of (48) in a manner analogous to what is done for the problem of uniform motion.

7. Solution of the two-dimensional problem. In what follows we wish to describe briefly one of the possible methods of solution of this problem, namely that by L. Schwarz [67]. We shall subsequently indicate how to utilize this method for an approximate solution of the three-dimensional problem.

Introducing the (unessential) restriction of simple harmonic motion we set

$$w_L = \bar{w}_L e^{it}, \qquad \gamma = \bar{\gamma} e^{it}, \qquad \Gamma = \bar{\Gamma} e^{it} \tag{51}$$

where the barred quantities are functions of the space coordinates at most. Equation (48) can then be written in the following form

$$\bar{w}_L(x) = -\frac{1}{\pi}\int_{-1}^{1}\frac{\bar{\gamma}(\xi)d\xi}{x-\xi} + \frac{ik}{\pi}\bar{\Gamma}\int_{1}^{\infty}\frac{e^{ik(1-\xi)}}{x-\xi}d\xi. \tag{52}$$

Equation (52) is solved by means of a pair of inversion formulas of the form

$$g(x) = -\frac{1}{\pi}\int_{-1}^{1}\frac{f(\xi)}{x-\xi}d\xi, \ f(1) \text{ finite,} \tag{53a}$$

$$f(x) = \frac{1}{\pi}\left(\frac{1-x}{1+x}\right)^{1/2}\int_{-1}^{1}\left(\frac{1+\xi}{1-\xi}\right)^{1/2}\frac{g(\xi)}{x-\xi}d\xi \tag{53b}$$

which may be considered as a result of two-dimensional potential theory, as discussed most fully by H. Söhngen.[3]

Application of (53) to (52) leads to the following expression for $\bar{\gamma}$

$$\begin{aligned}\bar{\gamma}(x) = \frac{1}{\pi}\left(\frac{1-x}{1+x}\right)^{1/2}\Bigg\{&\int_{-1}^{1}\left(\frac{1+\xi}{1-\xi}\right)^{1/2}\frac{\bar{w}_L(\xi)}{x-\xi}d\xi \\ &+ ik\bar{\Gamma}\int_{1}^{\infty}\left(\frac{\xi+1}{\xi-1}\right)^{1/2}\frac{e^{ik(1-\xi)}}{x-\xi}d\xi\Bigg\}.\end{aligned} \tag{54}$$

The main difficulty from here on is the calculation of the pressure $\bar{p}_i$ which according to (50) is given by

$$\frac{\bar{p}_i}{\rho_0 U} = -\left[ik\int_{-1}^{x}\bar{\gamma}(\xi)d\xi + \bar{\gamma}(x)\right]. \tag{55}$$

Before listing the result of this lengthy and somewhat devious calculation we may indicate the nature of the equation for $\bar{\Gamma}$ which occurs in (54). If we integrate both sides of (54) as follows:

$$\begin{aligned}\int_{-1}^{1}\bar{\gamma}dx = \frac{1}{\pi}&\int_{-1}^{1}\left(\frac{1+\xi}{1-\xi}\right)^{1/2}\left[\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{1/2}\frac{dx}{x-\xi}\right]\bar{w}_L(\xi)d\xi \\ &+\frac{ik}{\pi}\bar{\Gamma}\int_{1}^{\infty}\left(\frac{\xi+1}{\xi-1}\right)^{1/2}\left[\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{1/2}\frac{dx}{x-\xi}\right]e^{ik(1-\xi)}d\xi\end{aligned} \tag{56}$$

and take account of the formulas

$$\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{1/2}\frac{dx}{x-\xi} = -\pi, \qquad |\xi| < 1, \tag{57a}$$

---

[3] Math. Zeit. vol. 45 (1939) pp. 245–264.

$$(57\mathrm{b})\qquad \int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{1/2}\frac{dx}{x-\xi} = -\pi\left(1-\left(\frac{\xi-1}{\xi+1}\right)^{1/2}\right), \qquad 1 \leqq \xi,$$

then equation (56) can be written one of the forms

$$(58)\qquad \begin{aligned}\Gamma = &-\int_{-1}^{1}\left(\frac{1+\xi}{1-\xi}\right)^{1/2}\bar{w}_L(\xi)d\xi \\ &- ike^{ik}\Gamma\int_{1}^{\infty}\left(\left(\frac{\xi+1}{\xi-1}\right)^{1/2}-1\right)e^{-ik\xi}d\xi,\end{aligned}$$

$$(59)\qquad \Gamma = \frac{-\displaystyle\int_{-1}^{1}\left(\frac{1+\xi}{1-\xi}\right)^{1/2}\bar{w}_L(\xi)d\xi}{1+ike^{ik}\displaystyle\int_{1}^{\infty}\left(\left(\frac{\xi+1}{\xi-1}\right)^{1/2}-1\right)e^{-ik\xi}d\xi}\,.$$

It is at this stage that a combination of Bessel functions makes its appearance in the theory. The integral in the denominator of (59) is expressible in terms of Hankel functions, as follows:

$$(60)\qquad \begin{aligned}&\int_{1}^{\infty}\left(1-\left(\frac{\xi+1}{\xi-1}\right)^{1/2}\right)e^{-ik\xi}d\xi \\ &\qquad = \frac{\pi}{2}\left[H_1^{(2)}(k)+iH_0^{(2)}(k)\right]+\frac{e^{-ik}}{ik}\,.\end{aligned}$$

In view of (60) Γ can be written in the alternate form

$$(61)\qquad \Gamma = \frac{\dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\left(\frac{1+\xi}{1-\xi}\right)^{1/2}\bar{w}_L(\xi)d\xi}{2^{-1}ike^{ik}\left[H_1^{(2)}(k)+iH_0^{(2)}(k)\right]}\,.$$

Our purpose in outlining in some detail the steps leading from (52) to (61) has been to indicate the nature of some of the more simple transformations in the calculation of the pressure distribution on oscillating airfoils. Considerable care is necessary to arrange the analysis in such a manner that advantage is taken of all possible simplifications. In this way there is found the following expression for $\bar{p}_i$ of equation (55)

$$(62)\qquad \begin{aligned}\frac{\bar{p}_i}{\rho_0 U} = &\frac{1}{\pi}\int_{-1}^{1}\left(\left(\frac{1-x}{1+x}\right)^{1/2}\left(\frac{1+\xi}{1-\xi}\right)^{1/2}\frac{1}{x-\xi}-ik\Lambda\right)\bar{w}_L(\xi)d\xi \\ &+\frac{C(k)-1}{\pi}\left(\frac{1-x}{1+x}\right)^{1/2}\int_{-1}^{1}\left(\frac{1+\xi}{1-\xi}\right)^{1/2}\bar{w}_L(\xi)d\xi.\end{aligned}$$

In (62) the function Λ is given by

$$(63)\qquad \Lambda(x, \xi) = \frac{1}{2} \ln \frac{1 - x\xi + (1 - x^2)^{1/2}(1 - \xi^2)^{1/2}}{1 - x\xi - (1 - x^2)^{1/2}(1 - \xi^2)^{1/2}}$$

and the function C, first introduced by Theodorsen [73], is of the form

$$(64)\qquad C(k) = \frac{H_1^{(2)}(k)}{H^{(2)}(k) + iH_0^{(2)}(k)} \cdot$$

The results outlined in this section find their main application in the analysis of airplane flutter.[4] For this purpose explicit expressions have been obtained by Cicala [7], Kuessner [42], Theodorsen [73], and others for lift and moment amplitudes $\overline{L}$ and $\overline{M}_a$ defined by

$$(65a)\qquad \overline{L} = 2b \int_{-1}^{1} \bar{p}_s dx, \qquad \overline{M}_a = 2b^2 \int_{-1}^{1} (x - a)\bar{p}_s dx$$

and for control-surface hinge moments $\overline{M}_c$ defined by

$$(65c)\qquad \overline{M}_c = 2b^2 \int_{c}^{1} (x - c)\bar{p}_s dx$$

for various appropriate forms of $\bar{w}_L$.

Plots of representative pressure distributions for various values of the reduced frequency k and for some of the more important forms of w_L may be found in a recent paper by Postel and Leppert [55].

We may conclude this section with some remarks concerning the solution of the problem for non-oscillatory motion.

It may readily be seen that the results for simple harmonic motion may be used for Laplace transform analysis by replacing wherever it occurs ik by $-q$ whereupon equations (61) and (62) become relations between Laplace transforms. For applications of the Laplace transform method in this field reference may be made to work by I. E. Garrick [17, 18] and W. R. Sears [70].

Another form of the results consists in integro-differential equations for Γ, L and M_a, without any assumption concerning the form of the solution. The nature of these results may be seen from the simplest of them, the equation determining Γ. Omitting all but the first two of the integrals on the right of (48) we may obtain the following relation

---

[4] Briefly, the problem of flutter is the determination of those flight speeds at which self-sustained oscillations of a component of the airplane become possible due to the aerodynamic forces produced by an oscillatory motion of this component.

$$\begin{aligned}(66)\quad &\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{1/2} w_L(x,t)dx \\ &\quad = -\frac{1}{\pi}\int_{-1}^{1}\gamma(\xi,t)\left\{\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{1/2}\frac{dx}{x-\xi}\right\}d\xi \\ &\quad\quad + \frac{1}{\pi}\int_{-\infty}^{t}\frac{d\Gamma}{d\tau}\left\{\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{1/2}\frac{dx}{x-1-(t-\tau)/k}\right\}d\tau.\end{aligned}$$

After evaluating the inner integrals on the right of (66) we are left with an equation of the form

$$\begin{aligned}(67)\quad &\Gamma(t)+\int_{-\infty}^{t}\frac{d\Gamma}{d\tau}\left[\left(\frac{t-\tau+2k}{t-\tau}\right)^{1/2}-1\right]d\tau \\ &\quad\quad = -\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{1/2} w_L(x,t)dx.\end{aligned}$$

A formal solution of (67) by Fourier series or Laplace transform methods is again readily obtained. A special case of such a solution is given by (61). Practical applications, especially of the Laplace transform solution, are however not a simple matter, the reason for this being the occurrence of Hankel function combinations in the denominator of the transforms to be evaluated, and the possibility of solving (67) directly by machine methods would be of considerable advantage.

8. **Remarks on the problem of tunnel wall interference in the two-dimensional theory.** A further problem of some interest concerning the two-dimensional theory of oscillating airfoils is the effect of tunnel walls on the pressure distribution $\bar{p}_i$. For a wing of chord $2b$ located at the center of a tunnel of height $2h$ one finds, using the method of images, that (52) is replaced by the following equation [58]

$$(68)\quad \bar{w}_L(x) = -\frac{1}{\pi}\int_{-1}^{1}\frac{\lambda\bar{\gamma}(\xi)d\xi}{\sinh[\lambda(x-\xi)]}+\frac{ik}{\pi}\Gamma\int_{1}^{\infty}\frac{\lambda e^{ik(1-\xi)}d\xi}{\sinh[\lambda(x-\xi)]}$$

where the parameter λ is given by

$$(69)\quad \lambda = \pi b/2h.$$

Equation (69) may be transformed into an equation of the form (61) by means of the following substitutions[5]

---

[5] This transformation has been used in a study of the corresponding problem of steady flow, where the second integral on the right of (68) is absent, by L. Lees and H. S. Tsien, Journal of the Aeronautical Sciences vol. 12 (1945) pp. 173–187, 202.

$$\kappa z = \tanh \lambda x, \qquad \kappa\zeta = \tanh \lambda\xi, \qquad \kappa = \tanh \lambda. \tag{70}$$

The result is of the form

$$\begin{aligned}\frac{\bar{w}_L^*(z)}{(1-\kappa^2 z^2)^{1/2}} = {} & -\frac{1}{\pi}\int_{-1}^{1} \frac{\bar{\gamma}^*(\zeta)}{(1-\kappa^2\zeta^2)^{1/2}}\,\frac{d\zeta}{z-\zeta} \\ & + \frac{ik}{\pi}\Gamma\int_{1}^{1/\kappa} \frac{\exp\left\{ik[1-(\tanh^{-1}\kappa\zeta)/\lambda]\right\}}{(1-\kappa^2\zeta^2)^{1/2}\,(z-\zeta)}\,d\zeta.\end{aligned} \tag{71}$$

The solution of (71), which involves elliptic integrals, has not yet been given. An approximate solution, valid for sufficiently small values of λ, has been obtained in the following way [58]. We set in the interval $|\xi| \leqq 1$

$$\frac{\lambda}{\sinh \lambda(x-\xi)} \approx \frac{1}{x-\xi} - \frac{\lambda^2}{6}(x-\xi). \tag{72}$$

The limits of integration in the second integral on the right of (68) preclude the direct use of (72). This difficulty is overcome by writing

$$\begin{aligned}&\int_1^\infty \frac{\lambda e^{-ik\xi}d\xi}{\sinh\,[\lambda(x-\xi)]} \\ &\qquad = \int_1^\infty \frac{e^{ik\xi}d\xi}{x-\xi} + \int_1^\infty e^{ik\xi}\left[\frac{\lambda}{\sinh\,[\lambda(x-\xi)]} - \frac{1}{x-\xi}\right]d\xi\end{aligned} \tag{73}$$

and by splitting the second integral on the right of (73) into two integrals as follows, $\int_1^\infty = \int_x^\infty - \int_x^1$. The integral $\int_x^\infty$ leads to the function

$$f(\mu) = \int_0^\infty e^{-i\mu x}\left(\frac{1}{x} - \frac{1}{\sinh x}\right)dx \tag{74}$$

which has been tabulated. In the integral $\int_x^1$ one may introduce the approximation (72). In this way the following approximate equation, which takes the place of (68), is obtained:

$$\begin{aligned}\bar{w}_L(x) = {} & -\frac{1}{\pi}\int_{-1}^{1}\frac{\bar{\gamma}(\xi)d\xi}{x-\xi} + \frac{ik}{\pi}\Gamma\int_1^\infty \frac{e^{ik(1-\xi)}}{x-\xi}\,d\xi \\ & + \frac{\lambda^2}{6\pi}\left\{\int_{-1}^{1}\bar{\gamma}(\xi)(x-\xi)d\xi + \Gamma\left[\frac{1-e^{ik(1-x)}}{ik} + (1-x)\right]\right\} \\ & + \frac{ik}{\pi}\Gamma f\left(\frac{k}{\lambda}\right)e^{ik(1-x)}.\end{aligned} \tag{75}$$

It is of some interest to observe that the effect of the presence of tunnel walls may, for moderate values of k, be appreciably larger than the corresponding effect in the steady-state solution for which $k=0$. It was found in one representative example that when $k=0.25$ the wall effect was twice what it was when $k=0$. This increase of the wall effect will occur when at the same time k is (1) large enough for the trailing surface of discontinuity to be of importance and (2) small enough for the characteristic wavelength in the trailing surface of discontinuity to be appreciably greater than the wing chord.

9. **Approximate theory for lifting surfaces of finite span.** We now propose to discuss a method of approximate solution of the integral equation (48) for three-dimensional flow from the following point of view. Our object is to reduce (48), which contains double integrals, to such a form that a solution of single integral equations only is required. We shall show that this is possible, provided the "aspect ratio" s is sufficiently large, in such a way that what remains to be found is the solution of a problem of the kind encountered in the two-dimensional theory and the solution of a problem of the kind encountered in the determination of the spanwise lift distribution for a wing in uniform motion according to Prandtl's lifting-line theory. In this analysis we shall restrict attention to the case of simple harmonic motion in the sense of (51). Equation (48) then assumes the following form

$$\begin{aligned}\bar{w}_L(x, y) = &-\frac{1}{\pi}\int_{-1}^{1}\frac{\bar{\gamma}(\xi, y)}{x-\xi}\,d\xi + \frac{ik}{\pi}\Gamma(y)\int_{1}^{\infty}\frac{e^{ik(1-\xi)}}{x-\xi}\,d\xi \\ &-\frac{1}{2\pi}\int_{-s}^{s}\int_{-1}^{1}\frac{\partial\bar{\gamma}}{\partial\eta}\left\{\frac{1}{y-\eta} + K(x-\xi, y-\eta)\right\}d\xi d\eta \\ &+\frac{ik}{2\pi}\int_{-s}^{s}\int_{1}^{\infty}\frac{d\Gamma}{d\eta}e^{ik(1-\xi)}\left\{\frac{1}{y-\eta}\right. \\ &\left.\qquad\qquad + K(x-\xi, y-\eta)\right\}d\xi d\eta.\end{aligned} \tag{76}$$

Our first step is to observe that the function K as given by equation (40) behaves, for $|(x-\xi)/(y-\eta)| \ll 1$, like $2^{-1}(x-\xi)/|y-\eta| \cdot (y-\eta)$ and is therewith in this region small compared with the remaining part of the kernel, $1/(y-\eta)$. If it is now assumed that $1 \ll s$, then K is small compared with $1/(y-\eta)$ over most of the region of integration and may over this part of the region be disregarded. There remains the immediate neighborhood of the line $\eta=y$ where this order of magnitude relation does not hold. In order to disregard the contribution due to K in this region it is observed that K is an odd function of $y-\eta$ and that we expect $\partial\bar{\gamma}/\partial\eta$ to be a slowly varying function of η. In view of this we expect the contribution due to both K and $(1/(y-\eta)$ to be negligible. We thus assume that

$$\int_{-s}^{s}\int_{-1}^{1}\frac{\partial\bar{\gamma}}{\partial\eta}\left\{\frac{1}{y-\eta}+K\right\}d\xi d\eta\approx\int_{-s}^{s}\int_{-1}^{1}\frac{\partial\bar{\gamma}}{\partial\eta}\frac{d\xi d\eta}{y-\eta}\tag{77}$$

and assume that the approximation is justified for "sufficiently" large values of s.

Obviously the above argument is not very satisfactory, from a mathematical point of view, but no more rigorous argument has yet been given to justify this, from a practical standpoint rather satisfactory, result. Possibly, equation (77) represents the first step in an asymptotic development in powers of $1/s$ but this has not yet been proved.

The argument leading to (77) cannot directly be applied to the last integral in (76) since in this integral we do not have the fact that $|(x-\xi)/(y-\eta)|$ is small compared to 1 over most of the region of integration. We proceed instead as follows. Write, with $\xi-x=\lambda$,

$$\begin{aligned}\int_{1}^{\infty}e^{-ik\xi}\left\{\frac{1}{y-\eta}+K(x-\xi,\,y-\eta)\right\}d\xi&=\int_{x}^{\infty}-\int_{x}^{1}\\&=\int_{0}^{\infty}e^{-ik(\lambda+x)}\left\{\frac{1}{y-\eta}-K(\lambda,\,y-\eta)\right\}d\lambda\\&\quad-\int_{x}^{1}e^{-ik\xi}\left\{\frac{1}{y-\eta}+K(x-\xi,\,y-\eta)\right\}d\xi.\end{aligned}\tag{78}$$

In the second integral on the right we can again neglect the contribution due to K, and thus we may write

$$\begin{aligned}\int_{1}^{\infty}e^{-ik\xi}\left\{\frac{1}{y-\eta}+K\right\}d\xi&\approx\frac{e^{-ik}-e^{-ikx}}{ik(y-\eta)}\\&\quad+e^{-ikx}\int_{0}^{\infty}e^{-ik\lambda}\left\{\frac{1}{y-\eta}-\frac{(\lambda^{2}+(y-\eta)^{2})^{1/2}-|y-\eta|}{(y-\eta)\lambda}\right\}d\lambda.\end{aligned}\tag{79}$$

In view of (79) we now define a function F by the relation

$$F(z)=\frac{|z|}{z}\int_{0}^{\infty}e^{-\lambda}\left\{\frac{1}{|z|}+\frac{1}{\lambda}-\left(\frac{1}{z^{2}}+\frac{1}{\lambda^{2}}\right)^{1/2}\right\}d\lambda\tag{80}$$

and combine (77), (79), and (80) in order to obtain from (76) the following approximate integral equation of the problem:

$$
\begin{aligned}
\bar{w}_L(x, y) = &-\frac{1}{\pi}\int_{-1}^{1}\frac{\bar{\gamma}(\xi, y)d\xi}{x-\xi} + \frac{ik}{\pi}\Gamma(y)\int_{1}^{\infty}\frac{e^{ik(1-\xi)}}{x-\xi}d\xi \\
&-\frac{e^{ik(1-x)}}{2\pi}\int_{-s}^{s}\frac{d\Gamma}{d\eta}\left\{\frac{1}{y-\eta} - ikF[k(y-\eta)]\right\}d\eta.
\end{aligned}
\tag{81}
$$

Equation (81) is the result which it was intended to obtain. The solution of (81) proceeds as follows. One first obtains $\bar{\gamma}(x, y)$, just as in the two-dimensional theory except that now $\bar{\gamma}$ also depends on $d\Gamma/d\eta$. One then obtains, by integration of $\bar{\gamma}$, an equation for the determination of Γ which is, as was previously stated, of the type of the equation for the determination of the spanwise lift-distribution for a wing in uniform motion according to the lifting-line theory. Finally, just as in the two-dimensional theory, an expression is obtained for the pressure distribution p_i.

It is readily shown, for instance by the procedure leading from (66) to (67), that the equation for Γ is of the following form:

$$
\Gamma(y) + \mu(k)\int_{-s}^{s}\frac{d\Gamma}{d\eta}\left\{\frac{1}{y-\eta} - ikF[k(y-\eta)]\right\}d\eta = \Gamma^{(2)}(y)
\tag{82}
$$

where $\Gamma^{(2)}$ is the value of Γ according to the two-dimensional theory as given by (61) and where the function μ is defined by

$$
\mu(k) = \frac{J_0(k) - iJ_1(k)}{\pi k[H_0^{(2)}(k) - iH_1^{(2)}(k)]}\,.
\tag{83}
$$

An equation corresponding to (82) had been obtained by Cicala [9] on the basis of entirely different considerations involving the effect of lifting lines and horseshoe vortices. Equation (82) as it stands is different from Cicala's to the extent of a difference in the expression for μ. It turns out that Cicala's result can be obtained from (81) by omitting the factor e^{-ikx} in front of the last integral in (81). In aerodynamical language this means that "the downwash induced by the spanwise variation of circulation" is assumed uniform across the chord. This is indeed correct, as has long been known, for the case of uniform motion for which $k=0$. Evidently lifting line considerations do not permit us to determine the chordwise variation of downwash referred to above and in this respect the integral equation method as outlined goes further.

Further work along the lines indicated leads to the result that the effect of three-dimensionality of the flow as determined by the foregoing approximate theory may be incorporated into equation (62) for the pressure distribution p_i of the two-dimensional theory by merely changing the function $C(k)$ into $C(k)+\sigma$. The term σ depends on the data of a given problem in the following manner:

$$\sigma = \left[\frac{\Gamma(y)}{\Gamma^{(2)}(y)} - 1\right]\left[C(k) + \frac{iJ_1(k)}{J_0(k) - iJ_1(k)}\right]. \tag{84}$$

The foregoing theory, up to equation (83) and including expressions for lift $\overline{L}$ and moment $\overline{M}_a$ as defined by (65a, b), is essentially that of [56]. In this reference there are also included discussions of earlier work of a related nature by R. T. Jones [33], Kuessner [44], von Borbely [5], and Sears [69]. Extension of the theory to surfaces of non-rectangular plan form and modification of p_s by means of the function σ was first presented in [57]. Prior to this M. W. Hunter had established that the results of [56] permitted incorporation of the effect of finite span into the expressions for $\overline{L}$, $\overline{M}_a$, and $\overline{M}_c$ by means of the function σ [26].

A considerable simplification of the present developments as compared with those in [56] and [57] is due to the fact that the basic integral equation of the problem is here taken in the form (76), which is a direct consequence of (38), rather than in that form which corresponds to equation (36).

Methods of analysis and numerical examples of application of this theory may be found in [61]. It may further be mentioned that the analogue of equation (81), for subsonic compressible flow, has recently been obtained [59].

A shortcoming of the approximate theory as discussed is the following. One would expect from an exact solution of the problem that the pressure p_s tends to zero, and therewith also circulation Γ, lift L, and moments M, as the tip sections are approached. This is indeed the case for lifting surfaces with zero tip chord, such as the elliptical surface. For lifting surfaces with finite tip chord, such as the rectangular surface, one finds however that only Γ_{tip} vanishes as it should whereas L_{tip} and M_{tip}, while smaller than according to the two-dimensional theory, can not in general be made to vanish. The reason for this difficulty is to be found in the form of the approximate equation (81) in which the effect of spanwise variation of $\bar{\gamma}$ appears solely by way of the average Γ of $\bar{\gamma}$. A more refined approximate theory undoubtedly requires inclusion of the effects of weighted averages of $\bar{\gamma}$ as well, such as $\int_{-1}^{1}x\bar{\gamma}dx$ and $\int_{-1}^{1}x^2\bar{\gamma}dx$, in the integral equation of the problem. With such a refined theory one has reason to expect that tip conditions can be satisfied to a greater degree of approximation than by the present theory, in particular, it will be possible to insure that L_{tip} and M_{tip} vanish.

Development of such a refined approximate theory is believed to be one of the worthwhile future tasks in this field. An application of this thought to the problem of *uniform* motion may be found in a note in Proc. Nat. Acad. Sci. U. S. A. vol. 35 (1949) pp. 208–215.

REFERENCES AND LITERATURE

1. M. A. Biot, *Some simplified methods in airfoil theory*, Journal of the Aeronautical Sciences vol. 9 (1942) pp. 185–190.

2. M. A. Biot and C. T. Boehnlein, *Aerodynamic theory of the oscillating wing of finite span*, GALCIT Report No. 5 to the Army Air Forces (1942).

3. W. Birnbaum, *Das ebene Problem des schlagenden Flügels*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 4 (1924) pp. 277–292.

4. S. von Borbely, *Mathematischer Beitrag zur Theorie der Flügelschwingungen*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 16 (1936) pp. 1–4.

5. ———, *Über einen Grenzfall der instationären räumlichen Tragflügelströmung*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 18 (1938) pp. 319–342.

6. ———, *Über die Luftkrafte die auf einen harmonisch schwingenden zweidimensionalen Flügel bei Überschallgeschwindigkeit wirken*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 22 (1942) pp. 190–205.

7. P. Cicala, *L'azione aerodinamiche sui profili di ala oscillante in presenza di corrente uniforme*, Memorie della Reale Accademia delle Scienze di Torino (2) vol. 68 (1934–1935) pp. 73–98.

8. ———, *Su moto non stazionario di un'ala di allungamento finito*, Rendiconti della R. Accademia Nazionale dei Lincei vol. 25 (1937) pp. 97–102.

9. ———, *La teoria e l'espirienza nel fenomeno della vibrazioni alari*, L'Aerotechnica vol. 18 (1938) pp. 412–433 [Also NACA T.M. No. 887 (1939)].

10. ———, *Zuschriften an den Herausgeber*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 19 (1939) p. 384 and vol. 20 (1940) p. 184.

11. ———, *Lo stato attuale delle ricerche sul moto instazionario di una superficie portante*, L'Aerotechnica vol. 21 (1941) pp. 671–685, 759–773.

12. F. Dietze, *Zur Berechnung der Auftriebskraft am schwingenden Ruder*, Luftfahrtforschung vol. 14 (1937) pp. 361–362.

13. ———, *Die Luftkräfte der harmonisch schwingenden in sich verformbaren Platte (Ebenes Problem)*, Luftfahrtforschung vol. 16 (1939) pp. 84–96.

14. G. Ellenberger, *Bestimmung der Luftkrafte auf einen ebenen Tragflügel mit Querruder*, Zeitschrift fur Angewandte Mathematik und Mechanik vol. 16 (1936) pp. 199–226.

15. ———, *Luftkräfte bei beliebiger instationärer Bewegung eines Tragflügels mit Querruder und bei Vorhandensein von Boen*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 18 (1938) pp. 173–176.

16. I. E. Garrick, *Propulsion of a flapping and oscillating airfoil*, NACA T.R. No. 567 (1936).

17. ———, *On some Fourier transforms in the theory of non-stationary flows*, Proceedings of the Fifth International Congress on Applied Mechanics, 1938, pp. 590–593.

18. ———, *On some reciprocal relations in the theory of non-stationary flows*, NACA T.R. No. 629 (1938).

19. I. E. Garrick and S. I. Rubinov, *Flutter and oscillating airforce calculations for an airfoil in two-dimensional supersonic flow*, NACA T.N. No. 1158 (1946).

20. ———, *Theoretical study of airforces on an oscillating or steady thin wing in a supersonic main stream*, NACA T.N. No. 1383 (1947).

21. R. Glaser, *Über die Berechnung der Koefficienten einer in der instationären Tragflügeltheorie auftretenden unendlichen Matrix*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 23 (1943) pp. 279–289.

22. H. Glauert, *The force and moment on an oscillating airfoil*, British A.R.C., R. and M. No. 1242 (1929).

23. J. M. Greenberg, *Some considerations on an airfoil in an oscillating stream*, NACA T.N. No. 1372 (1947).

24. M. D. Haskind, *Oscillations of a wing in a subsonic gas flow* (in Russian), Prikl. Math. i Mekh. vol. 11 (1947) pp. 120–146. [Translation A9-T-22 Air Materiel Command].

25. H. Hoenl, *Über das Schallfeld einer gleichförmig-translatorisch bewegten punktförmigen Schallquelle*, Annalen der Physik vol. 43 (1943) pp. 437–464.

26. M. W. Hunter, Jr., *Calculation of the aerodynamic span effect in flutter analysis*, M.S. Thesis, Massachusetts Institute of Technology (1944).

27. R. Isaacs, *Airfoil theory for flows of variable velocity*, Journal of the Aeronautical Sciences vol. 12 (1945) pp. 113–117.

28. ———, *Airfoil theory for rotary wing aircraft*, Journal of the Aeronautical Sciences vol. 13 (1946) pp. 218–220.

29. K. Jaeckel, *Über die Kräfte auf beschleunigt bewegte veränderliche Tragflügelprofile*, Ingenieur-Archiv vol. 9 (1938) pp. 371–395.

30. ———, *Über die Bestimmung der Zirkulationsverteilung für den zwei-dimensionalen Tragflügel bei beliebigen periodischen Bewegungen*, Luftfahrtforschung vol. 16 (1939) pp. 135–138.

31. ———, *Zur Theorie der tragenden Linie im Instationären*, Luftfahrtforschung vol. 19 (1942) pp. 57–63.

32. R. T. Jones, *The unsteady lift of a finite wing*, NACA T.N. No. 682 (1939).

33. ———, *The unsteady lift of a wing of finite aspect ratio*, NACA T.R. No. 681 (1940).

34. W. P. Jones, *Aerodynamic forces on wings in simple harmonic motion*, British A.R.C., R. and M. No. 2026.

35. ———, *Theoretical airload and derivative coefficients for rectangular wings*, British A.R.C., R. and M. No. 2145.

36. ———, *Wind tunnel interference effect on the values of experimentally determined derivative coefficients for oscillating airfoils*, British A.R.C., R. and M. No. 1912.

37. W. P. Jones and S. S. Skan, *Calculations of derivatives for rectangular wings of finite span by Cicala's method*, British A.R.C., R. and M. No. 1920.

38. T. von Kármán and W. R. Sears, *Airfoil theory for non-uniform motion*, Journal of the Aeronautical Sciences vol. 5 (1938) pp. 379–390.

39. S. N. Karp, S. S. Shu and H. Weil, *Aerodynamics of the oscillating airfoil in compressible flow*, Air Materiel Command T.R. No. F-TR-1167-ND (1947) [Also describes unpublished work by Biot, von Borbely, Dietze, Eichler, Frazer, Jordan, Kuessner, Schade, Schwarz, and Wieland].

40. E. A. Krasilschickowa, *Disturbed motion of air caused by vibration of a wing moving at supersonic speed* (in Russian), Prikl. Math. i Mekh. vol. 11 (1947) pp. 147–164.

41. H. Krienes and T. Schade, *Theorie der schwingenden kreisformigen Tragflache auf potentialtheoretischer Grundlage*, Luftfahrtforschung vol. 19 (1942) pp. 282–291.

42. H. G. Kuessner, *Zusammenfassender Bericht uber den instationären Auftrieb von Tragflügeln*, Luftfahrtforschung vol. 13 (1936) pp. 410–424.

43. ———, *Das zwei-dimensionale Problem der beliebig bewegten Tragfläche unter Berücksichtigung von Partialbewegungen der Flüssigkeit*, Luftfahrtforschung vol. 17 (1940) pp. 355–361.

44. ———, *Allgemeine Tragflächentheorie*, Luftfahrtforschung vol. 17 (1940) pp. 370–378 [Also NACA T.M. No. 979].

45. ———, *Zuschriften an den Herausgeber*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 19 (1939) p. 384 and vol. 20 (1940) p. 184.

46. H. G. Kuessner and L. Schwarz, *Der schwingende Flugel mit aerodynamisch ausgeglichenem Ruder*, Luftfahrtforschung vol. 17 (1940) pp. 337–354 [Also NACA T.M. No. 991].

47. C. C. Lin, E. Reissner, and H. S. Tsien, *On two-dimensional non-steady motion of a slender body in a compressible fluid*, Journal of Mathematics and Physics vol. 27 (1948) pp. 220–231.

48. J. W. Miles, *The aerodynamic forces on an oscillating airfoil at supersonic speeds*, Journal of the Aeronautical Sciences vol. 14 (1947) pp. 351–357.

49. ———, *Harmonic and transient motion of a swept wing in supersonic flow*, Journal of the Aeronautical Sciences vol. 15 (1948) pp. 343–346.

50. R. M. Morris, *The two-dimensional theory of moving airfoils*, IV, Proc. Roy. Soc. London Ser. A vol. 188 (1947) pp. 439–463.

51. I. A. Panichkin, *Forces acting on an oscillating profile in a supersonic gas flow* (in Russian), Prikl. Math. i Mekh. vol. 11 (1947) pp. 165–170 [Translation A9-T-21 Air Materiel Command].

52. C. Possio, *L'azione aerodinamica sul profilo oscillante alle velocità ultrasonore*, Acta Pontificia Accademia delle Scienze vol. 1 (1937) pp. 93–106.

53. ———, *L'azione aerodinamica sul profilo oscillante in un fluido compressible a velocità iposonora*, L'Aerotechnica vol. 18 (1938) pp. 441–458.

54. ———, *L'azione aerodinamica su una superficie portante in moto oscillatoria*, Atti dell'Accademia Nazionale Lincei, Rendiconti (VI) vol. 28 (1938) pp. 194–200.

55. E. E. Postel and E. L. Leppert, Jr., *Theoretical pressure distributions for a thin airfoil in incompressible flow*, Journal of Aeronautical Sciences vol. 15 (1948) pp. 486–492.

56. E. Reissner, *On the general theory of thin airfoils for non-uniform motion*, NACA T.N. No. 946 (1944).

57. ———, *Effect of finite span on the airload distributions for oscillating wings* I, NACA T.N. No. 1194 (1947).

58. ———, *Wind tunnel corrections for the two-dimensional theory of oscillating airfoils*, Cornell Aeronautical Laboratory Report No. SB-218-5-3 (1947).

59. ———, *Oscillating airfoils of finite span in subsonic compressible flow*, NACA T.N. to be published (1949).

60. E. Reissner and S. Sherman, *Compressibility effects in flutter*, Curtiss-Wright Research Lab. Report No. SB-240-S-1 (1944).

61. E. Reissner and J. E. Stevens, *Effect of finite span on the airload distribution for oscillating wings* II, NACA T.N. 1195 (1947).

62. T. Schade, *Theorie der schwingenden kreisförmigen Tragfläche auf potentialtheoretischer Grundlage* I, Luftfahrtforschung vol. 17 (1940) pp. 387–400.

63. W. Schmeidler, *Schwingenflug mit Wirbelablosung*, Zeitschrift fur Angewandte Mathematik und Mechanik vol. 17 (1937) p. 361.

64. ———, *Vortrieb und Widerstand*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 19 (1939) pp. 65–86.

65. ———, *Zur Theorie des Schwingenfluges*, Sitzungsberichte der Berliner Mathematischen Gesellschaft vols. 38, 39 pp. 64–71.

66. L. Schwarz, *Berechnung der Funktionen $V_1(s)$ und $V_2(s)$ für grössere Werte von s*, Luftfahrtforschung vol. 17 (1940) pp. 362–369.

67. ———, *Berechnung der Druckverteilung einer harmonisch sich verformenden Tragfläche in ebener Stromung*, Luftfahrtforschung vol. 17 (1940) pp. 379–386.

68. ———, *Untersuchung einiger mit den Zylinderfunktionen nullter Ordnung verwandter Funktionen*, Luftfahrtforschung vol. 20 (1943) pp. 341–372.

69. W. R. Sears, *A contribution to airfoil theory for non-uniform motion*, Proceedings of the Fifth International Congress on Applied Mechanics, 1938, pp. 483–487.

70. ———, *Operational methods in the theory of airfoils in non-uniform motion*, Journal of the Franklin Institute vol. 230 (1940) pp. 95–111.

71. H. Söhngen, *Bestimmung der Auftriebsverteilung für beliebige instationäre Bewegungen (Ebenes Problem)*, Luftfahrtforschung vol. 17 (1940) pp. 401–419.

72. G. Temple and H. A. Jahn, *Flutter at supersonic speeds*, British A.R.C., R. and M. No. 2140.

73. T. Theodorsen, *General theory of aerodynamic instability and the mechanism of flutter*, NACA T.R. No. 496 (1935).

74. T. Theodorsen and I. E. Garrick, *Non-stationary flow about a wing-aileron tab combination including aerodynamic balance*, NACA T.R. No. 732 (1942).

75. R. Timman, *Beschouwingen over de luchtkrachten op trillende vliegtuigvleugels*, Dissertation, Technische Hogeschool Delft, 1946, 154 pp.

76. O. (Taussky) Todd, *On some boundary value problems in the theory of the non-uniform supersonic motion of an aerofoil*, British A.R.C., R. and M. No. 2141.

77. ———, *A boundary value problem for a hyperbolic differential equation arising in the theory of the non-uniform supersonic motion of an aerofoil*, Courant Anniversary Volume, 1948, pp. 421–435.

78. H. Wagner, *Über die Entstehung des dynamischen Auftriebs von Tragflügeln*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 5 (1925) pp. 17–35.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

CONVEXITY THEOREMS

R. SALEM

In spite of the generality of our title, we do not intend to give here a survey of all convexity theorems. Most of them, like the three circles theorem of Hadamard, are too classical to be commented on here. We shall confine ourselves to Marcel Riesz's convexity theorem, which is one of the very powerful tools of modern analysis, and to certain of its recent extensions.

1. **Marcel Riesz's theorem.** Let

$$f = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$$

be a bilinear form, where the constants a_{ij} are real or complex, and the variables x_i, y_j are essentially supposed to be complex. Let $\alpha \geqq 0$, $\beta \geqq 0$ be real numbers, and denote by $M(\alpha, \beta)$ the maximum of $|f|$ under the conditions

$$\sum_{1}^{m} | x_i |^{1/\alpha} \leqq 1, \qquad \sum_{1}^{n} | y_j |^{1/\beta} \leqq 1.$$

(If $\alpha = 0$ the first condition means that $|x_i| \leqq 1$ for $i = 1, 2, \cdots, m$, and the same remark applies to the second condition.) Then $\log M(\alpha, \beta)$ is a convex function of α, β in the quadrant $\alpha \geqq 0, \beta \geqq 0$; in other words if $0 < t < 1$ and if

$$\alpha = \alpha_1 t + \alpha_2(1 - t), \qquad \beta = \beta_1 t + \beta_2(1 - t),$$

then

$$M(\alpha, \beta) \leqq M^t(\alpha_1, \beta_1) M^{1-t}(\alpha_2, \beta_2).$$

This is M. Riesz's fundamental theorem. (See M. Riesz [5][1] and a different proof in Paley [4]; see also a generalization of the theorem in L. C. Young [11].) M. Riesz's argument proved the convexity only in the triangle $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, $\alpha + \beta \geqq 1$. The extension to the whole quadrant is due to Thorin [9]. We shall not give the proof of the theorem here, since we intend to sketch later on the proof of a more general result. Let us only point out that if we restrict the

---

An address delivered before the New York Meeting of the Society on October 30, 1948, by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors November 23, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

variables to be real, the convexity theorem can not be extended beyond the domain $\alpha+\beta\geqq 1$, $\alpha\geqq 0$, $\beta\geqq 0$. For the proof of this result, we refer the reader to Thorin [10]. We add that in the classical applications of the theorem the convexity in Riesz's triangular domain is sufficient.

In its applications, M. Riesz's theorem is mostly used in a different form. Let us write

$$X_j(x) = \sum_{i=1}^{m} a_{ij}x_i \qquad (j = 1, 2, \cdots, n).$$

Let now $0\leqq\gamma\leqq 1$ and $\alpha\geqq 0$, and denote by $\mathfrak{M}(\alpha, \gamma)$ the maximum of

$$\left(\sum_1^n |X_j|^{1/\gamma}\right)^\gamma$$

when

$$\sum_1^m |x_i|^{1/\alpha} \leqq 1.$$

It is a classical consequence of Hölder's theorems that, if $\beta=1-\gamma$ and $M(\alpha, \beta)$ has the same meaning as above,

$$\mathfrak{M}(\alpha, \gamma) = \max \frac{\left(\sum_1^n |X_j|^{1/\gamma}\right)^\gamma}{\left(\sum_1^m |x_i|^{1/\alpha}\right)^\alpha} = M(\alpha, \beta)$$

and thus $\mathfrak{M}(\alpha, \gamma)$ is convex in the domain $\alpha\geqq 0$, $0\leqq\gamma\leqq 1$. (If the variables are real, the theorem is true in the domain $\alpha\geqq 0$, $0\leqq\gamma\leqq 1$, $\gamma\leqq\alpha$.)

Finally, it is hardly worth mentioning that the theorem remains valid if we consider, under the same conditions for the exponents, the maximum of

$$\frac{\left(\sum_1^n \sigma_j |X_j|^{1/\gamma}\right)^\gamma}{\left(\sum_1^m \rho_i |x_i|^{1/\alpha}\right)^\alpha}$$

where $\rho_i>0$, $\sigma_j>0$.

2. The applications of M. Riesz's theorem. The most general application of M. Riesz's theorem, which is due to M. Riesz himself, can be stated as follows.

Let us denote generally by $L^{r,\phi}(t_1, t_2)$, for $r \geqq 1$, the class of (complex-valued) functions $f(t)$ such that the Lebesgue-Stieltjes integral $\int_{t_1}^{t_2} |f|^r d\phi$, taken with respect to the nondecreasing function $\phi(t)$, is finite. By $\mathfrak{N}_{r,\phi}[f]$ we shall denote the norm $(\int_{t_1}^{t_2} |f|^r d\phi)^{1/r}$. Let $\phi(t)$ $(t_1 \leqq t \leqq t_2)$ and $\psi(u)$ $(u_1 \leqq u \leqq u_2)$ be two nondecreasing functions. Let $a \geqq 1$, $c \geqq 1$ and suppose that there exists a linear operation T associating to every $f(t) \in L^{a,\phi}(t_1, t_2)$ a function $T(f) = g(u) \in L^{c,\psi}(u_1, u_2)$ such that the ratio

$$\frac{\mathfrak{N}_{c,\psi}[g]}{\mathfrak{N}_{a,\phi}[f]}$$

is bounded independently of f. The operation T will be called of the type (a, c). The least value of this ratio, that is, the modulus of the operation, will be denoted by $M(\alpha, \gamma)$, where $\alpha = 1/a$, $\gamma = 1/c$.

Now let ϕ and ψ be fixed, and let T be an operation which is simultaneously of the types (a_1, c_1) and (a_2, c_2). Write $\alpha_i = 1/a_i$, $\gamma_i = 1/c_i$. Then, given any point (α, γ) on the segment joining $P_1(\alpha_1, \gamma_1)$ and $P_2(\alpha_2, \gamma_2)$, the operation T can be extended so as to become also of the type (a, c) where $a = 1/\alpha$, $c = 1/\gamma$. Moreover $\log M(\alpha, \gamma)$ is convex on the segment P_1P_2. (The domain of convexity is to be reduced to the triangle $0 \leqq \alpha \leqq 1$, $0 \leqq \gamma \leqq \alpha$ if the functions are real-valued.)

For the proof we refer the reader to M. Riesz's original memoir [5] or to Zygmund [13], whose convenient notations we have adopted here. Let us only point out that the result is essentially a consequence (1°) of M. Riesz's theorem; (2°) of the fact that the operation T is linear; (3°) of the fact that since the set of step functions is everywhere dense in the class $L^{r,\phi}$ $(1 \leqq r < \infty)$, we can approximate the functions of the class by step functions, and thus consider, instead of integrals, finite sums.

The most familiar consequence of this result is the Young-Hausdorff theorem, in the general form given by F. Riesz. Let $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ be an orthonormal set of complex-valued functions in (a, b), uniformly bounded, $|\phi_n| \leqq M$, and let $\int_a^b f\bar{\phi}_n dt$ be the Fourier coefficient of the complex-valued function $f(t)$ with respect to ϕ_n. If we are given any sequence of numbers $\{c_n\}$ such that $\sum |c_n|^2 < \infty$, we know that there exists an $f(t) \in L^2$ having the c_n's as Fourier coefficients; moreover $\int_a^b |f|^2 dt = \sum |c_n|^2$. If, in addition, $\sum |c_n| < \infty$, f having the same meaning, one has $|f| \leqq M \sum |c_n|$.

$f(t)$ is obtained from $\{c_n\}$ by a linear operation of the type (2, 2) with modulus 1, and of the type $(1, \infty)$ with modulus M. Hence, if

$$\begin{aligned} \alpha &= \lambda/2 + 1\cdot(1-\lambda) = 1 - \lambda/2, \\ \beta &= \lambda/2 + 0\cdot(1-\lambda) = \lambda/2, \end{aligned} \qquad (0 < \lambda < 1)$$

the operation is also of the type $(2/(2-\lambda), 2/\lambda)$ with modulus $M^{1-\lambda}$. Setting $2/(2-\lambda) = p$, $2/\lambda = p'$, one has

$$1 < p < 2,\ 1/p + 1/p' = 1,\ 1 - \lambda = 2/p - 1$$

and

$$\left(\int_a^b |f|^{p'} dt\right)^{1/p'} \leqq M^{(2-p)/p}\left(\sum |c_n|^p\right)^{1/p}.$$

The proof of the other part of the Young-Hausdorff theorem is similar.

The Young-Hausdorff theorem is valid only if the orthogonal system is uniformly bounded. A generalization of the theorem in the case of an unbounded system has been given by Marcinkiewicz and Zygmund [3]. Assuming that the functions ϕ_n satisfy the inequality

$$\left(\int_a^b |\phi_n|^{\nu} dt\right)^{1/\nu} \leqq M_n$$

for a certain $\nu > 2$, and applying in the same way as above M. Riesz's theorem, the authors get, instead of the preceding inequality

$$\left(\int_a^b |f|^{p'} dt\right)^{1/p'} \leqq \left(\sum_1^\infty M_n^{2-p} |c_n|^p\right)^{1/p} \quad (1 \leqq p \leqq 2)$$

but then p' is given by the equation

$$\frac{2-\mu}{p} + \frac{\mu}{p'} = 1$$

where $1/\mu + 1/\nu = 1$; besides, the series $\sum |c_n|^2$ is supposed to be convergent. (This condition is necessary only if the interval (a, b) is infinite.) The second part of the Young-Hausdorff theorem is generalized in a similar fashion.

Let us finally quote the following theorem, concerning Radamacher's functions and due to Zygmund [14]. Starting from the inequality

$$\left(\int_0^1 |f(t)|^k dt\right)^{1/k} \leqq k^{1/2}\left(\sum_0^\infty |c_n|^2\right)^{1/2}$$

where $f(t) = \sum_0^\infty c_n\phi_n(t)$, $\phi_n(t)$ being the nth Radamacher function, an inequality which holds whenever $\sum |c_n|^2 < \infty$ and $k \geqq 2$, and using the relation

$$|f(t)| \leqq \sum_0^\infty |c_n|,$$

the author gets

$$\left(\int_0^1 |f(t)|^k dt \right)^{1/k} \leqq k^{1/r'} \left(\sum_0^\infty |c_n|^r \right)^{1/r}$$

whenever $\sum |c_n|^r < \infty$, $1 \leqq r \leqq 2$, $k \geqq 1$, and r' is the complementary exponent to r $(1/r + 1/r' = 1)$.

3. **Thorin's generalization of M. Riesz's theorem.** Thorin's first generalization [9] not only allows the extension, which we have already mentioned, of M. Riesz's triangular domain of convexity to the whole quadrant, but, what is much more important for new applications, it shows that instead of the maximum of a bilinear form, we can consider the maximum of any entire function of n complex variables.

The theorem of Thorin can be stated as follows: Let $f(z_1, z_2, \cdots, z_n)$ be an entire function of the n complex variables $z_1, z_2, \cdots, z_n$. Let V be a bounded domain in the n-dimensional euclidean space with coordinates $v_1, v_2, \cdots, v_n$. Let us denote by $\alpha_1, \alpha_2, \cdots, \alpha_n$ nonnegative exponents, and by $M(\alpha_1, \alpha_2, \cdots, \alpha_n)$ the maximum of $|f(z_1, \cdots, z_n)|$ under the conditions

$$|z_1| = v_1^{\alpha_1}, \cdots, |z_n| = v_n^{\alpha_n} \qquad (v_1, v_2, \cdots, v_n) \in V.$$

Then $\log M(\alpha_1, \cdots, \alpha_n)$ is convex in the domain $\alpha_k \geqq 0$ $(k = 1, 2, \cdots, n)$.

The original proof of Thorin [9] was rather long. In 1944, Tamarkin and Zygmund [8] proved in a very elegant way that Thorin's theorem was a simple consequence of the maximum modulus theorem. We shall sketch here a short proof of Thorin's result which has been published recently [6].

Suppose first that $0 < A \leqq v_k \leqq B < \infty$ $(k = 1, \cdots, n)$ and let us write $z_k = e^{\alpha_k \xi_k + i\eta_k}$ $(k = 1, \cdots, n)$ where the η_k are arbitrary and the point $(\xi_1 \cdots \xi_n)$ belongs to the bounded domain D corresponding to $(e^{\xi_1}, \cdots, e^{\xi_n}) \in V$. Let $\alpha_k = a_k + \lambda_k \log t$ where the a_k and λ_k are real and fixed, and t is a positive real variable. $M(\alpha_1, \cdots, \alpha_n)$ becomes a function of t, $M(t)$, and we have to show that $\log M(t)$ is a convex function of $\log t$. Write, for $p \geqq 1$,

$$I_p(t) = \int \left| f(e^{\xi_1(a_1+\lambda_1 \log t)+i\eta_1}, \cdots, e^{\xi_n(a_n+\lambda_n \log t)+i\eta_n}) \right|^p \cdot d\xi_1 \cdots d\xi_n d\eta_1 \cdots d\eta_n,$$

the integral being extended to the domain

$$0 \leqq \eta_k \leqq 2\pi, \qquad k = 1, \cdots, n; \ (\xi_1 \cdots \xi_n) \in D.$$

If we consider now t as a complex variable, $t=\rho e^{i\sigma}$, $\rho>0$, the integral is a logarithmically subharmonic function of t, and so $I_p(t)$ is, for a determined choice of the logarithm, a subharmonic logarithmically function of t. Moreover this function is uniform, and $I_p(t)=I_p(\rho)$, thus depending only on the modulus of t. For:

$$\xi_k(a_k + \lambda_k \log \rho + \lambda_k i\sigma) + i\eta_k = \xi_k(a_k + \lambda_k \log \rho) + i(\eta_k + \xi_k\lambda_k\sigma)$$

and our assertion is justified by the fact that the integral is extended to all possible values of η_k, mod 2π. Now $\log I_p(\rho)$ being subharmonic, $\log I_p(\rho)$ is a convex function of $\log \rho$; and since $M(t) = \lim_{p\to\infty} [I_p(t)]^{1/p}$ it is sufficient to let p increase infinitely to obtain the asserted result.

The argument remains valid without change if, instead of supposing that $f(z_1, \cdots, z_n)$ is analytic, we assume only that $\log |f|$ is subharmonic, that is, that $|f(z_1 \cdots z_n)|$ is logarithmically subharmonic (in the paper quoted in [6] the word "logarithmically" has been omitted and should be restored). According to Lelong (Ann. École Norm. vols. 61–62 (1945) pp. 301–338) $g(z_1 \cdots z_n)$ is subharmonic if $g[\phi_1(t) \cdots \phi_n(t)]$ is subharmonic for analytic functions $\phi_1(t) \cdots \phi_n(t)$.

As Thorin has pointed out, the convexity holds if we suppose only $v_k \geqq 0$ $(k=1, \cdots, n)$ provided we restrict the domain of convexity from $-\infty<\alpha_k<\infty$ to $\alpha_k \geqq 0$ $(k=1, \cdots, n)$.

In his recent thesis [10] Thorin has given further extensions of the theorem, especially to analytic and subharmonic functionals. We refer the reader to Thorin's paper for these extensions which would require too much space to be quoted here with accuracy.

4. **Applications of Thorin's generalization.** In order to understand the most interesting applications of Thorin's generalized convexity theorem, let us recall the following notations. A function $f(z)$ of the complez variable z, regular for $|z|<1$, is said to belong to H^p $(p>0)$ if $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ is bounded as $r\to 1$. One writes H instead of H^1. A series (Σ) of the form $\sum_0^\infty c_n e^{ni\theta}$, which is formally $S+iT$, where S and T are conjugate trigonometric series, is said to belong to H^p if $f(z)=\sum_0^\infty c_n z^n \in H^p$. If S (or T) is the Fourier series of a function of

the class L^p $(p>1)$, then T(or S) belongs also to L^p and $\Sigma\in H^p$; this, however, is false for $p=1$. If $\Sigma\in H^p$ $(p\geqq 1)$, both S and T are Fourier series of the class L^p. In other words the properties H^p for Σ and L^p for S are equivalent for $p>1$ but not for $p=1$.

Now, certain theorems are true for power series of the class H but not for Fourier series of the class L. As an important example, we quote the following theorem due to Zygmund [12]. If $S_n(\theta)$ is the partial sum of order n of the series $f(e^{i\theta})\sim\sum_0^\infty c_\nu e^{i\nu\theta}\in H$, then

$$\int_0^{2\pi}\frac{|S_{n(\theta)}(\theta)|}{\log[n(\theta)+2]}d\theta\leqq C_1\int_0^{2\pi}|f(e^{i\theta})|\,d\theta$$

the constant C_1 being independent of f, of the (measurable) function $n(\theta)$ $(0\leqq n(\theta)\leqq N)$, and of N. No such theorem is true for Fourier series of the class L. On the other hand, if $f\in H^2$, one has [1]

$$\int_0^{2\pi}\frac{|S_{n(\theta)}(\theta)|^2}{\log[n(\theta)+2]}d\theta\leqq C_2\int_0^{2\pi}|f(e^{i\theta})|^2d\theta.$$

If we could apply a convexity theorem here, we would deduce

$$\int_0^{2\pi}\frac{|S_{n(\theta)}(\theta)|^p}{\log[n(\theta)+2]}\leqq C_p\int_0^{2\pi}|f(e^{i\theta})|^pd\theta$$

for $f\in H^p$ and $1\leqq p\leqq 2$, C_p depending on p only; and the theorem would be also true, by what we have said, for Fourier series of the class L^p, but only for $1<p\leqq 2$. Such a theorem is indeed true; in an important paper [2] Littlewood and Paley have given an extremely difficult proof of it. Hence it is of interest to prove the last inequality (which has important consequences in the theory of convergence of Fourier series) in a simple way by a sort of convexity theorem.

The difficulty of the problem lies in the fact that, although the passage from $f(e^{i\theta})$ to $S_{n(\theta)}(\theta)$ is linear, Marcel Riesz's theorem cannot be applied in the same manner as in §2 above, for the following reason: f is not *any* Fourier series with complex coefficients; it is a power series, and for this reason cannot be approximated by step functions, but only by polynomials of the form $\sum_0^n c_\nu e^{i\nu\theta}$.

It is, however, possible to overcome this difficulty, and we have the following general theorem [7]:

Let $M(\alpha,\beta)$ $(\alpha>0,\beta\geqq 0)$ denote the maximum of the modulus of the complex bilinear form

$$\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jk}x_jy_k$$

under the conditions $\sum_0^n |y_k|^{1/\beta} \leqq 1$, $\int_0^{2\pi} |f(re^{i\theta})|^{1/\alpha} \leqq 1$ $(0 \leqq r < 1)$ when $f(z) = \sum_0^\infty x_\nu z^\nu$ belongs to $H^{1/\alpha}$ and has $x_0, x_1, \cdots, x_m$ as its first $m+1$ expansion coefficients. Then, if $\alpha = t\alpha_1 + (1-t)\alpha_2$, $\beta = t\beta_1 + (1-t)\beta_2$, one has

$$M(\alpha, \beta) \leqq C(\alpha_1, \alpha_2) M^t(\alpha_1, \beta_1) M^{1-t}(\alpha_2, \beta_2),$$

$C(\alpha_1, \alpha_2)$ being a finite constant depending on α_1 and α_2 only.

This is not exactly a convexity theorem, owing to the introduction of the constant $C(\alpha_1, \alpha_2)$; but its usefulness is the same since it asserts the finiteness of $M(\alpha, \beta)$ when (α, β) lies on the segment $(\alpha_1, \beta_1) - (\alpha_2, \beta_2)$ and when $M(\alpha_1, \beta_1)$ and $M(\alpha_2, \beta_2)$ are both finite.

We shall indicate briefly the ideas underlying the proof. We start from:

$$\left| \sum_{j=0}^{m} \sum_{k=0}^{n} a_{jk} x_j y_k \right| \leqq M(\alpha, \beta) \left[\int_0^{2\pi} |f(e^{i\theta})|^{1/\alpha} d\theta \right]^\alpha \left[\sum_0^n |y_k|^{1/\beta} \right]^\beta$$

valid for $\alpha = \alpha_1$, $\beta = \beta_1$ and $\alpha = \alpha_2$, $\beta = \beta_2$, and we want to prove that a similar inequality holds for $\alpha = \alpha_1 t + (1-t)\alpha_2$, $\beta = \beta_1 t + (1-t)\beta_2$. The first natural idea is to prove the theorem when f is a polynomial of degree p, and to replace the integral by a sum of $p+1$ terms, which would allow us to take the elements of this sum (more exactly: the values of the polynomial at the points of subdivision of the interval $(0, 2\pi)$, values which are linear combinations of the x_j) as new variables. The difficulty lies in the fact that the ratio between the integral and the approximating sum lies between two constants only if the exponent $1/\alpha$ stays larger than a constant $q > 1$. Hence we use the idea, which has been used first by Thorin in the proof of a particular result of the same kind [**10**, pp. 31–35], of writing $f(z) = G^k(z)$. By G we shall denote here a polynomial $\xi_0 + \xi_1 z + \cdots + \xi_p z^p$ and by k the smallest integer such that $k/\alpha_1 \geqq 2$, $k/\alpha_2 \geqq 2$. We can then apply the method just indicated of approximation of the integral by a finite sum, and Thorin's extension of M. Riesz's convexity theorem is essential here, because the x_j become functions $\phi_j(\xi_0, \cdots, \xi_p)$ of the variables $\xi_0, \cdots, \xi_p$ which are homogeneous and of degree k (and thus the function of the ξ's and y's is no longer bilinear).

The inequality once proved for $f = G^k(z)$, G being a polynomial, remains true when f is any function of the class $H^{1/\alpha}$ having no zeros for $|z| < 1$. One has afterwards to remember only that every $f \in H^{1/\alpha}$ is the sum of two zero-free functions f_1 and f_2 of the same class, with $|f_1| \leqq 2|f|$, $|f_2| \leqq 2|f|$.

Finally, if one writes $X_h = \sum_{j=0}^m a_{jh} x_j$ $(h = 0, \cdots, n)$ it is familiar that $M(\alpha, \beta)$ is also the maximum of the ratio

$$\left(\sum_0^n |X_k|^{1/\gamma}\right)^\gamma \bigg/ \left[\int_0^{2\pi} |f(e^{i\theta})|^{1/\alpha} d\theta\right]^\alpha$$

where $\gamma = 1-\beta$. From sums of powers of linear forms we can pass to integrals, and the theorem of Littlewood and Paley is the consequence of our result applied between $\alpha=\gamma=1/2$ and $\alpha=\gamma=1$.

The same general result is immediately applicable to the proof of a theorem of Hardy-Littlewood proved directly by Thorin in his thesis [10]. If $f \sim \sum_0^\infty c_\nu e^{i\nu\theta}$, one has:

$$\sum \frac{|c_\nu|}{\nu+1} \leqq A_1 \int_0^{2\pi} |f|\, d\theta \qquad (f \in H),$$

$$\left(\sum |c_\nu|^2\right)^{1/2} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f|^2 d\theta\right)^{1/2} \qquad (f \in H^2);$$

this means that

$$\left[\sum \frac{1}{(\nu+1)^2}\left[|c_\nu|\,(\nu+1)\right]^{1/\gamma}\right]^\gamma \leqq A(\alpha,\gamma)\left(\int_0^{2\pi} |f|^{1/\alpha} d\theta\right)^\alpha$$

is valid for $\alpha=1$, $\gamma=1$ with $A(1,1)=A_1$; and for $\alpha=1/2$, $\gamma=1/2$ with $A(1/2,1/2)=(2\pi)^{-1/2}$. Hence applying our result we find that, if $1 \leqq p \leqq 2$, there is a constant A_p depending on p only such that

$$\sum \frac{|c_\nu|^p}{(\nu+1)^{2-p}} \leqq A_p \int_0^{2\pi} |f|^p d\theta.$$

Of course, if $1 < p \leqq 2$ (but not for $p=1$) this is also true when $f \in L^p$ and the c_ν are the complex Fourier coefficients of f.

Finally, let us mention that, using his generalizations of Marcel Riesz's convexity theorem, Thorin, in his thesis [10], has given a new proof (and even an extension to several dimensions) of Hardy and Littlewood's inequality

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)g(y)}{|x-y|^{2-\alpha-\beta}}\, dx dy\right|$$

$$\leqq K(\alpha,\beta)\left(\int_{-\infty}^{\infty} |f(x)|^{1/\alpha} dx\right)^\alpha \left(\int_{-\infty}^{\infty} |g(y)|^{1/\beta} dy\right)^\beta$$

$$(\alpha+\beta > 1,\ \alpha < 1,\ \beta < 1,\ f \in L^{1/\alpha},\ g \in L^{1/\beta} \text{ in } (-\infty, \infty)).$$

This proof, however, is based not only on a convexity theorem but also on a deep result of Marcinkiewicz in the theory of Fourier series.

REFERENCES

1. G. H. Hardy and J. E. Littlewood, Proc. Cambridge Philos. Soc. vol. 40 (1944) pp. 103–107.

2. J. E. Littlewood and R. E. A. C. Paley, Proc. London Math. Soc. vol. 43 (1937) p. 105.

3. J. Marcinkiewicz and A. Zygmund, *Some theorems on orthogonal systems*, Fund. Math. vol. 28 (1937) pp. 309–335.

4. R. E. A. C. Paley, *A proof of a theorem on bilinear forms*, J. London Math. Soc. vol. 6 (1931).

5. M. Riesz, *Sur les maxima des formes bilineaires et sur les fonctionelles lineaires*, Acta Math. vol. 49 (1927).

6. R. Salem, *Sur une extension du théorème de convexité de M. Marcel Riesz*, Colloquium Mathematicum vol. 1 (1947) pp. 6–8.

7. R. Salem and A. Zygmund, *A convexity theorem*, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 443–447.

8. J. D. Tamarkin and A. Zygmund, *Proof of a theorem of Thorin*, Bull. Amer. Math. Soc. 50 (1944) pp. 279–282.

9. G. O. Thorin, *An extension of a convexity theorem due to M. Riesz*, Kungl. Fysiografiske Sallskapets i Lund Forhandlinger vol. 8 (1939) p. 14.

10. ———, *Convexity theorems*, Meddelanden fran Lunds Universitets Matematiska Seminarium, Thesis, Uppsala, 1948.

11. L. C. Young, *On an inequality of Marcel Riesz*, Ann. of Math. vol. 40 (1939).

12. A. Zygmund, Fund. Math. vol. 30 (1938) p. 190.

13. ———, *Trigonometrical series*, Warsaw, 1935.

14. ———, Prace Matematyczno-Fizyczne vol. 44 (1935).

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

THIRD SYMPOSIUM ON APPLIED MATHEMATICS

The Third Annual Symposium on Applied Mathematics of the American Mathematical Society was held at the University of Michigan, Ann Arbor, Michigan, Tuesday to Thursday, June 14–16, 1949. The subject of the Symposium was *Elasticity*. Attendance at the sessions ranged from 70 to more than 200. The following 133 persons, including 84 members of the Society, registered:

M. K. Al-Ghita, Maurice Anthony, R. C. F. Bartels, M. V. Barton, C. J. Blackall, F. E. Bothwell, F. C. Bragg, J. R. Britton, Bernard Budiansky, Jean Burnett, R. E. Carr, G. F. Carrier, K. V. Casey, Abraham Charnes, P. P. Chen, P. F. Chenea, R. V. Churchill, R. A. Clark, Nathaniel Coburn, C. J. Coe, H. G. Cohen, S. D. Conte, J. M. Crowley, N. C. Dahl, A. R. Decker, M. L. DeMoss, C. L. Dolph, L. H. Donnell, A. C. Downing, R. F. Dressler, D. C. Drucker, W. S. Ericksen, E. L. Eriksen, A. C. Eringen, F. D. Faulkner, G. S. Fenn, N. C. Fisk, J. S. Frame, D. M. Friedlen, K. O. Friedrichs, M. M. Frocht, I. M. Garfunkel, J. N. Goodier, L. E. Goodman, Lois Graham, D. F. Gunder, G. E. Hay, F. B. Hildebrand, T. H. Hildebrandt, H. H. Hilton, P. G. Hodge, D. L. Holl, C. C. Hurd, W. H. Ito, A. W. Jacobson, L. A. Jehn, Wilfred Kaplan, Leo Katz, Jean Kestens, W. M. Kincaid, J. T. Korkland, J. L. Lawrence, E. H. Lee, Harold Liebowitz, C. C. Lin, B. J. Lockhart, Duncan Luce, H. W. March, J. W. Marshall, H. L. Mason, E. F. Masur, Angelo Mennone, C. C. Miesse, Eugene Migotsky, R. D. Mindlin, M. V. Morkovin, E. J. Moulton, Manuel Murta, Douglas Muster, William Nash, E. A. Nordhaus, G. H. Nothmann, E. N. Oberg, W. R. Osgood, T. P. Palmer, W. H. Pell, Clay Perry, Hillel Poritsky, William Prager, Y. D. Prian, P. M. Quinlan, G. Y. Rainich, Herbert Reissmann, Eric Reissner, H. J. Reissner, M. M. Resnikoff, J. K. Riess, Felix Rosenthal, E. H. Rothe, T. R. Running, M. A. Sadowsky, Ward Sangren, Theodore Schiffman, H. M. Schwartz, B. R. Seth, H. S. C. Sharp, S. S. Shu, K. M. Siegel, Richard Skalak, W. K. Smith, I. S. Sokolnikoff, George Sonnemann, R. V. Southwell, F. L. Spitzer, Eli Sternberg, Marvin Stippes, J. J. Stoker, A. C. Sugar, P. C. Sweetland, P. S. Symonds, G. L. Thompson, C. J. Thorne, S. P. Timoshenko, H. M. Trent, Andrew Van Tuyl, E. G. Volterra, W. G. Warnoch, Alexander Weinstein, C. P. Wells, E. T. Welmers, A. W. Wundheiler, Dana Young, G. S. Young.

The Symposium was co-sponsored by the Applied Mechanics Division of the American Society of Mechanical Engineers which held its Fifteenth Applied Mechanics Division Conference at the University of Michigan, Monday to Wednesday, June 13–15.

The West Quadrangle Residence Halls and the Michigan Union of the University of Michigan together with the Cafeteria and Dining Room of the Michigan Union were available to those attending the Symposium and to their families.

Those attending the Symposium and the Applied Mechanics Division Conference and their families were guests of the Department of Mathematics at a lawn party near the campus on Tuesday evening.

Professor R. C. F. Bartels of the University of Michigan was in charge of local arrangements.

The program of the Symposium consisted of one sixty-minute address, four forty-minute addresses, twelve twenty-minute addresses, all presented by invitation of the Committee on Program and Arrangements, and four thirty-minute addresses, one by title, contributed from the program of the Conference of the Applied Mechanics Division of the American Society of Mechanical Engineers. The programs on Tuesday and Thursday consisted of one morning and one afternoon session on each day. On Wednesday there were two morning sessions and one afternoon session. All three sessions on Wednesday were held jointly with the Applied Mechanics Division Conference. Each paper was followed by a discussion.

The twenty-one papers were presented in the following order:

1. Eric Reissner: *On axi-symmetrical deformations of thin shells of revolution.* (40 minutes)
2. H. W. March: *The elastic stability of the facings of sandwich columns.* (20 minutes)
3. J. J. Stoker: *Pre-stressing of a plane circular plate to stiffen it against buckling.* (20 minutes)
4. I. S. Sokolnikoff: *An approximate method of solution of contact problems in anisotropic elasticity.* (40 minutes)
5. Alexander Weinstein: *New methods for the estimation of torsional rigidity.* (20 minutes)
6. F. B. Hildebrand: *On asymptotic integration in shell theory.* (20 minutes)
7. T. J. Jaramillo: *Deflections and moments due to a concentrated load on a cantilever plate of infinite length.* (30 minutes) Presented by Professor Eli Sternberg.
8. C. C. Miesse: *Determination of the buckling load for columns of variable stiffness.* (30 minutes)
9. L. H. Donnell and C. C. Wan: *Effect of imperfections on buckling of thin cylinders and columns under axial compression.* (30 minutes) Presented by L. H. Donnell.
10. S. Dokos: *A force applied to the median plane at the center of a circular disk in a plate.* (Presented by title)
11. Sir Richard Southwell: *On the computation of strain and displacement in a prism plastically strained by torsion.* (60 minutes)
12. William Prager and P. S. Symonds: *Stress analysis in elastic-plastic structures.* (40 minutes) Presented by William Prager.
13. K. O. Friedrichs: *Kirchhoff's boundary condition and the edge effect for elastic plates.* (20 minutes)
14. G. E. Hay: *Beams under concentrated loading.* (20 minutes)

15. B. R. Seth: *Some recent applications of finite elastic deformation.* (40 minutes)

16. G. F. Carrier and F. S. Shaw: *Some problems in the bending of thin plates.* (20 minutes) Presented by G. F. Carrier.

17. Nathaniel Coburn: *A graphical method for solving problems in plane plasticity.* (20 minutes)

18. D. L. Holl: *Dynamic loads on thin plates on elastic foundation.* (20 minutes)

19. C. L. Perry: *The bending of thin elliptic plates.* (20 minutes)

20. Hillel Poritsky: *Torsion problems with axial symmetry.* (20 minutes)

21. E. H. Lee: *On stress discontinuities in plane plastic flow.* (20 minutes)

The presiding officers at the sessions were, respectively: Professors I. S. Sokolnikoff, William Prager, Dr. W. R. Osgood, Professors J. N. Goodier, D. L. Holl, K. O. Friedrichs, and Eric Reissner.

At the close of the Thursday afternoon session a resolution of thanks and appreciation to the University of Michigan for its hospitality was presented by Professor William Prager and unanimously approved.

All papers listed above except numbers 3 and 7 to 11 will be published as Vol. 3 of the Proceedings of a Symposia in Applied Mathematics. References will be given to the six papers not published in this volume.

R. V. CHURCHILL,
Chairman of the Committee on Program and Arrangements

J. W. T. YOUNGS,
Associate Secretary of the Society

BOOK REVIEWS

Les méthodes modernes du calcul des probabilités et leur application au problème de la turbulence. By J. Bass. (Groupement français pour le développement des Recherches Aéronautiques, rapport technique no. 28.) Paris, 1946, 241 pp.

This is a technical report on methods of probability theory and their application to the problem of turbulence. It completely fulfills its purpose which is to give to the physicists a thorough picture of the theories involved as they stood in 1944. It contains also indications of the contributions of the author and of other French mathematicians in 1945 but nothing about the contributions of Kolmogoroff and his school, unknown in France at the date of publication (1946). The author does not intend to give complete and rigorous proofs but mostly indicates the lines along which such proofs have been obtained.

The first part deals with probability theory. In the first four chapters the author treats stochastic processes with independent increments and Markoff chains. Chaps. V to VIII are concerned with random functions: stochastic differentiation and integration, and the stationary case.

The second part deals with the statistical theories of turbulence. It starts with the Navier equations and the Reynolds theory, examines the Taylor theory, then the Dedebant and Wehrlé theory, and finishes with the spectrum and the levels of perturbation.

M. Loève

Rings and ideals. By N. H. McCoy. (Carus Mathematical Monographs, no. 8.) Buffalo, Mathematical Association of America, 1948. 12+216 pp.

A ring is a set of elements which forms an abelian group under addition and is closed under multiplication which is associative and distributive. Postulates for a ring differing only slightly from the above were given by Fraenkel in 1914.

The modern theory of linear associative algebras dates from Wedderburn's thesis of 1907. A linear associative algebra is a ring which is specialized by possessing a finite basis relative to a field. This basis plays an essential part in Wedderburn's treatment and in most of the work which has followed it. Several attempts have been made, one by Wedderburn himself, to extend the theory to algebras with an infinite basis. Probably the most successful generalization was made by Artin in 1927 by ignoring the basis and extending the main features of the Wedderburn structure theory to rings satisfying the chain conditions. Since then the theory of rings has developed steadily through the researches of many writers such as Noether, Krull, Köthe, von Neumann, Albert, Brauer, Jacobson, and McCoy, to mention only a few. That so many detailed theorems can be obtained from such general hypotheses is indeed remarkable, and attests the power of the abstract analysis.

The concept of ideal had its genesis in the study of algebraic numbers, where it was introduced by Kummer and Dedekind. Wedderburn based his theory of algebras on the concept of invariant subalgebra but later writers, recognizing the similarity of the two concepts, have used the word ideal in both connections.

The book under review is an excellent introduction to this modern branch of algebra. Seldom does one have the pleasure of reading a book which has been written with so much care and expository skill. The difficulties have been so carefully anticipated that the reader is apt to get a false impression of the simplicity of the material.

There are nine chapters in the course of which the pace is gradually increased. The first two involve well known material freshly presented. In Chapter III on *Ideals and homomorphisms* some of the fundamental properties of finite fields are very simply obtained. Chapter IV deals with the imbedding of one ring in another, particularly with quotient rings. The plot begins to thicken in Chapter V, *Prime ideals in commutative rings*. Use is made of Zorn's Lemma, which McCoy calls the Maximum Principle. It is a form of Zermelo's axiom.

Chapter VI is entitled *Direct and subdirect sums*. This is material in the development of which the author has taken part. A result startlingly similar to the Wedderburn decomposition theorem for linear algebras holds in a commutative ring R. If $\mathfrak{y} \neq R$ is the Jacobson radical, $R/\mathfrak{y}$ is isomorphic to a subdirect sum of fields.

Chapter VII is devoted to Boolean rings and the algebra of logic, and their generalizations, the p-rings. Chapter VIII is concerned with rings of matrices whose elements belong to an arbitrary ring R. For the introduction of determinants it becomes necessary to assume that R is commutative with more than one element, and that the unit element 1 is present. With these assumptions most of the theory of determinants and systems of equations carry over, but not without serious modifications in the usual theory. The extension of the concept of rank is particularly ingenious, but the reviewer will refrain from revealing the plot. The reader may find it on page 159. The author's own researches on the characteristic ideal and the null ideal of a matrix are developed in this chapter.

The concluding chapter is entitled *Further theory of ideals and commutative rings* and is fairly concentrated. It is concerned with Noetherian rings and algebraic manifolds and is designed, probably, to leave the reader in a more humble frame of mind.

It is surprising that this book and the one by Jacobson (*The theory of rings*, Mathematical Surveys, no. 2, New York, 1943) overlap so little. Doubtless McCoy planned it that way. Jacobson had more space at his disposal and his book has not been supplanted for reference purposes. But the McCoy book has many novel points of view and some more recent material, and as an introduction to the powerful and highly abstract method of thinking which now characterizes modern algebra, it is a gem.

C. C. MACDUFFEE

Mathematical analysis of binocular vision. By R. K. Luneburg. Princeton University Press, 1947. 6+104 pp. $2.50.

An attempt is made in this study to derive a metric for the psychological space of binocular vision. It is first shown that the recognition of greater and smaller and of greater and smaller contrast uniquely determines the psychometric coordination of numbers to sensation within the limits of a linear transformation. In order to proceed there is then introduced a rather strong hypothesis which is suggested by some experimental observations.

Let an observer first view a point P with head fixed. For this situation a convenient set of coordinates is γ, ϕ, θ, where γ is the angle of convergence (angle LPR, L and R representing the eyes), ϕ is a lateral angular deviation $(PLR/2-PRL/2)$, and θ is the angular elevation of the plane PRL from the horizontal plane. Next let the observer be permitted to rotate his head about a vertical axis so that the eyes converge symmetrically on P. For this situation another set of coordinates γ^*, ϕ^*, θ^* is introduced. The corresponding angles are very similarly defined. In the transformation from cartesian coordinates to these starred coordinates the distance d' between the line through the eyes and the axis of rotation enters as a parameter.[1]

Now suppose we have two different configurations of object points, the first being viewed with fixed head, the second being viewed with rotating head. If there is a correspondence between the two such that γ for the first equals γ^* for the second and similarly $\phi=\phi^*$, $\theta=\theta^*$, then the hypothesis is that the two configurations lead to the same sensation. Since elements of length appear to be equal, then if the apparent size is determined by a quadratic differential form, it follows that

---

[1] It would seem that d' could be experimentally made to take on values from zero, or even less, to many times the normal value. If so it would be of interest to determine whether or not the hypothesis would hold when the effect is accentuated by choosing extreme values of d'.

$$g_{11}d\gamma^2 + g_{22}d\phi^2 + g_{33}d\theta^2 + \cdots = g_{11}d\gamma^{*2} + g_{22}d\phi^{*2} + g_{33}d\theta^{*2} + \cdots,$$

where the coefficients $g_{ik}(\gamma, \phi, \theta)$ on the left are the same functions as $g_{ik}(\gamma^*, \phi^*, \theta^*)$ on the right. This leads to a non-Euclidean metric in general. The observation that two line elements on the same line of view which give the same $d\gamma$, $d\phi$ are seen as parallel, together with considerations of symmetry, leads to the quadratic differential form

$$ds^2 = M^2(\gamma)(\sigma^2 d\gamma^2 + d\phi^2 + \cos^2 \phi d\theta^2),$$

in which σ is a parameter of the observer which measures the relative effectiveness of convergence as compared with angular displacement in the estimation of length. Further considerations suggest that $M(\gamma) = 1/\sin h\sigma(\gamma+\mu)$, in which case the metric is hyperbolic.

Some of the topics which are treated in considerable detail are the horopter problem (geodesic lines), the alley problem, and rigid transformations of the hyperbolic visual space. In the alley problem with walls sensed as parallel it is pointed out that the meaning of "parallel" is ambiguous. Two cases are treated which correspond to different instructions to the subject. These in general give different results, the difference depending on the geometry chosen. Thus, while it is possible to account for apparently conflicting experimental results, it is also possible to make use of the experimental data to obtain restrictions on the metric. Using the hyperbolic metric, the author calculates the shape of distorted rooms which are congruent to rectangular rooms, that is, rooms with distorted walls and windows which appear (under fixed conditions) to be identical to rectangular rooms with rectangular windows.

Even if further experiments show that the metric derived is not adequate to account for the data, the author's efforts will greatly facilitate the task of determining a better approximation. The point of view developed, together with the many suggestions given, should prove to be of great help in determining the direction to be taken for further theoretical studies and experimental observations.

H. D. LANDAHL

Leçons de géometrie différentielle. Vol. 1. By G. Vranceanu. Bucarest, Rotativa, 1947. 422 pp.

This volume, which is the first of two proposed volumes, is divided into two parts. The first part (Chapters I, II, III) deals with the author's topics of predilection; namely congruences, finite continuous groups, and equivalence problems of congruences. The second part (Chapters IV, V, VI) may be thought of as a report on metric, affine and projective differential geometry of curved spaces. It is written in a style which reminds one of the charm, as well as of the disadvantages, of Felix Klein's lithographed lectures. It may be labelled as one of the most attractive books on geometry, written in the style of pre-epsilon-delta era.

In the first chapter an informal approach is given to the fundamental concepts necessary for the calculus of congruences: Tensor properties, Pfaffians and the corresponding bilinear covariants. Poisson's brackets, and so on.

The second chapter deals with Lie groups. The author starts with a one-parameter Lie group and develops the theory of an r-parameter Lie group with all its fundamental concepts. The classical Lie theorems are explained in an intuitive way using the notion of the neighborhood of identity. The structural tensors C_{ab}^{e}, C_{ab}^{b}, C_{ab}^{d} C_{de}^{b} are investigated and corresponding topics, such as derived groups, center, characteristic equation, are introduced. A complete study of a G_3 (canonical forms) is presented. The framework of this chapter is strictly a classical one. The modern idea due to Chevalley (Pontrjagin) of considering a Lie group as an entity is not adopted, and consequently the Cartan-Schouten geometry of group space is not studied.

The last section of this chapter is devoted to the concept of primitivity and transitivity of a group with subsequent developments and applications on G_3.

The third chapter is devoted to the problem of equivalence of congruences. The author uses throughout this chapter the calculus with linear Pfaffian forms and gives many examples which illustrate this theory. In particular, he ends this chapter by a thorough study of the invariants of an equation $y'' = F(x, y, y')$.

In the fourth chapter the author introduces the elementary concepts dealing with an affine curved space L_n, covariant derivative, parallelism,[1] curvature, and torsion. Moreover, as is his predilection, he introduces a topic which may be described as a geometry of L_n+ennuples.

The investigation is sometimes rather superficial (pp. 224–225), sometimes is dealt with by an unusually thorough treatment (pp. 249–254). A relatively large part is devoted to the Cartan spaces with covariant constant curvature tensor. The statement on "Fermi's theorem" (p. 219) is erroneous—for there is always a coordinate system $\bar{x}^a$ along a given open curve C for which $\bar{\Gamma}^a_{bc}=0$ along C *even in the case if C is not a path.*

---

[1] G. Hessenberg may be thought of as a godfather to the idea of parallelism which was discovered in 1917 by Levi-Civita and independently of Levi-Civita by J. A. Schouten in 1918 (reviewer's remark).

In the fifth chapter the author confines himself to just classical results about a Riemann space: geodesics, curvature, the embedding of a V_n in a euclidean space E_N and the Levi-Civita deformation of geodesics. The embedding problem does not go beyond the classical elementary standpoint, where all the osculating spaces, as well as all the normal spaces, are mixed up. Incidentally, the E. E. Levi invariants $I_{klr}\ldots$ are not invariant with respect to parameter transformations of the V_n. The more recent and deeper results of W. Mayer, Tucker, T. Y. Thomas, Schouten-Struik are not discussed. In the last section the author reproduces in his own way the Levi-Civita deformation theory of geodesics. The resulting necessary and sufficient conditions (say $L_2^\nu=0$) are differential equations of second order. The reviewer would like to remark that in the case of a congruence they may be replaced by a system of order 1, say $L_1^\nu=0$.[2]

It seems to the reviewer that the topics being too familiar, the author hastened too much in their treatment, for there are some points in this chapter which have to be corrected: The equation (3), p. 298 is misleading. The formulae (4′) p. 299 and (5) p. 302 lead in the case of a non-definite metric to $|\cos\theta|>1$ (!). Page 305: "En désignant . . . les differentielles des arcs sur deux congruences nulles. . . . " The nonzero arc is uniquely defined (p. 300) only for non-null curves. The last formula on p. 317 is misprinted and the first formula on p. 318 is wrong but this fact does not affect the right result.[3]

The last chapter deals with the projective connection. There are different treatments of this topic: that of the Princeton school including Veblen's and T. Y. Thomas' theories, Cartan's method of the "repère mobile," Schouten-v. Dantzig algorithm which uses homogeneous coordinates. The author confines himself to the introductory elements of the Princeton school and Cartan's method and uses the ennuple calculus to investigate thoroughly the one- and two-dimensional spaces. The approach to the projective connection is very intuitive but not a satisfactory one: The author obtains the projective connection "En considérant . . . les espaces euclidiens E_P and E_Q de deux points voisins P et Q et en désignant avec X_0^i les coordonnées de Q dans $E_P \cdots$" (p. 372). This kind of heuristic[4] assumption has to be avoided in a modern mathematical book.[5]

---

[2] If dealing with a congruence, we have $L_2^\nu=DL_1^\nu+L_1^\alpha D_\alpha i^\nu$, where D stands for covariant derivative along the curve to be deformed and i^ν is its unit tangential vector. (See Hlavatý, *Les courbes de la variété générale à n dimensions*, Mémorial des Sciences Mathématiques, no. 43, Paris, 1934, pp. 47–63.) Hence $L_1^\nu=0$ is a sufficient but not necessary condition.

[3] Some minor mistakes or misprints may be easily checked and corrected by the reader himself.

The book is clearly meant for specialists. Any specialist will enjoy it for its large number of topics.

V. HLAVATÝ

---

[4] The assumption mentioned in the text above is mathematically incomprehensible to the reviewer. It seems to be described fully by the following argument: "Let Q be so close to P that it is incident with E_P, in spite of the fact that $E_P \neq E_Q$." No exact mathematical proof may be based on such contradictory requirements.

[5] The same remark holds for the author's approach to the affine connection.

ABSTRACTS OF PAPERS

The abstracts below are abstracts of papers presented by title at the Fifty-Fifth Summer Meeting of the American Mathematical Society. Abstracts of papers presented in person at that meeting will be included in the report of the meeting which will be published in the November issue of this BULLETIN.

Abstracts are numbered serially throughout this volume.

ALGEBRA AND THEORY OF NUMBERS

445. W. V. Parker: *On matrices whose characteristic equations are identical.*

Two generalizations of an earlier theorem (Bull. Amer. Math. Soc. vol. 55 (1949) p. 115) are given. (1) Let A be an $n \times m$ matrix of rank $r<n$ and let C be an $m \times n$ matrix such that $ACA=kA$ (k a scalar). If B is an $m \times n$ matrix, the characteristic equation of AB is $x^{n-r}\phi(x)=0$ and the characteristic equation of $A(B+C)$ is $x^{n-r}\phi(x-k)=0$. (2) If M and N are square matrices such that MN (or NM) $=N^2=0$, then M and $M+N$ have the same characteristic equation. (Received July 5, 1949.)

446. H. J. Ryser: *A note on a combinatorial problem.*

Let v elements be arranged into v sets such that each set contains exactly k distinct elements and such that every pair of distinct sets has exactly λ elements in common $(0<\lambda<k<v)$. It is shown that these hypotheses imply that $\lambda=k(k-1)/(v-1)$. It then follows readily that in the given arrangement each element must occur exactly k times and every pair of elements must occur exactly λ times. For further results concerning the above combinatorial problem see the recent abstracts of Chowla and Ryser entitled *Combinatorial problems* I and II. (Received June 12, 1949.)

ANALYSIS

447. Garrett Birkhoff: *Group theory and differential equations.*

Let an nth order system of ordinary differential equations Γ be invariant under a solvable Lie group having m-dimensional sets of transitivity. Then the integration of Γ can be reduced to the integration of an $(n-m)$th order system, and quadratures. (Received May 5, 1949.)

448. S. H. Chang: *A generalization of a theorem of Hille and Tamarkin with applications.*

Hille and Tamarkin (Acta Math. vol. 57 (1931) pp. 1–75, p. 46) proved that if $\partial^\rho K(x, y)/\partial x^\rho = K^{(\rho)}(x, y)$ $(\rho=1, 2, \cdots, s-2)$ be continuous and $K^{(s-1)}(x, y) = \int_a^x g(t, y)dt+c(y)$ where $g(x, y) \subset L^2$, so that $\|g(x, y)\|^2 = \int_a^b\int_a^b |g(x, y)|^2 dxdy<+\infty$, then the set of characteristic values $\{\mu_k\}$ of $K(x, y)$ satisfies $1/|\mu_k| = o(k^{-s-1/2})$. In this paper the author proves that under the same hypothesis, we also have $1/|\lambda_k| = o(k^{-s-1/2})$, where $\{\lambda_k\}$ denotes the set of singular values of $K(x, y)$, that is, E. Schmidt's characteristic values of unsymmetric kernels. From this result, the author gets a new simpler proof of Hille and Tamarkin's theorem and also proves that if a kernel satisfies Hille and Tamarkin's condition, then there is a decomposition of $K(x, y)$ into at least $2s$ L^2 factors: $K(x, y)=K_1K_2 \cdots K_{2s}(x, y)$, where $K_1K_2(x, y) =\int_a^b K_1(x, s)K_2(s, y)ds$, $K_1K_2K_3(x, y)=(K_1K_2)K_3(x, y)$, and so on. In the appendix, the author gives another very simple proof that if $\partial^\rho K(x, y)/\partial x^\rho$ $(\rho=1, 2, \cdots, s)$ is continuous in $a\leqq x$, $y\leqq b$, then $1/|\mu_k|=o(h^{-s-1/2})$ and $1/|\lambda_k|=o(h^{-s-1/2})$; it is a generalization of a theorem of Weyl. (Received June 7, 1949.)

449. S. H. Chang: *A relation between the characteristic values and singular values of integral equations.*

It is known that an unsymmetric L^2 kernel $K(x, y)$ may have no characteristic values while the corresponding kernel $KK^*(x, y)=\int_a^b K(x, s)K(y, s)ds$ has at least one characteristic value and may even have infinitely many characteristic values. Now in this paper, assuming that $K(x, y)$ has characteristic values, and letting $\{\mu_k\}$ be the set of characteristic values of $K(x, y)$ and $\{\lambda_k^2\}$ be the set of characteristic values of the kernel $KK^*(x, y)$ such that $|\mu_1|\leqq|\mu_2|\leqq\cdots$, $|\lambda_1|\leqq|\lambda_2|\leqq\cdots$, the author gives two proofs that $|\lambda_1|\leqq|\mu_1|$ with some other interesting deductions. A parallel theorem about matrices can be easily deduced from a theorem of A. Loewy and R. Brawer (Tôhoku Math. J. vol. 32 (1929) pp. 45–49). By a parallel theorem is meant a theorem which plays a role in the Fredholm solution of linear integral equations which is analogous to Cramer's rule for solving linear systems of algebraic equations. However, it is evident that theorems about matrices cannot cover the situations encountered in integral equations. (Received June 7, 1949.)

450. S. H. Chang: *On the distribution of the characteristics values and singular values of linear integral equations.*

In the first part, a new method is given to determine the order of magnitude of characteristic values and singular values of linear integral equations, as given by the following theorem: Let $\{\mu_k[K]\}$ and $\{\lambda_k[K]\}$ be the sequences of characteristic and singular values, respectively, of the real L^2 kernel $K(x, y)$ and let $D_{kk'}(\lambda) =\sum_{n=0}^\infty (-1)\, c_n\lambda^n$ be the Fredholm determinant of $KK'(x, y)=\int_a^b K(x, s)K(y, s)ds$. Then the convergence of the series $\sum_{k=1}^\infty 1/|\lambda_k[K]|^\tau$ $(\tau>0)$ implies the convergence of the series $\sum_{k=1}^\infty 1/|\mu_k[K]|^\tau$. The converse is not true. Also a necessary and sufficient condition for the convergence of the series $\sum_{k=1}^\infty 1/|\lambda_k[K]|^\tau$ $(0<\tau<2)$ is the convergence of the series $\sum_{k=1}^\infty |C_k|^{\rho/2k}$. In the second part, the author proves that for the composite kernel $K(x, y)=K_1K_2\cdots K_m(x, y)$ where $K_1K_2(x, y)=\int_a^b K_1(x, s)K_2(s, y)ds$, $K_1K_2K_3(x, y)=(K_1K_2)K_3(x, y)$, and so on, each $K_i(x, y)\subset L^2$, the series $\sum_{k=1}^\infty 1/|\lambda_k[K]|^{2/m}$ and $\sum_{k=1}^\infty 1/|\mu_k[K]|^{2/m}$ are both convergent. Also for any Marty kernel $K(x, y)=QL(x, y)$, where $Q(x, y)$ is a semidefinite continuous symmetric kernel, the series $\sum_{k=1}^\infty 1/|\lambda_k[K]|^{2/3}$ is convergent. Finally in the appendix, the author proves that if $K(x, y)$ is a real symmeric kernel such that $\partial^r K(x, y)/\partial x^r$ is continuous in $a\leqq x$, $y\leqq b$, then $K(x, y)$ has a decomposition into at least $2r$ factors: $K(x, y)=K_1K_2\cdots K_{2r}(x, y)$. (Received May 6, 1949.)

451. S. H. Chang: *On the integral equations with normal kernels.*

A. Loewy and R. Brawer proved that for a matrix V such that $V\bar{V}'=\bar{V}'V_3$ the characteristic roots of the matrix $W=V'\bar{V}$ are of the form $\tau_i\bar{\tau}_k$, where $\tau_1, \tau_2, \cdots, \tau_n$ are the characteristic roots of the matrix V (Tôhoku Math. J. vol. 32 (1929) pp. 45–49). In this paper, the author proves that for normal kernels of integral equations, that is, kernels satisfying the condition $KK^*(x, y) = K^*K(x, y)$ or $\int_a^b K(x, s)\overline{K(y, s)}ds = \int_a^b \overline{K(s, x)}K(s, y)ds$, the set of characteristic values $\{\lambda_k^2\}$ of $KK^*(x, y)$ and those $\{\mu_k\}$ of $K(x, y)$, arranged in the usual order $|\lambda_1| \leqq |\lambda_2| \leqq \cdots$, $|\mu_1| \leqq |\mu_2| \leqq \cdots$, must satisfy the relation $|\mu_k| = |\lambda_k|$ $(k = 1, 2, \cdots)$. The author also obtains in the paper: (i) a proof that any normal kernel has at least one characteristic value; (ii) a generalization of a theorem of Heywood and Goursat to the effect that if $K(x, y) = \text{L.I.M.} \sum_{k=1}^{\infty} K_k(x, y)$, where $K_iK_j(x, y) = 0$, $i \neq j$; $i, j = 1, 2, \cdots$, and $\|K(x, y)\|^2 = \sum_{k=1}^{\infty} \|K_k(x, y)\|^2 < +\infty$, then $D_K^*(\lambda) = \prod_{k=1}^{\infty} D_{K_k}^*(\lambda)$; here $D_K^*(\lambda)$ is the modified Fredholm determinant in Carleman's sense of $K(x, y)$; (iii) a general solution of integral equations with normal kernels, and related results. (Received June 7, 1949.)

452. V. F. Cowling: *On the analytic continuation of Newton series.*

Let $h > 0$ and $0 < B \leqq \pi$. Let $a(w)$ be regular in the region $|\text{Arg } (w-h| \leqq B$, and there satisfy the condition $|a(h+Re^{i\psi})| \leqq R^k \exp(-LR \sin \psi)$ for some k and some L, $0 < L < 2\pi$, and all large R. Then if $f(s) = \sum_{n=0}^{\infty} a(n)(-1)^n(C_{s-1,n})$ has an abscissa of convergence $\sigma < +\infty$, it is entire. (Received June 16, 1949.)

453. I. I. Hirschman and J. A. Jenkins: *On lacunary Dirichlet series.*

This paper considers the relation between the degree of lacunarity of a Dirichlet series and the allowable modular orders of its zeros on the abscissa of holomorphy. A special case of these results is that if f is represented as the sum of a Dirichlet series whose exponents have exponent of convergence $\nu < 1$, and if f has an exponential zero of order μ with $\mu > \nu/(1-\nu)$, then $f \equiv 0$. (Received April 5, 1949.)

454. E. J. McShane: *Linear functionals on certain Banach spaces.*

By using the property of uniform convexity, a simple derivation is obtained for the form of the general linear functional on spaces L_p, the integrals being over any space on whose subsets a measure function is defined, and also for the linear functional on spaces $L_p(B)$ of functions $f(\cdot)$ with values in a Banach space B and having $\|f(\cdot)\|^{p-1}f(\cdot)$ integrable in the sense of Bochner. (Received March 29, 1949.)

455. Leopoldo Nachbin: *A theorem of the Hahn-Banach type for linear transformations.*

A (real) normed space E is said to have the *extension property* if, for any normed space X and any vectors subspace $S \subset X$, every continuous linear transformation $f: S \rightarrow E$ has a continuous linear extension $F: X \rightarrow E$ with $\|F\| = \|f\|$. A necessary and sufficient condition for this is that the collection of all (closed) spheres of E should have the following property: every subcollection, any two members of which intersect, has a nonvoid intersection. Every complete vector lattice E, with an order unity e, can be made into a normed space in a natural way, the norm being $\|t\| = \inf \{\lambda: -\lambda e \leqq t \leqq \lambda e\}$. Then E has the extension property and e is an extreme point of the unity sphere. Conversely, if a normed space E has the extension property and contains e as an extreme point of the unity sphere, there is one and only one way of making E into a complete vector lattice with e as an order unity such that the norm deduced from the ordering be identical to the given one. By using some results established by S. Kakutani (Ann. of Math. vol. 42 (1941)) and by M. H. Stone (Canadian Journal of Mathematics vol. 1 (1949)), a representation theorem of a normed space with the extension property as the system of all continuous real functions over certain compact Hausdorff spaces and a relation to the theory of complete Boolean algebras are given. The question, involved in these results, as to whether every normed space with the extension property must contain some extreme point in its unity sphere is left undecided. (Received May 17, 1949.)

456. Jane C. Rothe: *The existence of multiple solutions of elliptic differential equations.*

By using topological methods, existence theorems are obtained for the differential equation (1) $F(x, y, z, p, q, r, s, t)=\psi_0(x, y)$ having in a region K an initial solution $z_0(x, y)$ which is elliptic relative to F. More precisely, if ϕ_0 is the boundary value of z_0, it is proved that for ϕ_1 and ψ_1 sufficiently close to ϕ_0 and ψ_0, equation (1) has a solution z_1 in region K when ψ_1 is substituted for ψ_0 in (1); and z_1 has boundary value ϕ_1. The theorems are new in that it is not necessary to assume that the Jacobi differential equation associated with (1) has only the zero solution in K. That is, the main assumption used in the classical-treatment is dropped. Also the existence of several distinct solutions is demonstrated for certain quasi-linear elliptic equations: $A(x, y)\partial^2z/\partial x^2+B(x, y)\partial^2z/\partial x\partial y+C(x, y)\partial^2z/\partial y^2+f(x, y, z, p, q)=\psi(x, y)$. (Received June 23, 1949.)

457. C. F. Stephens: *Nonlinear q-difference equations.*

The author applies to q-difference systems a method used for difference systems in an earlier paper (Trans. Amer. Math. Soc. vol. 64 (1948) pp. 268–282), with suitable modifications, and obtains results similar to those found for difference systems. (Received June 21, 1949.)

458. J. L. Walsh and H. Margaret Elliott: *Polynomial approximation to harmonic and analytic functions: generalized continuity conditions.*

Zygmund has shown that the uniform condition (*) $|f(s+h)+f(s-h)-2f(s)| \leqq L|h|$ for a continuous periodic function of the real variable s is significant in the study of approximation of $f(s)$ by trigonometric polynomials. The authors investigate this condition in the complex domain and show that if $f(z)$ is analytic interior to an analytic Jordan curve C, continuous in the closed interior $\overline{C}$, and if $f^{(k)}(z)$ exists on C and satisfies there a condition (*), then $f(z)$ can be approximated in $\overline{C}$ by polynomials in z with degree of approximation $1/n^{k+1}$, and conversely. Let $g(x, y)$ denote Green's function for the exterior of C with pole at infinity, and let C_R denote the level curve $g(x, y)=\log R\ (>0)$; if $f^{(k)}(z)$ exists on C_R and satisfies there a condition (*), then $f(z)$ can be approximated in $\overline{C}$ by polynomials in z with degree of approximation $1/R^n n^{k+1}$; if $f(z)$ can be approximated in C with degree of approximation $1/R^n n^{k+2}$, then $f^{(k)}(z)$ exists on C_R and satisfies (*). If a Jordan curve C is sufficiently smooth, if $f(z)$ is analytic interior to C and continuous in $\overline{C}$, and if (*) is satisfied on C, then the condition $|f(z+\Delta z)+f(z-\Delta z)-2f(z)| \leqq L'|\Delta z|$ is valid uniformly for z, $z-\Delta z$, $z+\Delta z$ in $\overline{C}$. (Received May 20, 1949.)

459. D. V. Widder: *An inversion of the Lambert transform.*

The Lambert transform is defined by the equation $F(x)=\int_0^\infty a(t)/[e^{xt}-1]dt$. It is shown that the transform can be inverted by the symbolic operator $(1/\zeta(D)(1/\Gamma(D))$ applied to the function $F(e^{-s})$. Here $\zeta(s)$ is the zeta function of Riemann, $\Gamma(s)$ is the familiar function of Euler, and D stands for the operation of differentiation. The symbolic inversion operator is realized by the combination of the operation $\sum_0^\infty \mu(n)F(nx)$, which converts the Lambert transform into a Laplace transform, and the Post-Widder formula, which is known to invert the latter. Here $\mu(n)$ is the Möbius function. (Received May 28, 1949.)

LOGIC AND FOUNDATIONS

460. J. B. Rosser and Hao Wang: *Nonstandard models for formal logics.*

A model for a logic is called nonstandard if it fails to have certain properties which are apparently called for by the axioms of the logic. Thus, if a logic L contains the ordinal numbers and in a model M of L the ordinal numbers of L are represented by a subset of M which is not well-ordered, then M is a nonstandard model of L. For the system of logic known as Quine's New Foundations, it is shown that there are no standard models. For a certain logic L with a simple theory of types, the existence of a standard model is shown, and it is further shown that within various other formal logics (including one with a simple theory of types) one can prove that L has a standard model. It is further shown for logics in general that if they are ω-consistent, then they cannot prove about themselves that if they are consistent, then they must have a standard model. Since the usual theorems about modelling can be proved within such logics (a proof of this is sketched), it follows that the property of having a standard model is quite exceptional, and so may fail for other logics besides Quine's New Foundations. (Received April 26, 1949.)

TOPOLOGY

461. Mary E. Estill: *Concerning abstract spaces.*

Let Axiom 1_3 denote the axiom resulting from the omission of condition (4) in the statement of Axiom 1 of R. L. Moore's *Foundations of point set theory*. Let Axiom 1'' denote the axiom obtained by replacing condition (4) of Axiom 1 by the statement: if $g_1, g_2, \cdots$ is a sequence such that, for each n, g_n is a region of G_n containing g_{n-1}, then $g_1, g_2, \cdots$ have a point in common. In his paper *Concerning separability* Moore has shown that if Axioms 0 and 1 hold true and there do not exist uncountably many mutually exclusive domains, then space is separable. In the present paper it is shown that this proposition does not remain true if Axiom 1 is replaced by Axiom 1_3. The relationship between several modifications of Axiom 1 are considered. In particular it is shown that there is a space satisfying Axioms 0 and 1_3 which is not a subspace of any space satisfying Axioms 0 and 1''and also that there is a space satisfying Axioms 0 and 1'' which is not a subspace of any space satisfying Axioms 0 and 1. (Received May 31, 1949.)

NOTES

It has been announced that in connection with "Jubile Scientifique du Professor Maurice Fréchet" one or two prizes will be awarded to an author, or authors, of a memoir on general analysis. The prize winners will be selected by a committee appointed by the Council of the Société Mathématique de France. The rules of the competition are as follows: Manuscripts in a foreign language should be accompanied by a typewritten summary in French. All manuscripts should be sent to the President of the Société Mathématique, Institut H. Poincaré, 11, rue Pierre Curie, Paris 5, before March 1, 1950. They should treat of general analysis (theory of abstract spaces, transformations of abstract elements into abstract elements) or its applications. Manuscripts should bear the name and address of the author. If preferred, authors who desire to remain anonymous can write an identifying sentence on the manuscript, which is reproduced on a sealed envelope containing the name and address of the author.

Professors Emil Artin of Princeton University, Garrett Birkhoff of Harvard University, André Weil of the University of Chicago, and Oscar Zariski of Harvard University will participate in a colloquium on Algebra and the Theory of Numbers (sponsored by the Centre National de la Recherche Scientifique) to be held in Paris, September 23–October 1, 1949.

Professor L. V. Kantorovich of the University of Leningrad has been awarded a Stalin prize for his work on functional analysis.

Professor A. D. Ross of the University of Western Australia received King's Birthday Honors.

Dr. S. A. Schelkunoff of the Bell Telephone Laboratories has been awarded the Stuart Ballantine Medal of the Franklin Institute for his outstanding contributions to the extension of the electromagnetic wave theory.

Dr. E. E. Moise of the University of Michigan has been awarded a National Research Fellowship and will study at the Institute for Advanced Study.

Professor Selig Brodetsky of Leeds University has been appointed to the presidency of Hebrew University.

Professor Alberto Dias Coimbra of Lyceum, Evora, Portugal, has been appointed to an assistant professorship at the University of Coimbra, Coimbra, Portugal.

Mr. Balthasar van der Pol of the Advisory Committee for Radio Communications, Eindhoven, Netherlands, has accepted a position with the International Telecommunication Union, Geneva, Switzerland.

Dr. Peter Thullen of the Colombian Insurance Company, Bogotá, Colombia, has accepted a position as chief actuary with the Social Security Department of the Republic of Panama.

Dr. Milton Abramowitz of the Computation Laboratory of the National Bureau of Standards is now associated with the Numerical Mathematics Service of New York City.

Mr. R. V. Andree of the University of Wisconsin has been appointed to an assistant professorship at the University of Oklahoma.

Professor Max Astrachan of Antioch College is on leave and has been appointed to a professorship at the U. S. Air Forces Institute of Technology, Dayton, Ohio.

Dr. W. E. Barnes of Cornell University has been appointed to an assistant professorship at the College of William and Mary.

Associate Professor J. W. Beach of the New Mexico School of Mines has been appointed to an assistant professorship at the University of New Mexico.

Assistant Professor R. H. Bing of the University of Wisconsin has been appointed to an acting professorship at the University of Virginia.

Dr. J. H. Blau of the Massachusetts Institute of Technology has been appointed to an assistant professorship at Pennsylvania State College.

Mr. Joseph Blum, who has been employed in the Department of the Army in Washington, D. C., has accepted a position as staff member at the Los Alamos Scientific Laboratories.

Dr. C. E. Bures of the University of Oregon has been appointed to an assistant professorship of philosophy and psychology at the California Institute of Technology.

Dr. Sarvadaman Chowla of the Institute for Advanced Study has been appointed to an acting professorship at the University of Kansas.

Mr. R. F. Cotellesa of Columbia University has accepted a position as engineer with the Allen B. DuMont Laboratories, New York, N. Y.

Assistant Professor V. F. Cowling of Lehigh University has been appointed to an associate professorship at the University of Kentucky.

Mr. N. J. Divinsky of the University of Chicago has accepted a position as research associate with the Cowles Commission for Research in Economics, Chicago, Ill.

Mr. J. H. Engel of the University of Wisconsin has accepted a position as staff member, Operations Evaluation Group, Massachusetts Institute of Technology.

Mr. Robert Fitzpatrick of the University of Pittsburgh has accepted a position as project director with the American Institute for Research, Pittsburgh, Pa.

Dr. M. M. Flood of the American Statistical Association has accepted a position as project officer with the Rand Corporation, Santa Monica, Calif.

Mr. E. E. Floyd of Princeton University has been appointed to an assistant professorship at the University of Virginia.

Mr. J. R. Foote of the Massachusetts Institute of Technology has been appointed to an assistant professorship at the Iowa State College of Agriculture and Mechanic Arts.

Dr. A. W. Goodman of Rutgers University has been appointed to an associate professorship at the University of Kentucky.

Professor E. D. Hellinger of Northwestern University is on leave and has been appointed to a visiting professorship at the Illinois Institute of Technology.

Mr. W. C. Hoffman of the Iowa State College of Agriculture and Mechanic Arts has accepted a position as assistant scientist with the Rand Corporation, Santa Monica, Calif.

Assistant Professor M. Gweneth Humphreys of Newcomb Memorial College, Tulane University of Louisiana, has been appointed to an associate professorship at Randolph Macon Woman's College.

Professor W. R. Hutcherson of Northwestern State College, Natchitoches, La., has been appointed to a professorship at the University of Florida.

Professor J. B. Jeffries of the Agricultural and Technical College of North Carolina has accepted a position as assistant director with the Midway Technical School, New York City.

Mr. C. E. Langenhop of Princeton University has been appointed to an assistant professorship at the Iowa State College of Agriculture and Mechanic Arts.

Dr. Lee Lorch of City College, New York City, has been appointed to an assistant professorship at Pennsylvania State College.

Professor H. B. Mann of Ohio State University is on leave and has been appointed to a visiting professorship at the University of California.

Associate Professor C. G. Maple of North Texas State Teachers College has been appointed to an associate professorship at the Iowa State College of Agriculture and Mechanic Arts.

Assistant Professor Paul Meier of Lehigh University has accepted a position as statistician with the Philadelphia Tuberculosis and Health Association.

Dr. B. C. Meyer of Stanford University has been appointed to an assistant professorship at the University of Arizona.

Assistant Professor H. J. Miser of Williams College has accepted a position as operations analyst with the U. S. Air Forces.

Mr. J. D. Newburgh of the Massachusetts Institute of Technology is now at the Institute for Advanced Study.

Dr. Paul Olum of the Institute for Advanced Study has been appointed to an assistant professorship at Cornell University.

Mr. C. L. Perry of the University of Michigan has been appointed to an assistant professorship at the University of Arkansas.

Dr. R. C. Prim of Princeton University has accepted a position as mathematician with the Bell Telephone Laboratories.

Mr. Frank Proschan of George Washington University has accepted a position as statistician with the Atomic Energy Commission, New York City.

Mr. Gordon Raisbeck of the Massachusetts Institute of Technology has accepted a position as staff member with the Acoustics Division, Bell Telephone Laboratories.

Professor W. E. Roth of Sampson College has been appointed to a professorship at the University of Tulsa.

Dr. H. J. Ryser of the Institute for Advanced Study has been appointed to an assistant professorship at Ohio State University.

Assistant Professor Alfred Schild of the University of Toronto has been appointed to an assistant professorship at Carnegie Institute of Technology.

Dr. W. R. Scott of the University of Michigan has been appointed to an assistant professorship at the University of Kansas.

Mr. K. M. Siegel has been appointed to the position of research associate at the Aeronautical Research Center, University of Michigan, Willow Run Airport, Ypsilanti, Mich.

Associate Professor H. E. Stelson of Michigan State College is on leave and has been appointed to a visiting professorship at the University of Hawaii.

Dr. D. B. Sumner of the University of Toronto has been appointed to an associate professorship at the Louisiana State University and Agricultural and Mechanic College.

Professor Otto Szasz of the University of Cincinnati is on leave and will be at the Institute for Numerical Analysis, National Bureau of Standards, University of California at Los Angeles, for the academic year beginning September 1949.

Dr. Deonisie Trifan of Case Institute of Technology has been appointed to an assistant professorship at the University of Arizona.

Dr. Bryant Tuckerman of Cornell University has been appointed to an assistant professorship at Oberlin College.

Assistant Professor G. L. Walker of Purdue University is on leave and has been appointed to a visiting professorship at Cornell University.

Dr. V. M. Wolontis of Harvard University has been appointed to an assistant professorship at the University of Kansas.

Professor T. O. Yntema of the University of Chicago has accepted a position as Vice President with the Ford Motor Company, Birmingham, Mich.

The following promotions are announced:

H. A. Heilbronn, University of Bristol, to a professorship.

E. A. Machado, University of LaPlata, to an assistant professorship.

Daniel Pedoe, Westfield College, University of London, to reader in pure mathematics.

V. W. Adkisson, University of Arkansas, to Dean of the Graduate School.

D. B. Ames, Rensselaer Polytechnic Institute, to a professorship.

J. D. Bankier, McMaster University, to an associate professorship.

H. D. Block, Iowa State College of Agriculture and Mechanic Arts, to an assistant professorship.

T. A. Botts, University of Virginia, to an assistant professorship.

P. B. Burcham, University of Missouri, to an associate professorship.

Paul Civin, University of Oregon, to an associate professorship.

C. L. Clark, Oregon State College, to a professorship.

J. E. Eaton, Queens College, to an assistant professorship.

Ky Fan, University of Notre Dame, to an associate professorship.

R. E. Gaskell, Iowa State College of Agriculture and Mechanic Arts, to an associate professorship.

N. G. Gunderson, University of Rochester, to an assistant professorship.

P. R. Halmos, University of Chicago, to an associate professorship.

Eugene Isaacson, New York University, to an assistant professorship.

V. L. Klee, University of Virginia, to an assistant professorship.

Mary A. Lee, Sweet Briar College, to an associate professorship.

J. G. Lemmer, West Baden College, Loyola University, to an associate professorship.

F. J. Murray, Columbia University, to a professorship.

H. K. Riess, Physics Department, Tulane University of Louisiana, to an associate professorship.

R. A. Rosenbaum, Reed College, to a professorship.

J. A. Schumaker, MacMurray College for Women, to an assistant professorship.

W. T. Scott, Northwestern University, to an associate professorship.

E. B. Shanks, Vanderbilt University, to an associate professorship.

Andrew Sobczyk, Boston University, to an associate professorship.

C. W. Topp, Fenn College, to an associate professorship.

M. L. Vest, West Virginia University, to an associate professorship.

S. I. Vrooman, Rensselaer Polytechnic Institute, to an assistant professorship.

Frances Weisbecker, Milwaukee-Downer College, to an assistant professorship.

R. M. Whitmore, Southwestern University, to an associate professorship.

Jacob Wolfowitz, Columbia University, to a professorship of mathematical statistics.

The following appointments to instructorships are announced: Institute of Technology, Delft, Netherlands: Dr. N. H. Kuiper; Allegheny College: Mr. T. T. Tanimoto; Brown University: Dr. Gerald Freilich; Cambridge Junior College: Mr. Carl Cohen; Clark University: Mr. R. R. Christian; Clarkson College of Technology, Malone, N. Y.: Mr. Leon Benson; Clemson Agricultural College, Clemson, S. C.: Mr. R. B. Johnson; Duke University: Miss Mary E. Estill; Goucher College: Dr. Verena H. Haefeli; University of Hawaii: Mr. C. H. Murphy; Illinois Institute of Technology: Mr. E. F. Trombley; Massachusetts Institute of Technology: Dr. E. A. Coddington, Mr. B. W. Lindgren; University of Michigan: Mr. W. J. LeVeque, Mr. A. J. Lohwater; University of Minnesota: Miss Margaret Owchar; University of Missouri: Mr. J. W. Gaddum; Montana State College: Mr. J. E. Whitesitt; New York University: Dr. G. E. Noether; Oklahoma Agricultural and Mechanical College: Mr. C. J. Clark; University of Pennsylvania: Miss Anne M. Whitney; Pennsylvania State College: Miss Leila R. Raines; Princeton University: Mr. F. H. Brownell; Reed College: Mr. D. V. V. Wend; Temple University: Miss L. Aileen Hostinsky; University of Washington (Department of Philosophy): Mr. E. C. Schlesinger; Wilson College: Miss Joan E. Robinson; Yale University: Dr. D. T. Finkbeiner.

Dr. J. K. L. MacDonald of the Naval Ordnance Test Station, China Lake, Calif. died February 3, 1949, at the age of forty-three years.

Professor Nelle Miller of the University of Arizona died June 20, 1949, at the age of sixty-eight years.

Professor A. H. Mowbray of the University of California died on January 7, 1949, at the age of sixty-seven years.

Dean Emeritus R. G. D. Richardson of Brown University died July 17, 1949, at the age of seventy-two years. He was a member of the Society for forty-four years.

NEW PUBLICATIONS

Ambrose, A., and Lazerowitz, M. *Fundamentals of symbolic logic.* New York, Rinehart, 1948. 10+310 pp. $5.00.

Behnke, H., and Sommer, F. *Klassische Funktionentheorie.* Vol. 2. Münster, Aschendorff, 1948. 8+234 pp. 6DM

Betz, A. *Konforme Abbildung.* Berlin, Springer, 1948. 8+539 pp. 36DM.

Blum, F. See van der Waerden, B. L.

Colombo, S. See Humbert, P.

Coolidge, J. L. *The mathematics of great amateurs.* Oxford University Press, 1949. 8+211 pp. $6.00.

Description of a relay calculator. (Annals of the Computation Laboratory of Harvard University, vol. 24.) Cambridge, Harvard University Press, 1949. 18+366 pp. $8.00.

Esnault-Pelterie, R. *L'analyse dimensionnelle.* Paris, Gauthier-Villars, 1948. 236 pp. 1400 francs.

Gröbner, W. *Moderne algebraische geometrie. Die idealtheoretischen Grundlagen.* Vienna, Springer, 1949. 12+212 pp. $5.70.

Guillemin, E. A. *The mathematics of circuit analysis.* New York, Wiley, 1949. 14+590 pp. $7.50.

Hamel, G. *Integralgleichungen, Einführung in Lehre und Gebrauch.* Berlin, Springer, 1949. 8+166 pp. 15.60DM.

Harvey, J. *Elementary calculus and allied geometry.* London, Hutchinson, 1949. 8+493 pp. 25s.

Heath, T. *Mathematics in Aristotle.* Oxford University Press, 1949. 14+291 pp. 21s.

Helmer, O. See Weyl, H.

Houtermans, C. See Wentzel, G.

Humbert, P., and Colombo, S. *Introduction mathématique à l'étude des théories électromagnetiques.* Part 1: *Analyse vectorielle, transformation conforme, théorie due potentiel.* Paris, Gauthier-Villars, 1949. 7+149 pp. 1200 francs.

Jauch, J. M. See Wentzel, G.

Johnson, P. O. *Statistical methods in research.* New York, Prentice-Hall, 1949 18+377 pp. $5.00.

Lazerowitz, M. See Ambrose, A.

Parodi, M. *Applications physiques de la transformation de Laplace.* Paris, Gauthier-Villars, 1948. 8+178 pp.

Rainville, E. D. *A short course in differential equations.* New York, Macmillan, 1949. 10+210 pp. $3.00.

Schmidt, F. K. *Vektorrechnung.* Vol. 2. Münster, Aschendorff, 1948. 8+244 pp. 6 DM.

Smail, L. L. *Calculus.* New York, Appleton-Century-Crofts, 1949. 16+592 pp. $4.50.

Sommer, F. See Behnke, H.

Table of sines and cosines to fifteen decimal places at hundredths of a degree. Prepared by the Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards. Washington, Government Printing Office, 1949. 8+95 pp. $.40.

van der Waerden, B. L. *Modern Algebra.* Vol. 1. Translated from the second revised German edition by F. Blum with revisions and additions by the author. New York, Ungar, 1949. 12+264 pp. $5.50.

Wentzel, G. *Quantum theory of fields.* Translated from the German by C. Houtermans and J. M. Jauch. New York, Interscience, 1949. 10+224 pp. $6.00.

WEYL, H. *Philosophy of mathematics and natural science.* Revised and augmented English edition based on a translation by Olaf Helmer. Princeton University Press, 1949. 10+311 pp. $5.00.

WHITTAKER, SIR EDMUND. *From Euclid to Eddington.* The Tarner Lectures, 1947. Cambridge University Press, 1949. 10+212 pp.

WILDER, R. L. *Topology of manifolds.* (American Mathematical Society Colloquium Publications, vol. 32.) New York, American Mathematical Society, 1949. 10+397 pp. $7.00.

SOME PROBABILITY LIMIT THEOREMS

A. M. MARK

1. Introduction. We prove the following theorems:

Let $X_1, X_2, \cdots$ be independent, identically distributed random variables, each having mean 0 and standard deviation 1 and let $S_k = X_1 + X_2 + \cdots + X_k$.

I. If $N = [\alpha n]$ where $0 < \alpha < 1$, then:

$$\lim_{n\to\infty} \text{Prob}\left\{\min(S_{N+1}, S_{N+2}, \cdots, S_n) > \beta n^{1/2}\right\} = V_1(\beta)$$

where

$$V_1(\beta) = \frac{1}{\pi}\int_{u=\beta\alpha^{-1/2}}^{\infty} e^{-u^2/2}du \int_{t=0}^{(u\alpha^{1/2}-\beta)(1-\alpha)^{-1/2}} e^{-t^2/2}dt.$$

II. If $p(t)$ is continuous in $0 \leqq t \leqq 1$ and has at most a denumerable set of zeros in the interval, then:

$$\lim_{n\to\infty} \text{Prob}\left\{\frac{1}{n^2}\sum_{j=1}^{n} p(j/n)S_j^2 < \beta\right\} = V_2(\beta)$$

where the characteristic function of $V_2(\beta)$ is given by

$$\int_{-\infty}^{\infty} e^{i\beta\xi}dV_2(\beta) = (D(2i\xi))^{-1/2}$$

and $D(\lambda)$ is the Fredholm determinant associated with the integral equation

$$\lambda\int_0^1 \min(s, t)p(t)f(t)dt = f(s).$$

In the case where $p(t) > 0$ in $0 \leqq t \leqq 1$, we have

$$\int_{-\infty}^{\infty} e^{i\beta\xi}dV_2(\beta) = \prod_{j=1}^{\infty}(1 - 2i\xi\lambda_j)^{-1/2}$$

where $\lambda_1, \lambda_2, \cdots$ are the eigenvalues of

$$g''(s) + p(s)g(s) = 0$$

subject to the conditions

Presented to the Society, September 5, 1947; received by the editors June 29, 1948.

$$g(0) = g'(1) = 0.$$

We also prove the following:

III. Let $X_{n1}, X_{n2}, \cdots, X_{nn}$ be independent, identically distributed random variables, each having standard deviation 1 and mean μ_n where $\lim_{n\to\infty} n^{1/2}\mu_n = u$. Let $S_{nj} = X_{n1} + X_{n2} + \cdots + X_{nj}$. Then:

$$\lim_{n\to\infty} \text{Prob} \left\{ \frac{1}{n^2} \sum_{j=1}^{n} S_{nj}^2 < \beta \right\} = V_3(\beta)$$

where the characteristiç function $V_3(\beta)$ is given by

$$\int_0^\infty e^{i\beta\xi} dV_3(\beta) = (\text{sech } (-2i\xi)^{1/2})^{1/2} \exp \left\{ \frac{\mu^2}{2} \left(\frac{\tanh (-2i\xi)^{1/2}}{(-2i\xi)^{1/2}} - 1 \right) \right\}.$$

The pattern of proof is essentially the same for the three theorems. We show first that the limiting distribution, if it exists, is independent of the distribution of the X's. A particular distribution is then chosen so that the limiting distribution can be conveniently calculated. This method was first used in a joint paper by Erdös and Kac [1].[1] Since our independence proofs differ only slightly from those in the latter paper, the details will not be given here.[2] Moreover, it will easily be seen that the requirement of identically distributed variables may be relaxed since it is only essential that the central limit theorem apply to the variables in question.

From I, the following corollary is obtained:

$$\lim_{n\to\infty} \text{Prob} \left\{ \min (S_{N+1}, \cdots, S_n) > 0 \right\} = \frac{1}{\pi} \sin^{-1} \alpha^{1/2}.$$

If, in addition, we let $M = [\gamma n]$, $0 < \gamma < \alpha < 1$, it can be shown by a method similar to the one used in proving I that

$$\lim_{w\to 0} \lim_{n\to 0} \text{Prob} \left\{ \min (S_{M+1}, S_{M+2}, \cdots, S_N) > 0 \,\middle|\, |S_n| < wn^{1/2} \right\}$$
$$= \frac{1}{\pi} \sin^{-1} \left(\frac{\gamma(1-\alpha)}{\alpha(1-\gamma)} \right)^{1/2}.$$

These special results had previously been derived by Lévy [3]. As stated, his theorems apply only to normally distributed X_i although he seems to have been aware that his theorems were true for the general class of distributions considered here.

Theorems II and III are generalizations of the third theorem in [1]. The calculation made in II for the special case where $p(t) > 0$ is similar to that made by Kac and Siegert [4, pp. 392–393]. There is a footnote in the latter paper that in the general case we must resort to the Fredholm determinant but the details are not carried out there. The results of the special case of positive $p(t)$ in II and also III had been obtained by Cameron and Martin [5, 6] in their work on Wiener space. Their results are equivalent to ours for the case of normally distributed X's.

---

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] The details may also be seen in another paper by the same authors [2].

The theorems can easily be interpreted in terms of a one-dimensional random walk. Theorems I and II would apply to a free particle and III would apply to a particle in a constant force field.

2. Proof of I. Let

$$P_n(\beta) = \text{Prob}\ \{\min (S_{N+1}, \cdots, S_n) > \beta n^{1/2}\}.$$

Let k be a positive integer and let

$$n_j = N + [(n - N)j/k)] \qquad (j = 1, 2, \cdots, k)$$

and

$$P_{n,k}(\beta) = \text{Prob}\ \{\min (S_{n_1}, S_{n_2}, \cdots, S_{n_k}) > \beta n^{1/2}\}.$$

Then, for any $\epsilon > 0$, we have

$$P_{n,k}(\beta + \epsilon) - 1/k\epsilon^2 < P_n(\beta) < P_{n,k}(\beta).$$

Letting $n \to \infty$ and using the multidimensional central limit theorem, we have

$$\begin{aligned} &\text{Prob}\ \{\min (\bar{R}_1, \bar{R}_2, \cdots, \bar{R}_k) > (\beta + \epsilon) k^{1/2}\} - 1/k\epsilon^2 \\ (1) \qquad &\qquad \leqq \liminf_{n \to \infty} P_n(\beta) \leqq \limsup_{n \to \infty} P_n(\beta) \\ &\qquad \leqq \text{Prob}\ \{\min (\bar{R}_1, \cdots, \bar{R}_k) > \beta k^{1/2}\} \end{aligned}$$

where

$$\begin{aligned} \bar{R}_1 &= (1 + (k - 1)\alpha)^{1/2} G_1, \\ \bar{R}_j &= \bar{R}_1 + (1 - \alpha)^{1/2}(G_2 + G_3 + \cdots + G_j) \qquad (j \geqq 2) \end{aligned}$$

and $G_1, G_2, \cdots$ are independent, normally distributed random variables each with mean 0 and standard deviation 1.

We now consider the particular case in which $X_1, X_2, \cdots$ are independent random variables such that

$$\text{Prob}\ \{X_j = 1\} = \{\text{Prob}\ X_j = -1\} = 1/2$$

for each j. We next obtain estimates on $P(n, l)$ for large n where

$$P(n, l) = \text{Prob}\ \{\min (S_1, S_2, \cdots, S_n) > -l\}$$

and l is an integer. For $l > 0$, we make use of a reflection argument to show that

$$\text{Prob}\{S_1 > -l, S_2 > -l, \cdots, S_{n-1} > -l, S_n = -l + p\} = \text{Prob}\{S_n = -l + p\} - \text{Prob}\{S_n = -l - p\}$$

and it will easily follow that

$$P(n, l) = 2\sum_{j=0}^{l} \text{Prob}\{S_n = j\} - \text{Prob}\{S_n = 0\} - \text{Prob}\{S_n = l\}.$$

We have

$$\begin{aligned}\text{Prob}\{S_n &= -l + p\}\\ &= \text{Prob}\{S_1 > -l, \cdots, S_{n-1} > -l, S_n = -l + p\}\\ &\quad + \text{Prob}\{S_n = -l + p \text{ with } S_i = -l \text{ for at least one } i < n\}.\end{aligned}$$

We consider the third probability in this equality and we suppose that S_i is the first such sum for which $S_i = -l$. Since

$$\text{Prob}\{X_{i+1} + X_{i+2} + \cdots + X_n = p\} = \text{Prob}\{X_{i+1} + \cdots + X_n = -p\},$$

we have

$$\text{Prob}\{S_n = -l + p \text{ with } S_i = -l \text{ for at least one } i < n\} = \text{Prob}\{S_n = -l - p\}$$

and the desired result follows immediately.

We now make use of the following well known result [7, p. 135]:

$$\text{Prob}\{S_n = j\} = \begin{cases}(2/\pi n)^{1/2}e^{-j^2/2n} + \Delta & \text{if } n \equiv j \pmod 2,\\ 0 & \text{otherwise}\end{cases}$$

where $|\Delta| < 3n^{-3/2}$ provided that $n \geqq 100$. Thus, for sufficiently large n, we have[3]

$$P(n, l) = 2\sum_{j=0}^{l}{}^{*}\left(\frac{2}{\pi n}\right)^{1/2} e^{-j^2/2n} + \epsilon_1 \qquad (l > 0)$$

where $|\epsilon_1| < 8ln^{-3/2} + 2n^{-1/2}$. Since

$$P(n, 0) = P(n - 1, 1)/2$$

we have

$$P(n, 0) < 7n^{-1/2}.$$

Moreover, since

$$\text{Prob}\{S_1 > -l\} = 0 \qquad (l \leqq -1),$$

---

[3] An asterisk on a summation sign will indicate that only those values of the index of summation are taken for which the index and the upper limit are congruent modulo 2.

we have

$$P(n, l) = 0 \qquad (l < 0).$$

Now

$$P_n(\beta) = \sum_{l=-N}^{N} \text{Prob}\,\{S_N = l, \min(S_{N+1} - S_N, S_{N+2} - S_N, \cdots, S_n - S_N) > -l + \beta n^{1/2}\}$$

$$= \sum_{l=-N}^{N} \text{Prob}\,\{S_N = l\} P(m, l - p)$$

where $m = n - N$ and $p = [\beta n^{1/2}]$. Using the previous approximations, we then have

$$(2) \qquad P_n(\beta) = \frac{1}{\pi} \sum_{l=-p}^{N}{}^{*} \left\{ 2N^{-1/2} e^{-l^2/2N} \sum_{j=0}^{l-p}{}^{*} 2m^{-1/2} e^{-j^2/2m} \right\} + \epsilon_2$$

where $|\epsilon_2| < Cn^{-1/2}$, C being bounded independently of n. The sums in (2) correspond to Riemann sums and, letting $n \to \infty$, we have

$$\lim_{n\to\infty} P_n(\beta) = \frac{1}{\pi} \int_{u=\beta\alpha^{-1/2}}^{\infty} e^{-u^2/2} du \int_{t=0}^{(u\alpha^{1/2}-\beta)(1-\alpha)^{-1/2}} e^{-t^2/2} dt = V_1(\beta).$$

The proof is now easily completed. Applying our particular result (1) we have

$$(3) \qquad \begin{aligned} \text{Prob}\,\{\min(\bar{R}_1, \bar{R}_2, \cdots, \bar{R}_k) > (\beta + \epsilon)k^{1/2}\} - 1/k\epsilon^2 \leqq V_1(\beta) \\ \leqq \text{Prob}\,\{\min(\bar{R}_1, \bar{R}_2, \cdots, \bar{R}_k) > \beta k^{1/2}\}. \end{aligned}$$

Replacing β by $\beta - \epsilon$, we have

$$\text{Prob}\,\{\min(\bar{R}_1, \bar{R}_2, \cdots, \bar{R}_k) > \beta k^{1/2}\} - 1/k\epsilon^2 \leqq V_1(\beta - \epsilon)$$

and replacing β by $\beta + \epsilon$ in (3), we have

$$V_1(\beta + \epsilon) \leqq \text{Prob}\,\{\min(\bar{R}_1, \cdots, \bar{R}_k) > (\beta + \epsilon)k^{1/2}\}$$

and applying these results to (1), we find that for the general case

$$\begin{aligned} V_1(\beta + \epsilon) - \frac{1}{k\epsilon^2} \leqq \liminf_{n\to\infty} P_n(\beta) \leqq \limsup_{n\to\infty} P_n(\beta) \\ \leqq V_1(\beta - \epsilon) + 1/k^2. \end{aligned}$$

The proof is completed by letting $k \to \infty$ while ϵ is held fixed and then letting $\epsilon \to 0$ and using the fact that $V_1(\beta)$ is continuous. The corollary follows by calculating $V_1(0)$ explicitly.

3. **Proof of II.** Letting

$$P_n(\beta) = \text{Prob}\left\{\frac{1}{n^2}\sum_{j=1}^{n} p(j/n)S_j^2 < \beta\right\},$$

and again following the method of Erdös and Kac, we find, for any $\epsilon > 0$ and all positive integers k, that

$$\begin{aligned} &\text{Prob}\left\{\frac{1}{k^2}\sum_{j=1}^{k} p(j/k)R_j^2 < \beta - \epsilon\right\} - C/\epsilon k^{1/2} \\ &\qquad \leqq \liminf_{n\to\infty} P_n(\beta) \leqq \limsup_{n\to\infty} P_n(\beta) \\ &\qquad \leqq \text{Prob}\left\{\frac{1}{k^2}\sum_{j=1}^{k} P(j/k)R_j^2 < \beta + \epsilon\right\} + C/\epsilon k^{1/2}, \end{aligned} \tag{4}$$

where C is a constant, $R_j = G_1 + G_2 + \cdots + G_j$, and $G_1, G_2, \cdots$ are independent, normally distributed random variables each having mean 0 and standard deviation 1. The remainder of the proof is essentially a calculation of $\lim_{k\to\infty} \phi_k(\xi)$ where $\phi_k(\xi)$ is the characteristic function of the distribution of $(1/k^2)\sum_{j=1}^{k} p(j/k)R_j^2$.

We consider first the case in which $p(t) > 0$, $0 \leqq t \leqq 1$, and for convenience we let $p_j = p(j/k)$. We then have

$$\begin{aligned} \phi_k(\xi) &= (2\pi)^{-k/2}\int_{-\infty}^{\infty}\cdots\int \exp\left\{i\xi/k^2\sum_{j=1}^{k} p_j\left(\sum_{i=1}^{j} x_i\right)^2 \right. \\ &\qquad\qquad \left. - \frac{1}{2}\sum_{j=1}^{k} x_j^2\right\} dx_1 \cdots dx_k \\ &= (2\pi)^{-k/2}\int_{-\infty}^{\infty}\cdots\int \exp\left\{\frac{i\xi}{k^2}\sum_{j=1}^{k} p_j y_j^2 - \frac{1}{2} y_1^2 \right. \\ &\qquad\qquad \left. - \frac{1}{2}\sum_{j=2}^{k} (y_j - y_{j-1})^2\right\} dy_1 \cdots dy_k. \end{aligned} \tag{5}$$

Let $A = ((a_{ij}))$ be the matrix of the quadratic form

$$y_1^2 + \sum_{j=2}^{k} (y_j - y_{j-1})^2$$

and let the transformation $z_j = p_j^{1/2} y_j$ take A into $B = ((b_{ij}))$, so that $b_{ij} = (p_i p_j)^{-1/2} a_{ij}$. We then have

$$\begin{aligned} \phi_k(\xi) = (2\pi)^{-k/2}(p_1 p_2 \cdots p_k)^{-1/2}\int_{-\infty}^{\infty}\cdots\int \exp&\left\{\frac{i\xi}{k^2}\sum_{j=1}^{k} z_j^2 \right. \\ &\left. - \frac{1}{2}\sum_{i,j=1}^{k} b_{ij} z_i z_j\right\} dz_1 \cdots dz_k, \end{aligned}$$

and if we denote the eigenvalues of B by $\mu_{1k}, \mu_{2k}, \cdots, \mu_{kk}$, it follows easily that

$$\phi_k(\xi) = \prod_{j=1}^{k} \left\{ p_j \left(\mu_{jk} - \frac{2i\xi}{k^2} \right) \right\}^{-1/2}.$$

Since $\phi_k(0) = 1$, we have

$$\phi_k(\xi) = \prod_{j=1}^{k} \left\{ 1 - \frac{2i\xi}{k^2 \mu_{jk}} \right\}^{-1/2}. \tag{6}$$

Now $B = C^{-1}$ where $C = ((p_i^{1/2} c_{ij} p_j^{1/2}))$ and $((c_{ij})) = ((\min \{i, j\}))$ $= A^{-1}$. It then follows that $1/\mu_{1k}, 1/\mu_{2k}, \cdots, 1/\mu_{kk}$ are the eigenvalues of C. From Hilbert's approach to Fredholm's theory [8, p. 14], it follows that $1/k^2\mu_{j,k} \to \lambda_j$ where $\lambda_1, \lambda_2, \cdots$ are the eigenvalues of the integral equation

$$\int_0^1 (p(s))^{1/2} \min (s, t)(p(t))^{1/2} f(t) dt = \lambda f(s).$$

Letting $g(s) = f(s)(p(s))^{-1/2}$, we obtain

$$\int_0^1 \min (s, t) p(t) g(t) dt = \lambda g(s)$$

and differentiating twice we find that

$$\lambda g''(s) + p(s) g(s) = 0. \tag{7}$$

Thus $\lambda_1, \lambda_2, \cdots$ are the eigenvalues of (7), subject to the conditions

$$g(0) = g'(1) = 0$$

which arise from the integral equation. Since $\min (s, t)$ is a positive definite kernel, it follows that $(p(s))^{1/2} \min (s, t)(p(t))^{1/2}$ is positive definite and by Mercer's theorem we have $\lambda_j > 0$ for each j and also $\sum_{j=1}^{\infty} \lambda_j$ converges. Hence we may pass to the limit in (6) and we have

$$\lim_{k \to \infty} \phi_k(\xi) = \prod_{j=1}^{\infty} (1 - 2i\xi\lambda_j)^{-1/2}, \tag{8}$$

the convergence being uniform in every finite ξ-interval.

We now remove the restriction that $p(t)$ be positive and we require that $p(t)$ have a most a denumerable set of zeros in the unit interval. Moreover, using a continuity argument, it is easily seen that we need only consider the case where the zeros of $p(t)$ are irrational. Letting $u = 2i\xi/k^2$, it follows from (5) that

$$\phi_k(\xi) = (2\pi)^{-k/2}\int_{-\infty}^{\infty}\cdots\int \exp\left\{-\frac{1}{2}\left(\sum_{i,j=1}^{k} a_{ij}y_iy_j - u\sum_{j=1}^{k} p_jy_j^2\right)\right\} dy_1\cdots dy_k.$$

Letting

$$J = \begin{pmatrix} a_{11}-up_1 & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22}-up_2 & \cdots & a_{2k} \\ \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ a_{k1} & & \cdots & a_{kk}-up_k \end{pmatrix}$$

it follows that

$$\phi_k(\xi) = |J|^{-1/2}.$$

Letting $H=((h_{ij}))$ where $h_{ij}=a_{ij}/p_i$, we have

$$|J| = p_1p_2\cdots p_k|H-uI|,$$

where I is the identity matrix. Moreover, if we denote the eigenvalues of $\lambda_{1k}, \lambda_{2k}, \cdots, \lambda_{kk}$, it follows that the eigenvalues of $H-uI$ are $\lambda_{1k}-u\lambda_{2k}-u, \cdots, \lambda_{kk}-u$. It then follows easily that

$$\phi_k(\xi) = \prod_{j=1}^{k}\{p_j(\lambda_{jk}-u)\}^{-1/2}.$$

Using the fact that $\phi_k(0)=1$, we have

$$\phi_k(\xi) = \prod_{j=1}^{k}\left(1-\frac{2i\xi}{k^2\lambda_{jk}}\right)^{-1/2}. \tag{9}$$

Moreover, it is clear that $1/\lambda_{1k}, 1/\lambda_{2k}, \cdots, 1/\lambda_{kk}$ are the eigenvalues of the matrix $G=((g_{ij}))=H^{-1}$ where

$$g_{ij} = p_j\min(i,j) = p(j/k)\min(i,j).$$

To evaluate $\lim_{k\to\infty}\phi_k(\xi)$ we again make use of Hilbert's considerations. We shall be concerned with the integral equation

$$f(s) - \lambda\int_0^1 \min(s,t)p(t)f(t)dt = 0 \tag{10}$$

and for convenience we write $K(s,t)=\min(s,t)\ p(t)$. Now let $K_{ij}=K(i/k, j/k)=\min(i,j)p_j/k$. Following Hilbert [8, p. 9] we define $D_k(l)$ to be the determinant

$$\begin{vmatrix} 1 - lK_{11} & -lK_{12} \cdots & -lK_{1k} \\ -lK_{21} & 1 - lK_{22} \cdots & -lK_{2k} \\ \vdots & & \\ -lK_{k1} & -lK_{k2} \cdots & 1 - lK_{kk} \end{vmatrix}$$

and it follows that

$$\lim_{k\to\infty} D_k(\lambda/k) = D(\lambda) \tag{11}$$

where $D(\lambda)$ is the Fredholm determinant associated with (10) and is defined over the entire complex λ-plane. Moreover, the convergence is uniform in every bounded region of the complex plane. It is easily seen that

$$D_k(l) = \left| J - \frac{l}{k} G \right|$$

and since the eigenvalues of G are $1/\lambda_{1k}, 1/\lambda_{2k}, \cdots, 1/\lambda_{kk}$, it easily follows that

$$D_k(l) = \prod_{j=1}^{k} \left(1 - \frac{l}{k\lambda_{jk}}\right),$$

and from (9) we have $\phi_k(\xi) = (D_k(2i\xi/k))^{-1/2}$. Letting $k \to \infty$ and using (11) we have

$$\lim_{k\to\infty} \phi_k(\xi) = (D(2i\xi))^{-1/2}. \tag{12}$$

Since $|\phi_k(\xi)| \leq 1$ for all k and for all real ξ, it follows that $D(2i\xi) \neq 0$ and $\lim_{k\to\infty} \phi_k(\xi)$ is defined for all real ξ. Furthermore, it is easily seen that (12) is equivalent to (8) when $p(t) > 0$. In the general case it follows readily that (10) is equivalent to (7) and it is known that (7) will have eigenvalues even if we remove the restriction that $p(t)$ be positive. However, in the general case we are unable to pass to the limit termwise as in (8) since our integral equation would not have a positive definite kernel. Therefore, we must resort to the Fredholm determinant.

As yet we have not shown the existence of the limiting distribution in the general case. We first consider the special case of normally distributed random variables. We have shown that $\phi_k(\xi)$ approaches a certain function $\phi(\xi)$ uniformly in every finite ξ-interval as k becomes infinite. Applying the continuity theorem for Fourier-Stieltjes transforms, we see that there exists a distribution function $V_2(\beta)$ such that

$$\lim_{k\to\infty} \text{Prob} \left\{ \frac{1}{k^2} \sum_{j=1}^{k} p(j/k) R_j^2 < \beta \right\} = V_2(\beta)$$

at every continuity point of $V_2(\beta)$. Furthermore, we have

$$\phi(\xi) = \int_{-\infty}^{\infty} e^{i\beta\xi} dV_2(\beta).$$

For $p(t) > 0$, we have

$$| \phi(\xi) | = \prod_{j=1}^{\infty} | 1 - 2i\xi\lambda_j |^{-1/2} \leqq \prod_{j=1}^{N} | 1 - 2i\xi\lambda_j |^{-1/2}$$

since $|1 - 2i\xi\lambda_j| \geqq 1$. Taking $N \geqq 4$, we see that $|\phi(\xi)|$ is integrable over $(-\infty, \infty)$ and it is evident from the inversion formula for Fourier transforms that $V_2(\beta)$ must be continuous at all points. The theorem then follows from (4). In the general case, however, we are unable to show that $V_2(\beta)$ is continuous everywhere and the theorem holds only at the continuity points of $V_2(\beta)$.

4. Proof of III. Letting

$$P_n(\beta) = \text{Prob}\left\{\frac{1}{n^2}\sum_{j=1}^{n} S_{nj}^2 < \beta\right\},$$

we find, for any $\epsilon > 0$ and for all positive integers k, that

$$\begin{aligned} &\text{Prob}\left\{\frac{1}{k^2}\sum_{j=1}^{k} R_{kj}^2 < \beta - \epsilon\right\} - C/\epsilon k^{1/2} \\ (13) \qquad &\leqq \liminf_{n\to\infty} P_n(\beta) \leqq \limsup_{n\to\infty} P_n(\beta) \\ &\leqq \text{Prob}\left\{\frac{1}{k^2}\sum_{j=1}^{k} R_{jk}^2 < \beta + \epsilon\right\} + C/\epsilon k^{1/2} \end{aligned}$$

where C is a constant, $R_{kj} = G_{k1} + G_{k2} + \cdots + G_{kj}$, and $G_{k1}, \cdots, G_{kj}$ are independent, normally distributed random variables each having mean $\mu k^{-1/2}$ and standard deviation 1. We now compute $\lim_{k\to\infty} \phi_k(\xi)$ where $\phi_k(\xi)$ is the characteristic function of the distribution of

$$\frac{1}{k^2}\sum_{j=1}^{k} R_{kj}^2.$$

We then have

$$\begin{aligned} \phi_k(\xi) = (2\pi)^{-k/2} \int_{-\infty}^{\infty} \cdots \int \exp\left\{\frac{i\xi}{k^2}\sum_{j=1}^{k}\left(\sum_{i=1}^{j} x_i\right)^2\right\} \\ \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{k}(x_j - \mu k^{-1/2})^2\right\} dx_1 \cdots dx_k, \end{aligned}$$

and making the substitution $y_j = \sum_{i=1}^{j} x_i$ we have

$$\phi_k(\xi) = (2\pi)^{-k/2} e^{-\mu^2/2} \int_{-\infty}^{\infty} \cdots \int \exp\left\{\frac{\mu y_k}{k^{1/2}} + \frac{i\xi}{k^2} \sum_{j=1}^{k} y_j^2\right\}$$
$$\cdot \exp\left\{-\frac{1}{2} y_1^2 - \frac{1}{2} \sum_{j=2}^{k} (y_j - y_{j-1})^2\right\} dy_1 \cdots dy_k.$$

We shall make use of matrices again and we shall denote the transpose of a matrix by a superscript "T." Let

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_k \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \mu k^{1/2} \end{bmatrix},$$

y and t being column matrices each with k rows. Furthermore, let A be the $k \times k$ matrix of the quadratic form

$$y_1^2 + \sum_{j=2}^{k} (y_j - y_{j-1})^2.$$

It then follows that

$$\phi_k(\xi) = (2\pi)^{-k/2} e^{-\mu^2/2} \int_{-\infty}^{\infty} \cdots \int \tag{14}$$
$$\cdot \exp\left\{\frac{i\xi}{k^2} y^T y - \frac{1}{2} y^T A y + t^T y\right\} dy_1 \cdots dy_k.$$

Let A have eigenvalues $\lambda_{1k}, \lambda_{2k}, \cdots, \lambda_{kk}$ and let C be an orthogonal matrix such that

$$C^T A C = D,$$

where D is the diagonal matrix

$$\begin{bmatrix} \lambda_{1k} & & & 0 \\ & \lambda_{2k} & & \\ & & \ddots & \\ 0 & & & \lambda_{kk} \end{bmatrix}.$$

Let

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_k \end{bmatrix},$$

and applying the transformation $y = Cz$ to (14) we have

$$\phi_k(\xi) = (2\pi)^{-k/2} e^{-\mu^2/2} \int_{-\infty}^{\infty} \cdots \int$$

$$\cdot \exp \left\{ \frac{i\xi}{k^2} z^T z - \frac{1}{2} z^T D z + u^T z \right\} dz_1 \cdots dz_k,$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_k \end{bmatrix} = C^T t.$$

We may now separate variables in the last integral and it follows easily that

$$\phi_k(\xi) = \left\{ \prod_{j=1}^{k} \left(\lambda_{jk} - \frac{2i\xi}{k^2} \right)^{-1/2} \right\}$$

$$\cdot \exp \left\{ -\frac{\mu^2}{2} + \frac{1}{2} \sum_{j=1}^{k} \frac{u_j^2}{\lambda_{jk} - 2i\xi/k^2} \right\}.$$

We define $Z^{1/2}$ in the plane with the negative real axis removed and satisfying the condition that $Z^{1/2}$ be positive for real and positive Z. Since $\phi_k(0) = 1$ for any μ, we consider the case $\mu = 0$ and we find that $\lambda_{1k}\lambda_{2k} \cdots \lambda_{kk} = 1$. We then have

$$\phi_k(\xi) = \left\{ \prod_{j=1}^{k} \left(1 - \frac{2i\xi}{\lambda_{jk} k^2} \right)^{-1/2} \right\} \exp \left\{ -\frac{\mu^2}{2} + \frac{1}{2} \sum_{j=1}^{k} \frac{u_j^2}{\lambda_{jk} - 2i\xi/k^2} \right\}.$$

Let

$$q = \sum_{j=1}^{k} \frac{u_j^2}{\lambda_{jk} + \alpha}$$

where

$$\alpha = -2i\xi/k^2$$

and it readily follows that

$$q = t^T (D + \alpha I)^{-1} t$$

where I is the identity matrix. We then find that

$$q = t^T (A + \alpha I)^{-1} t$$

and letting

$$B = (A + \alpha I)^{-1} = ((b_{ij}))$$

we find that

$$q = \frac{\mu^2}{k} b_{kk}.$$

We then have

$$\phi_k(\xi) = \left\{ \prod_{j=1}^{k} \left(1 - \frac{2i\xi}{\lambda_{jk}k^2}\right)^{-1/2} \exp\left\{\frac{\mu^2}{2}\left(\frac{b_{kk}}{k} - 1\right)\right\}\right\}. \tag{15}$$

We wish to let $k \to \infty$ in (15). It is known [1, p. 299] that

$$\lim_{k\to\infty} \prod_{j=1}^{k} \left(1 - \frac{2i\xi}{\lambda_{jk}k^2}\right)^{-1/2} = (\operatorname{sech} (-2i\xi)^{1/2})^{1/2} \tag{16}$$

and the convergence is uniform in every finite ξ-interval. Hence we need only consider $\lim_{k\to\infty} b_{kk}/k$. Let $F = A + \alpha I = ((f_{ij}))$ where $\alpha = -2i\xi/k^2$ and it readily follows that

$$b_{kk} = |F_{kk}| / |F|$$

where $|F|$ is the determinant of F and $|F_{kk}|$ is the co-factor of f_{kk}. To evaluate these determinants we make use of the following well known result:

Let

$$D_m(a) = \begin{vmatrix} 1 & a & & 0 \\ a & 1 & \ddots & \\ & \ddots & \ddots & a \\ 0 & & a & 1 \end{vmatrix}$$

where m is the order of the determinant. If $a^2 \neq 1/4$, we have

$$D_m(a) = \frac{r_1^{m-1} - r_2^{m-1}}{r_1 - r_2}$$

where r_1 and r_2 are the roots of $x^2 - x + a^2 = 0$.

We shall be interested in $D_m(a)$ as $m \to \infty$ and $a^2 \to 1/4$ and we shall find it more convenient to use $G_m(a)$ where

$$G_m(a) = 2^m \frac{r_1^{m-1} - r_2^{m-1}}{r_1 - r_2} = 2^m D_m(a).$$

Since F is the $k \times k$ matrix

$$\begin{bmatrix} 2+\alpha & -1 & & & 0 \\ -1 & 2+\alpha & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2+\alpha & -1 \\ 0 & & & -1 & 1+\alpha \end{bmatrix}$$

it follows that

$$F = a_2^k \begin{vmatrix} 1 & a & & 0 \\ a & \ddots & \ddots & \\ & \ddots & 1 & a \\ 0 & & a & a_1/a_2 \end{vmatrix} \tag{17}$$

where

$$a_1 = 1 + \alpha, \qquad a_2 = 2 + \alpha, \qquad a = -1/a_2.$$

Expanding the determinant in (17) by the last row, we find that

$$|F| = \frac{a_2^{k-2}}{2^{k-1}} (a_1 a_2 G_{k-1}(a) - 2G_{k-2}(a)).$$

Similarly we have

$$|F_{kk}| = \frac{a_2^{k-1}}{2^{k-1}} G_{k-1}(a),$$

and combining the last two equations we have

$$b_{kk} = \frac{a_2 G_{k-1}(a)}{a_1 a_2 G_{k-1}(a) - 2G_{k-2}(a)} \cdot \tag{18}$$

We next obtain estimates on $G_{k-1}(a)$ and $G_{k-2}(a)$ for large k and we shall use C to denote a number bounded independently of k. The various C's will not necessarily be equal. Since r_1 and r_2 are the roots of $x^2 - x + a^2 = 0$, we write

$$r_1 = (1 + \epsilon)/2$$

and

$$r_2 = (1 - \epsilon)/2$$

where

$$\epsilon = (1 - 4a^2)^{1/2}.$$

It is easily seen that

$$\epsilon = \frac{(2\xi_1)^{1/2}}{k} + \frac{C}{k^3} \tag{19}$$

where $\xi_1 = -i\xi$. Now

$$G_{k-1}(a) = 2^{k-1}\left(\frac{r_1^k - r_2^k}{r_1 - r_2}\right) = \sum_{j=1}^{k} (1 + \epsilon)^{j-1}(1 - \epsilon)^{k-j}.$$

Letting

$$Z_j = (1+\epsilon)^{j-1}(1-\epsilon)^{k-j},$$

it follows easily from (19) that

$$\log Z_j = -(2\xi_1)^{1/2} - \frac{1}{k}(\xi_1 + (2\xi_1)^{1/2}) + \frac{2j(2\xi_1)^{1/2}}{k} + C/k^2$$

and

$$Z_j = C/k^2 + \exp\left\{-(2\xi_1)^{1/2} - \frac{1}{k}(\xi_1 + (2\xi_1)^{1/2}) + \frac{2j}{k}(2\xi_1)^{1/2}\right\}.$$

Finally, we find that

$$G_{k-1}(a) = (k - \xi_1)\frac{\sinh (2\xi_1)^{1/2}}{(2\xi_1)^{1/2}} + \frac{C}{k}. \tag{20}$$

Since

$$G_{k-2}(a) = \sum_{j=1}^{k-1}(1+\epsilon)^{j-1}(1-\epsilon)^{k-j-1}$$

we have

$$G_{k-2}(a) = \frac{1}{1-\epsilon}\{G_{k-1}(a) - (1+\epsilon)^{k-1}\}.$$

It follows readily from (20) that

$$G_{k-2}(a) = (k - \xi_1)\frac{\sinh (2\xi_1)^{1/2}}{(2\xi_1)^{1/2}} - \cosh (2\xi_1)^{1/2} + \frac{C}{k}. \tag{21}$$

Combining (18), (20), and (21), we find

$$\frac{b_{kk}}{k} = \frac{2(k-\xi_1)\sinh (2\xi_1)^{1/2}/(2\xi_1)^{1/2} + C/k}{k(2\cosh (2\xi_1)^{1/2} + C/k)}$$

and letting $k\to\infty$, we have

$$\lim_{k\to\infty}\frac{b_{kk}}{k} = \frac{\tanh (2\xi_1)^{1/2}}{(2\xi_1)^{1/2}}.$$

Letting $k\to\infty$ in (15) and using (16), we find that

$$\begin{aligned}\lim_{k\to\infty}\phi_k(\xi) &= (\operatorname{sech}(-2i\xi)^{1/2})^{1/2} \\ &\quad\cdot\exp\left\{\frac{u^2}{2}\left(\frac{\tanh(-2i\xi)^{1/2}}{(-2i\xi)^{1/2}} - 1\right)\right\}\end{aligned} \tag{22}$$

and the convergence is uniform in every finite ξ-interval.

The proof is completed as in the previous section by applying the continuity theorem for Fourier-Stieltjes transforms. Thus there exists a distribution function $V_3(\beta)$ such that

$$\lim_{k\to\infty} \text{Prob}\left\{\frac{1}{k^2}\sum_{j=1}^{k} R_{kj}^2 < \beta\right\} = V_3(\beta)$$

at every continuity point of $V_3(\beta)$. Furthermore, if we let

$$\phi(\xi) = \lim_{k\to\infty} \phi_k(\xi)$$

we have

$$\phi(\xi) = \int_{-\infty}^{\infty} e^{i\beta\xi} dV_3(\beta).$$

It is easily seen from (22) that $\phi(\xi)$ approaches zero exponentially as $|\xi|$ becomes infinite and therefore $|\phi(\xi)|$ is integrable over $(-\infty, \infty)$. Hence $V_3(\beta)$ must be continuous at all points and the theorem follows easily from (13).

BIBLIOGRAPHY

1. P. Erdös and M. Kac, *On certain limit theorems of the theory of probability*, Bull Amer. Math. Soc. vol. 52 (1946) pp. 292–302.
2. ———, *On the number of positive sums of independent random variables*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1011–1029.
3. P. Lévy, *Un théorème d'invariance projective relatif au mouvement Brownien*, Comment. Math. Helv. vol. 16 (1943–1944) pp. 242–248.
4. M. Kac and A. J. F. Siegert, *On the theory of noise in radio receivers with square law detectors*, Journal of Applied Physics vol. 12 (1947) pp. 383–397.
5. R. H. Cameron and W. T. Martin, *The Wiener measure of Hilbert neighborhoods in the space of real continuous functions*, Journal of Mathematics and Physics vol. 23 (1944) pp. 195–209.
6. ———, *Evaluation of various Wiener integrals by use of certain Sturm-Liouville differential equations*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 73–89.
7. J. V. Uspensky, *Introduction tp mathematical probability*, New York, 1937.
8. D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Berlin and Leipzig, Teubner, 1912.

UNIVERSITY OF WISCONSIN

INTEGRAL DISTANCES IN BANACH SPACES

R. E. FULLERTON

In a recent paper,[1] Anning and Erdős have shown that in a finite-dimensional euclidean space a necessary and sufficient condition that an infinite set of points have the property that the distance between any two of them be an integer is that the points lie on a straight line. In this paper we shall investigate to what extent this theorem is true for general normed linear spaces of finite or infinite dimension. The theorem as stated is evidently not true for spaces of infinite dimension since in the space l^2 the infinite set of elements $z_i = (0, 0, \cdots, 2^{1/2}/2, 0, \cdots)$ with $2^{1/2}/2$ in the ith place and zero elsewhere have the property that $\|z_i - z_j\| = 1$ for $i \neq j$ and the z_i are linearly independent and not collinear. We shall prove that if a Banach space is not strictly convex there exists an infinite set of points each at an integral distance from the others which do not lie on a straight line. If the space is strictly convex, any such set has the property that if an infinite number of its points lie on a straight line then all the points of the set lie on the line. It is possible to prove certain theorems in analysis by applying these theorems to function spaces.

We shall define a *minimal set* connecting the points x_1 and x_2 in a Banach space X to be the set of all points of the space for which $\|x_1 - x\| + \|x_2 - x\| = \|x_1 - x_2\|$. A *straight line* containing x_1 and x_2 is the set of all points of the form $\alpha x_1 + \beta x_2$ where $\alpha + \beta = 1$. The set of points of the line for which α and β are non-negative is the *line segment* joining x_1 and x_2. If $\alpha + \beta = 1$, $\alpha \geqq 0$, $\beta \geqq 0$

$$\begin{aligned}\|x_1 - (\alpha x_1 + \beta x_2\| &+ \|x_2 - (\alpha x_1 + \beta x_2)\| \\ &= \|(1 - \alpha)x_1 - \beta x_2\| + \|(1 - \beta)x_2 - \alpha x_1\| \\ &= \|\beta x_1 - \beta x_2\| + \|\alpha x_2 - \alpha x_1\| = \beta\|x_2 - x_1\| + \alpha\|x_2 - x_1\| \\ &= (\alpha + \beta)\|x_2 - x_1\| = \|x_2 - x_1\|\end{aligned}$$

and the line segment joining x_1 and x_2 is in the minimal set determined by x_1 and x_2. A space in which the minimal set consists only of the line segment for any pair of points is called a *straight line space*. We shall show that such spaces are the strictly convex spaces.

THEOREM 1. *A necessary and sufficient condition that the minimal*

Presented to the Society, December 31, 1947; received by the editors April 15, 1948, and, in revised form, July 7, 1948.

[1] N. A. Anning and P. Erdős, *Integral distances*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 598–600.

set between any two points of a Banach space consist only of the line segment joining the points is that the space be strictly convex.

PROOF. Suppose that the unit sphere of X is not strictly convex and that there exist points x_1 and x_2, $\|x_1\| = \|x_2\| = 1$, and $\|\alpha x_1 + \beta x_2\| = 1$, $\alpha + \beta = 1$, $\alpha, \beta \geqq 0$. Then the line segment joining $-x_1$ and $-x_2$ is also on the boundary of the unit sphere. If the sphere is translated to the point $x_1 + x_2$ then $(x_1 + x_2) - x_2 = x_1$ and $(x_1 + x_2) - x_1 = x_2$ and the line segment joining $-x_1$ and $-x_2$ is translated to coincide with the segment joining x_2 and x_1. Hence the distance between $x_1 + x_2$ and a point on the segment is one, $\|(x_1 + x_2) - (\alpha x_1 + \beta x_2)\| = 1$. Also, $\|\alpha x_1 + \beta x_2\| = 1$. Thus

$$\|(x_1 + x_2) - (\alpha x_1 + \beta x_2)\| + \|\alpha x_1 + \beta x_2\| = \|x_1 + x_2\|$$

and the segment $\alpha x_1 + \beta x_2$ is in the minimal set between θ and $x_1 + x_2$. Hence the minimal set contains points other than those on the segment between θ and $x_1 + x_2$.

Let X be strictly convex and let M be the minimal set between θ and a point x_1. Consider the points $y \in M$ such that $\|y\| = d < \|x_1\|$. One such point y_0 is on the segment γx, $0 \leqq \gamma \leqq 1$. If there were another such point y_1, the line segment between y_0 and y_1 would be interior to both the spheres $\|x - x_1\| \leqq \|x_1\| - d$ and $\|x\| \leqq d$ since both these spheres are strictly convex. If y is a point of this segment, $y \neq y_0$, y_1, $\|x_1 - y\| + \|y\| < \|x_1\| - d + d = \|x_1\|$ which is not possible. This shows that y_0 is the only point of M at a given distance d from θ. Since the same argument may be used for any pair of points in X, X is a straight line space.

The above proof shows in particular that in a strictly convex space, metric definitions of a straight line coincide with algebraic definitions.

THEOREM 2. *If X is a Banach space which is not strictly convex there exists an infinite set of points $\{x_i\}$ such that $\|x_j - x_k\|$ is an integer for all j and k and the points are not all on the same straight line.*

PROOF.[2] Let S be the unit sphere in X, let $y_0 = 0$ and let $\{y_i\}$, $i = 1, 2, 3, \cdots$, be a set of points on the boundary of S which all lie on the same line segment L. Let $x_n = \sum_{i=0}^{n} y_i$, $n = 1, 2, 3, \cdots$. Evidently $\|x_0\| = 0$, $\|x_1\| = 1$, and $\|x_2\| = \|y_1 + y_2\| = 2\|(y_1 + y_2)/2\| = 2$ since $\|x_1/2 + x_2/2\| = 1$. Assume that $\|x_{n-1}\| = n - 1$ and that $(1/(n-1)) \cdot x_{n-1} \in L$. Then

$$\|x_n\| = \left\| \sum_{i=0}^{n} y_i \right\| = \|x_{n-1} + y_n\|$$

---

[2] This proof was suggested by the referee.

$$= n\left\| \frac{n-1}{n}\,\frac{1}{n-1}\,x_{n-1} + \frac{1}{n}\,y_n \right\| = n.$$

Hence $\|x_n\|$ is an integer and $(1/n)x_n \in L$. Since the above proof holds for any set y_i of L, we have

$$\|x_n - x_m\| = \left\| \sum_{i=m}^{n} y_i \right\| = \left\| y_0 + \sum_{i=m}^{n} y_i \right\| = | n - m |$$

and the distance between any two points of the set is an integer. The points do not all lie on the same line since in this case they would be of the form ny_i and $y_i = y_1$ for each i.

In case the set $\{y_i\}$ is a finite set, $i = 1, 2, \cdots, k$, the same procedure can be followed and we can define the points $y_{i+mk} = y_i$ for each i between 0 and k. For $n \geqq k$ the points x_n would be of the form

$$x_n = \sum_{i=0}^{m} (j+1)y_i + \sum_{i=m+1}^{k} jy_i = j\sum_{i=0}^{k} y_i + \sum_{i=0}^{m} y_i = ju + v_m$$

where $n = jk + m$ and $u = \sum_{i=0}^{k} y_i$, $v_m = \sum_{i=0}^{m} y_i$. Thus all points of the set in this case lie on the k lines $\gamma u + v_m$, $m = 1, 2, \cdots, k$, which are parallel in the sense that they are all translates of the line γu.

THEOREM 3. *Let the space X be strictly convex. Let $\{x_i\}$ be an infinite set of points of X such that $\|x_k - x_j\|$ is an integer for each j and k. Then if any infinite subset of this set consists of collinear points, all the points of the set are collinear.*

In the proof of this theorem we make use of two lemmas.

LEMMA 1. *Let X be strictly convex and let S_1 and S_2 be two spheres of different radii, r_1 and r_2, with centers x_1, x_2. Let x_3 be a point common to the boundaries of the two spheres lying on the line joining the two centers. Then x_3 is the only point common to the boundary of the two spheres and, except for x_3, S_1 is entirely interior to S_2, S_2 is interior to S_1, or the two spheres are entirely exterior to each other.*

PROOF. Suppose that x_4 is another point common to the boundaries of the two spheres. If x_4 is distinct from x_3, x_4 is not on the line of centers of the two spheres. If x_1 lies between x_2 and x_3 we have

$$\begin{aligned} r_2 = \|x_3 - x_2\| &= \|x_3 - x_1\| + \|x_1 - x_2\| = r_1 + \|x_1 - x_2\| \\ &= \|x_4 - x_1\| + \|x_1 - x_2\|. \end{aligned}$$

However, $r_2 = \|x_4 - x_2\|$ and this would imply that x_4 lies on the line of centers of the two spheres and gives a contradiction. In case x_3

lies between x_1 and x_2

$$\begin{aligned}\|x_1 - x_2\| &= \|x_1 - x_3\| + \|x_2 - x_3\| = r_1 + r_2 \\ &= \|x_1 - x_4\| + \|x_2 - x_4\|\end{aligned}$$

which implies again that x_4 lies on the line between x_1 and x_2. A similar argument holds if x_2 is between x_1 and x_3 and hence x_3 is the only point common to the two spheres.

To prove the second part of the lemma assume $r_1 \leqq r_2$, $x_1 \neq x_2$. Then either $\|x_1 - x_3\| + \|x_3 - x_2\| = \|x_1 - x_2\|$ or $\|x_1 - x_2\| + \|x_2 - x_3\| = \|x_1 - x_3\|$. Let x be any point of S_2. In the first case

$$\begin{aligned}\|x_1 - x\| + \|x_2 - x\| &\geqq \|x_1 - x_2\| \\ &= \|x_1 - x_3\| + \|x_3 - x_2\| = r_1 + r_2\end{aligned}$$

and $\|x_1 - x\| > r_1$ if $x \neq x_3$. Thus S_2 is entirely exterior to S_1 except for x_3. By a similar method it can be seen that in the second case S_2 is interior to S_1.

LEMMA 2. *Let a be a point of X and let r be a positive number $r < \|a\|$. Let H be a "hyperboloid" in X with foci at θ and a, that is, the set of points $x \in X$ such that $\|x\| - \|x - a\| = r$. Then a line through the focus θ has at most one point in common with H and a line through a has at most two points in common with H.*

PROOF. Let S_0 be the sphere with radius r and center θ. The set H is the locus of the centers of all spheres exterior to S_0 containing a on their boundaries and having one point in common with S_0. Let l be a line through θ and suppose that l has a point x_0 in common with H. Let $S(x_0)$ be the sphere exterior to S_0 with center x_0 having one point y in common with S_0 and containing a on its boundary. Evidently l contains the point y. Suppose that l intersects H in another point x_1. Then $S(x_1)$ must pass through y and hence must be exterior to $S(x_0)$, contain $S(x_0)$, or be contained in $S(x_0)$. In the first case y is exterior to the segment joining 0 and x_1 and hence $S(x_1)$ is not exterior to S_0. In the other two cases $S(x_1)$ cannot contain a on its boundary and hence x_1 cannot be in H.

Let l be a line through a. Suppose l intersects H in a point x_0. Then $S(x_0)$ has a on its boundary and also a point of S_0. If l intersects H in another point x_1, $S(x_1)$ contains $S(x_0)$, is contained in $S(x_0)$, or is exterior to $S(x_0)$. An argument similar to that of the first part of the proof shows that the first two cases cannot occur. Suppose that the third case occurs and that l intersects H in a third point x_2. Then $S(x_0)$, $S(x_1)$, and $S(x_2)$ all pass through a. By using Lemma 1 it can be seen that if x_0, x_1, and x_2 are distinct then $S(x_2)$ must either contain or be contained in $S(x_0)$ or $S(x_1)$. In any case it cannot have only one point in common with S_0 and hence l has at most two points in common with H.

It is evident that the above lemma holds for H defined by any two foci.

PROOF OF THE THEOREM. Suppose that an infinite set of the x_i lie on a line l. It can be assumed that $x_1=\theta$. Let a be a point of the set not on l. Consider the family of hyperboloids $\|x-a\|-\|x\|=n$ and $\|x\|-\|x-a\|=n$ for n an integer or zero. In the first case $\|a\|+\|x\| \geq \|x-a\|=n+\|x\|$ and in the second case $\|x\|=n+\|x-a\| \leq \|a\| + \|x-a\|$ and in both cases if the hyperboloid is to contain points not on the line between θ and a, n cannot exceed $\|a\|-1$. Since for each i, $\|x_i\|$ and $\|x_i-a\|$ are integers, $\|x_i-a\|-\|x_i\|$ or $\|x_i\|-\|x_i-a\|$ is an integer. Since l can intersect each of the first type of hyperboloids in at most two points and each of the second in at most one point, l cannot contain more than $3\|a\|-1$ points of the set. Hence if l contains an infinite subset of the given set it must contain all points of the set.

The above proof shows that if X is strictly convex and if x_i and x_j are two points of the set then any line through x_i but not through x_j contains not more than $3\|x_i-x_j\|-1$ points of the set. If the norm of the difference is one the integer n of the hyperboloids described above is zero and the only hyperboloid between the two points is the bisecting hyperplane of the segment joining the two points. Hence the line through x_i but not x_j can contain at most one other point. Applied to the set z_i in l^2 discussed at the beginning of the paper this shows that a line through any point of the set contains at most one other point of the set.

The above theorems give interesting results when applied to various well known strictly convex spaces. For example, let $\{x_i(t)\}$ be a set of functions in the space $L^p(a, b)$, $p>1$, such that $x_0(t)\equiv 0$ and $(\int_a^b |x_i(t)-x_j(t)|^p dt)^{1/p}$ is an integer for each i and j. The following corollaries are true.

COROLLARY 1. *Let the family* $\{x_i(t)\}$ *contain an infinite subfamily of the form* $\alpha_j x_1(t)$. *Then every function of the family is a multiple of* $x_1(t)$.

COROLLARY 2. *For any* k, *the number of multiples of* $x_k(t)$ *which may occur in the set cannot exceed* $3\cdot\min\,(\int_a^b |x_k(t)-x_j(t)|^p dt)^{1/p}-1$ *where the minimum is taken for* $j\neq k$ *and* $x_j(t)$ *not a multiple of* $x_k(t)$.

UNIVERSITY OF WISCONSIN

REAL ROOTS OF DIRICHLET L-SERIES

J. BARKLEY ROSSER

Let k be a positive integer. Let χ be a real, non-principal character (mod k) and

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

be the corresponding L-series, which converges uniformly for $R(s) \geqq \epsilon > 0$. If it could be shown that uniformly in k there is no real zero of $L(s, \chi)$ for

$$s \geqq 1 - \frac{A}{\log k},$$

where A is a constant, then the existing theorems on the distribution of primes in arithmetic progressions could be greatly improved (see [1]).[1] Moreover by Hecke's Theorem (see [2]) it would follow that uniformly in k

$$L(1, \chi) > \frac{B}{\log k}$$

where B is a constant. This would be a considerable improvement over Siegel's Theorem (see [3]), and would lead to an improved lower bound for the class number of an imaginary quadratic field.

In the present paper, we shall show that for $2 \leqq k \leqq 67$, $L(s, \chi)$ has no positive real zeros. By combining this information with the results of [1], we infer very sharp estimates on the distribution of primes in arithmetic progressions of difference k for $k \leqq 67$.

The methods used for $k \leqq 67$ certainly will suffice for many other k's greater than 67. They may possibly suffice for all k, but we can find no proof of this.[2]

In [5], S. Chowla has considered the positive real zeros of $L(s, \chi)$, and shown that for many explicit k's, no positive real zeros exist. However Chowla could not settle whether his methods would suffice

---

Presented to the Society, September 1, 1949; received by the editors August 30, 1948.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] These methods have been tried on all $k \leqq 227$ and it has been ascertained that except for the cases $k=148$ and $k=163$, $L(s, \chi)$ has no positive real zeros for $2 \leqq k \leqq 227$. Cases $k=148$ and $k=163$ are now being studied and any results obtained about them will appear in the Journal of Research of the National Bureau of Standards.

to handle the difficult cases $k=43$ and $k=67$. In [6], Heilbronn has shown that there exist values of k for which Chowla's methods are certainly inadequate.

THEOREM 1. *If* χ *is non-principal* (mod k) *and* $\chi(-1)=1$, *then for all* s

$$L(s,\chi)=\sum_{\alpha=1}^{\infty}\frac{2s(s+1)\cdots(s+2\alpha-1)}{4^{\alpha}(2\alpha)!\,k^{s+2\alpha}}(2^{s+2\alpha}-1)\zeta(s+2\alpha)\cdot\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha}.$$

PROOF. For $s>1$, we have

$$\begin{aligned}
L(s,\chi) &= 2^{s}\sum_{N=0}^{\infty}\sum_{n=1}^{k-1}\frac{\chi(n)}{(2kN+2n)^{s}}\\
&= 2^{s}\sum_{N=0}^{\infty}\frac{1}{k^{s}(2N+1)^{s}}\sum_{n=1}^{k-1}\chi(n)\left(1-\frac{k-2n}{k(2N+1)}\right)^{-s}\\
&= 2^{s}\sum_{N=0}^{\infty}\frac{1}{k^{s}(2N+1)^{s}}\sum_{n=1}^{k-1}\chi(n)\left\{1+s\frac{k-2n}{k(2N+1)}\right.\\
&\qquad+\frac{s(s+1)}{2!}\left(\frac{k-2n}{k(2N+1)}\right)^{2}\\
&\qquad\left.+\frac{s(s+1)(s+2)}{3!}\left(\frac{k-2n}{k(2N+1)}\right)^{3}+\cdots\right\}\\
&= 2^{s}\sum_{N=0}^{\infty}\frac{1}{k^{s}(2N+1)^{s}}\left\{\frac{s}{k(2N+1)}\sum_{n=1}^{k-1}\chi(n)(k-2n)\right.\\
&\qquad\left.+\frac{s(s+1)}{2!k^{2}(2N+1)^{2}}\sum_{n=1}^{k-1}\chi(n)(k-2n)^{2}+\cdots\right\}\\
&= \frac{s}{2k^{s+1}}\left(\sum_{N=0}^{\infty}\left(\frac{2}{2N+1}\right)^{s+1}\right)\sum_{n=1}^{k-1}\chi(n)(k-2n)\\
&\qquad+\frac{s(s+1)}{4(2!)k^{s+2}}\left(\sum_{N=0}^{\infty}\left(\frac{2}{2N+1}\right)^{s+2}\right)\sum_{n=1}^{k-1}\chi(n)(k-2n)^{2}+\cdots\\
&= \frac{s}{2k^{s+1}}(2^{s+1}-1)\zeta(s+1)\sum_{n=1}^{k-1}\chi(n)(k-2n)\\
&\qquad+\frac{s(s+1)}{4(2!)k^{s+2}}(2^{s+2}-1)\zeta(s+2)\sum_{n=1}^{k-1}\chi(n)(k-2n)^{2}+\cdots.
\end{aligned}$$

Since χ is non-principal, we have $k>2$, and so if k is even, we have $\chi([k/2])=\chi(k/2)=0$. Now since $\chi(-1)=1$,

$$\begin{aligned}\sum_{n=1}^{k-1}&\chi(n)(k-2n)^{2\alpha}\\ &=\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha}+\sum_{n=[k/2]+1}^{k-1}\chi(n)(2n-k)^{2\alpha}\\ &=\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha}+\sum_{n=k-[k/2]}^{k-1}\chi(n)(k-2(k-n))^{2\alpha}\\ &=\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha}+\sum_{n=1}^{[k/2]}\chi(k-n)(k-2n)^{2\alpha}\\ &=2\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha}.\end{aligned}$$

Similarly, we prove $\sum_{n=1}^{k-1}\chi(n)(k-2n)^{2\alpha+1}=0$.

Thus we infer that the equation stated is valid for $s>1$.

Now since

$$\left|\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha}\right|\leq\frac{k}{2}(k-2)^{2\alpha},$$

we see that the series on the right converges absolutely and uniformly for all s, and so our theorem follows by analytic continuation.

THEOREM 2. *If* χ *is non-principal* (mod k) *and* $\chi(-1)=-1$, *then for all* s

$$L(s,\chi)=\sum_{\alpha=0}^{\infty}\frac{s(s+1)\cdots(s+2\alpha)}{4^{\alpha}(2\alpha+1)!k^{s+2\alpha+1}}(2^{s+2\alpha+1}-1)\zeta(s+2\alpha+1)\cdot\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha+1}.$$

The proof is similar to the proof of Theorem 1.

Although these theorems hold for any non-principal χ, we shall use them only for real non-principal χ. We assume henceforth that χ is real and non-principal. We let Σ_M denote

$$\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{M}.$$

For sufficiently large M (certainly for $M\geqq k$), the initial term

$$\chi(1)(k-2)^{M}$$

of Σ_M dominates the remaining terms, and we infer that $\Sigma_M>0$. If by good chance $\Sigma_M\geqq 0$ for all $M\geqq 1$, then by Theorem 1 or Theorem 2 we infer that $L(s, \chi)>0$ for $s>0$, and hence that $L(s, \chi)$ has no positive real zeros. For $k\leqq 67$, this happens in a majority of cases.

When considering positive real zeros of $L(s, \chi)$ it suffices to restrict attention to primitive χ's (and to the k's for which there are primitive χ's. See [4, §125]). For primitive χ's, $\Sigma_M\geqq 0$ for $M\geqq 1$ for each $k\leqq 67$ except 43 and 67. Moreover for each such k, the proof of $\Sigma_M\geqq 0$ is easily accomplished by grouping the terms in groups, each of which is non-negative. Typical such groups are:

I. A^M-B^M, where $A>B$.

II. $A^M-B^M-C^M$, where $A\geqq B+C$.

III. $A^M-B^M-C^M+D^M$, where $A+D\geqq B+C$.

For $k=53$, there occurs the group $51^M-49^M-47^M+45^M-43^M+41^M+39^M-37^M$, which we show to be non-negative by writing it as $(44+7)^M-(44+5)^M-(44+3)^M+(44+1)^M-(44-1)^M+(44-3)^M+(44-5)^M-(44-7)^M$, and expanding each term by the binomial theorem.

For $k=43$ or 67, we have $\Sigma_3<0$, so that the series in Theorem 2 does not consist entirely of non-negative terms. However, we can show that the initial positive term outweighs the negative terms. We give the proof for $k=67$, since the proof for $k=43$ is similar and easier.

By the functional equation for $L(s, \chi)$ (see [4, §128]) it follows that if $L(s, \chi)$ has a zero ρ with $1/2<\rho<1$, then it has a zero ρ with $0<\rho<1/2$. As it is known that $L(s, \chi)>0$ for $1\leqq s$, it suffices to prove $L(s, \chi)>0$ for $0\leqq s\leqq 1/2$. So we take $k=67$ and $0\leqq s\leqq 1/2$. By Theorem 2,

$$L(s, \chi) = \frac{2^{s+1}-1}{67^s}\left\{\frac{s\zeta(s+1)}{67}\Sigma_1 + \frac{s(s+1)(s+2)}{3!(67)^3}\,\frac{2^{s+3}-1}{4(2^{s+1}-1)}\zeta(s+3)\Sigma_3+\cdots\right\},$$

where now $\Sigma_M=\sum_{n=1}^{33}\chi(n)(67-2n)^M$. For $s>0$,

$$\begin{aligned}\zeta(s+1)-\frac{1}{s} &= \sum_{n=1}^{\infty}\frac{1}{n^{s+1}}-\int_1^{\infty}\frac{dx}{x^{s+1}}\\ &= \sum_{n=1}^{\infty}\left\{\frac{1}{n^{s+1}}-\int_n^{n+1}\frac{dx}{x^{s+1}}\right\}\\ &> 0.\end{aligned}$$

So for $s\geqq 0$, $s\zeta(s+1)\geqq 1$. Also $\Sigma_1=67$. So

$$\frac{s\zeta(s+1)}{67}\Sigma_1 \geqq 1. \tag{1}$$

For $0 \leqq s$

$$s\,\frac{2^{s+2\alpha+1}-1}{4^{\alpha}(2^{s+1}-1)} < \frac{2s}{2-2^{-s}} \quad \text{and} \quad \frac{d}{ds}\left(\frac{2s}{2-2^{-s}}\right) > 0.$$

So for $0 \leqq s \leqq 1/2$

$$s\,\frac{2^{s+2\alpha+1}-1}{4^{\alpha}(2^{s+1}-1)} \leqq \frac{2(1/2)}{2-2^{-1/2}} < 0.77346.$$

Also

$$\frac{(s+1)(s+2)}{3!} \leqq \frac{(3/2)\cdot(5/2)}{3!} = \frac{5}{8}\,.$$

Since $\Sigma_3 = -102{,}845$, we infer

$$\begin{aligned} &\frac{s(s+1)(s+2)}{3!(67)^3}\,\frac{2^{s+3}-1}{4(2^{s+1}-1)}\,\zeta(s+3)\,\Sigma_3 \\ &\qquad \geqq -\frac{5}{8}\,\frac{1}{(67)^3}\,(0.77346)\zeta(3)(102{,}845) \\ &\qquad \geqq -\frac{5}{8}\,(0.77346)(1.20206)\,\frac{102{,}845}{300{,}763} \\ &\qquad > -0.199. \end{aligned} \tag{2}$$

Now for $M \geqq 1$,

$$\begin{aligned} \Sigma_M = {} & \{(57+8)^M - (57+6)^M - (57+4)^M + (57+2)^M - 57^M \\ & + (57-2)^M - (57-4)^M - (57-6)^M + (57-8)^M\} \\ & + \{(43+4)^M - (43+2)^M - 43^M - (43-2)^M + (43-4)^M\} \\ & + 37^M + 35^M + \cdots \\ > {} & -57^M + \frac{M(M-1)}{2!}\,57^{M-2}\{2\cdot 8^2 - 2\cdot 6^2 - 2\cdot 4^2 + 2\cdot 2^2\} \\ & + \frac{M(M-1)(M-2)(M-3)}{4!}\,57^{M-4}\{2\cdot 8^4 - 2\cdot 6^4 \\ & - 2\cdot 4^4 + 2\cdot 2^4\} + \cdots \\ & - 43^M + \frac{M(M-1)}{2!}\,43^{M-2}\{2\cdot 4^2 - 2\cdot 2^2\} + \cdots \end{aligned}$$

$$\geqq -57^M\left(1-\frac{16M(M-1)}{57^2}\right)-43^M\left(1-\frac{12M(M-1)}{43^2}\right).$$

In particular, if $\alpha \geqq 2$, then

$$\begin{aligned}\Sigma_{2\alpha+1} &\geqq -57^{2\alpha+1}\left(1-\frac{16(2\alpha+1)2\alpha}{57^2}\right)\\ &\quad -43^{2\alpha+1}\left(1-\frac{12(2\alpha+1)2\alpha}{43^2}\right)\\ &\geqq -57^{2\alpha+1}\left(1-\frac{320}{3249}\right)-43^{2\alpha+1}\left(1-\frac{240}{1849}\right)\\ &\geqq -57^{2\alpha+1}\frac{2929}{3249}-43^{2\alpha+1}\frac{1609}{1849}.\end{aligned}$$

So for $0 \leqq s \leqq 1/2$,

$$\sum_{\alpha=2}^{\infty}\frac{s(s+1)\cdots(s+2\alpha)}{(2\alpha+1)!(67)^{2\alpha+1}}\,\frac{2^{s+2\alpha+1}-1}{4^\alpha(2^{s+1}-1)}\zeta(s+2\alpha+1)\Sigma_{2\alpha+1}$$

$$\begin{aligned}&\geqq -\sum_{\alpha=2}^{\infty}\frac{s(s+1)\cdots(s+2\alpha)}{(2\alpha+1)!(67)^{2\alpha+1}}\,\frac{2^{s+2\alpha+1}-1}{4^\alpha(2^{s+1}-1)}\zeta(s+2\alpha+1)\\ &\qquad\cdot\left\{57^{2\alpha+1}\frac{2929}{3249}+43^{2\alpha+1}\frac{1609}{1849}\right\}\\ &\geqq -\sum_{\alpha=2}^{\infty}\frac{s(s+1)\cdots(s+4)}{5!(67)^{2\alpha+1}}\,\frac{2^{s+2\alpha+1}-1}{4^\alpha(2^{s+1}-1)}\zeta(5)\end{aligned}$$

$$\cdot\left\{57^{2\alpha+1}\frac{2929}{3249}+43^{2\alpha+1}\frac{1609}{1849}\right\} \tag{3}$$

$$\begin{aligned}&\geqq -\frac{63}{128}(0.77346)(1.03693)\sum_{\alpha=2}^{\infty}\left\{\left(\frac{57}{67}\right)^{2\alpha+1}\frac{2929}{3249}\right.\\ &\qquad\left.+\left(\frac{43}{67}\right)^{2\alpha+1}\frac{1609}{1849}\right\}\\ &\geqq -\frac{63}{128}(0.77346)(1.03693)\left\{\left(\frac{57}{67}\right)^5\frac{4489}{1240}\,\frac{2929}{3249}\right.\\ &\qquad\left.+\left(\frac{43}{67}\right)^5\frac{4489}{2640}\,\frac{1609}{1849}\right\}\\ &> -0.638.\end{aligned}$$

By (1), (2), and (3), for $0 \leqq s \leqq 1/2$,

$$L(s, \chi) \geqq \frac{2^{s+1} - 1}{67^s} \{1.000 - 0.199 - 0.638\} \geqq \frac{0.163}{(67)^{1/2}} \geqq 0.0199.$$

So $L(s, \chi) > 0$ for $0 \leqq s$.

When $\chi(-1) = -1$, Theorem 2 opens up further interesting possibilities. When $s \to 0$, the first term of the series is bounded away from zero, while the remaining terms approach zero. Thus one can always infer $L(s, \chi) > 0$ for $0 \leqq s \leqq \epsilon$, where ϵ depends on k. Even for ϵ as small as $A/\log k$, this would be a very worthwhile result, as remarked at the beginning of the paper.

For another possibility, let $s = 0$ and -2 in Theorem 2, and evaluate $L(0, \chi)$ and $L(-2, \chi)$ by the functional equation. We infer the known result

$$L(1, \chi) = \frac{\pi}{k^{3/2}} \Sigma_1 \tag{4}$$

and the result

$$L(3, \chi) = \frac{\pi^3}{6k^{7/2}} \{k^2\Sigma_1 - \Sigma_3\}. \tag{5}$$

From these follow

$$\Sigma_3 = k^{7/2}\left\{\frac{L(1, \chi)}{\pi} - \frac{6L(3, \chi)}{\pi^3}\right\}. \tag{6}$$

This gives

$$\Sigma_3 \geqq - k^{7/2} \frac{6L(3, \chi)}{\pi^3}.$$

If one could prove independently any appreciably better result, one could derive a sensational inequality for $L(1, \chi)$. For instance, if one could prove

$$\Sigma_3 \geqq - k^{7/2} \frac{4}{\pi^3} \geqq - k^{7/2} \frac{5L(3, \chi)}{\pi^3},$$

one could get by (6)

$$L(1, \chi) > \frac{L(3, \chi)}{\pi^2}.$$

Another possibility is that one can perhaps derive some connection between Σ_1 and Σ_3. For instance, if one could prove

$$\Sigma_3 \geqq -k^2 \log k \, \Sigma_1,$$

then by (4) and (6), we could infer

$$L(1, \chi) > \frac{6L(3, \chi)}{\pi^2(1 + \log k)}.$$

Even this would be a very worthwhile result, since the best known at present is, by Siegel's Theorem,

$$L(1, \chi) > \frac{L(3, \chi)}{k^\epsilon}$$

for $\epsilon > 0$ and large k.

BIBLIOGRAPHY

1. A. Page, *On the number of primes in an arithmetic progression*, Proc. London Math. Soc. (2) vol. 39 (1935) pp. 116–141.

2. E. Landau, *Über imaginärquadratische Zahlkörper mit gleicher Klassenzahl*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, 1918, pp. 277–284; *Über die Klassenzahl imaginärquadratischer Zahlkörper*, ibid. pp. 285–295.

3. C. L. Siegel, *Über die Classenzahl quadratischer Zahlkörper*, Acta Arithmetica vol. 1 (1935) pp. 83–86.

4. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 1, Leipzig, Teubner, 1909.

5. S. Chowla, *Note on Dirichlet's L-functions*, Acta Arithmetica vol. 1 (1935) pp. 113–114.

6. H. Heilbronn, *On real characters*, Acta Arithmetica vol. 2 (1937) pp. 212–213.

CORNELL UNIVERSITY

A NECESSARY AND SUFFICIENT CONDITION THAT A SUMMABILITY METHOD BE STRONGER THAN CONVERGENCE

ALBERT WILANSKY

Hill (Duke Math. J. vol. 9 (1942) p. 379), Agnew (Bull. Amer. Math. Soc. vol. 52 (1946) p. 128), and others have been concerned with finding conditions that summability methods should sum the convergent sequences and some others, or no others. We restrict ourselves to reversible matrix methods, that is, those in which the matrix $A=(a_{nk})$ transforms the sequence $x=\{x_n\}$ into the sequence $Ax=\{A_n(x)\}=\{\sum_{k=0}^{\infty} a_{nk}x_k\}$ and if y is any given convergent sequence, the equation $y=Ax$ has a unique solution x. Our results are particularly simple in the special case of normal matrices, that is, those with $a_{nk}=0$ for $k>n$, $a_{nn}\neq 0$.

We deal with conservative matrices, that is, those which transform every convergent sequence into a convergent sequence. The regular matrices form a subclass.

Consider, for the moment, normal matrices. What is known, and indeed trivial, is the following: a normal conservative matrix A sums only convergent sequences if and only if A^{-1} is conservative. Now, setting $\|A\|=\sup_n \sum_{k=0}^{\infty} |a_{nk}|$, A is conservative if and only if $\|A\|<\infty$ and in addition the sequences formed from its columns, and the sequence $\{\sum_{k=0}^{\infty} a_{nk}\}$, are all convergent (Silverman-Toeplitz-Kojima-Carmichael et al.). Thus the necessary and sufficient conditions are transparently displayed. What we propose to do is reduce these conditions by two, keeping the one which the author has found easiest to check in certain special cases.

According to Banach, *Theorie des operations lineaires*, Warsaw, 1932, p. 50, if A is reversible there is a bounded sequence $\{c_k\}$ and a matrix $B=(b_{nk})$ such that if y is convergent then $y=Ax$ where $x=\{x_n\}$, $x_n=c_n \lim_r y_r+\sum_{k=0}^{\infty} b_{nk}y_k$, the series being always convergent. Clearly if A is normal, $c_n=0$ and $B=A^{-1}$. As one of several possibilities let us define $\|A^{-1}\|$ as $\sup_n \{|c_n|+\sum_{k=0}^{\infty} |b_{nk}|\}$. This is consistent with the case when A is normal.

THEOREM. *A reversible conservative matrix A sums a divergent sequence if and only if* $\|A^{-1}\|=\infty$.

In particular if $\|A^{-1}\|<\infty$ we automatically conclude the convergence of the above mentioned sequences (formed from A^{-1}).

Received by the editors May 20, 1948, and, in revised form, July 28, 1948.

Sufficiency. Suppose $\|A^{-1}\|=\infty$. Since $\{c_n\}$ is bounded this means that $\|B\|=\infty$. By a now classical scheme we may construct a convergent sequence y such that By is unbounded. Let $x_n=c_n \lim_r y_r + \sum_{k=0}^{\infty} b_{nk}y_k$. The sequence x is also unbounded since $\{c_n\}$ is bounded but $Ax=y$ is convergent.

Necessity. Suppose $\|A^{-1}\|<\infty$. Let x be summable A. Then $y=Ax$ is convergent and $|x_n|=|c_n \lim y_r+\sum b_{nk}y_k| \leqq \|A^{-1}\| \cdot \sup |y_r| < \infty$ by hypothesis and since y is convergent. Thus x is a bounded sequence. The result is then immediate from the following lemma.

LEMMA. *If a reversible conservative matrix sums only bounded sequences it sums only convergent sequences.*

A proof is given for perfect matrices by Hill (Bull. Amer. Math. Soc. vol. 50 (1944) p. 229). The following proof for conservative matrices while still leaning on the methods of Mazur and Banach is along different lines.

Suppose first that A is an arbitrary reversible matrix. We denote its convergence field by (A). For a sequence x define $\|x\|=\sup_n |x_n|$. We shall call this the first metric. The second metric will be that defined by $\|x\|=\sup_n |A_n(x)|$. The one to one correspondence $x\leftrightarrow Ax$ between (A) and (c) (the class of convergent sequences) respectively is norm-preserving using the second and first metrics respectively and so (A) is a separable Banach space using the second metric. Moreover suppose $\{x^r\}$ is a sequence of elements of (A) with limit x in (A) using the second metric. We prove that

$$\lim_{r\to\infty} x_k^r = x_k \qquad \text{for each } k. \tag{1}$$

For a fixed t, define $F(y)=y_t$ where y is any sequence. Let $D=(d_{nk})$ where $d_{nk}=0$ for $k\neq t$, $d_{nt}=1$. Then the matrix D sums any sequence y whatsoever to the limit $y_t=F(y)$. Since $(D)\supseteq(A)$ the functional F is linear and continuous on (A) by use of the second metric by a result of Mazur. (Studia Math. vol. 2 (1930) p. 42. For this result Mazur uses only the hypothesis that (A) is a Banach space.) This proves (1).

The above results have been derived using only the reversibility of A.

Imposing the hypotheses of the lemma on A we next show that (A) is a Banach space using the first metric. We have only to show that it is complete so let us suppose that a sequence $\{x^r\}$ of elements of (A) is a Cauchy sequence using the first metric. It is readily verified that it is then a Cauchy sequence using the second metric. Thus $\{x^r\}$ converges to an element x of (A) using the second metric. It follows from (1) that the bounded sequence to which $\{x^r\}$ converges using the first metric is actually x. Thus (A) is a Banach space using either metric. It follows (Banach, loc. cit. Theorem 6, p. 41) that the two metrices are equivalent. But, as mentioned above, (A) is separable if we use the second metric and so it is separable if we use the first metric, and our lemma follows from a result of Agnew (Ann. of Math. (2) vol. 46 (1945) line 8 from foot of p. 99) that the bounded field of a matrix is not separable using the first metric unless the matrix sums no bounded divergent sequences.

LEHIGH UNIVERSITY

A PROBLEM OF ROBINSON

P. R. GARABEDIAN

We propose to prove in this paper a conjecture of R. M. Robinson which generalizes to domains of connectivity $n \geqq 3$ a result which he proved [3][1] for doubly-connected domains.

Let D be a finite domain of the z-plane bounded by $n \geqq 2$ simple, closed, disjoint analytic curves, $C_1, \cdots, C_n$, whose sum we denote by C. Let z_0 be a point of D. We consider the class Ω of functions $F(z)$ in D with at most one simple pole

$$F(z) = \frac{a_F}{z - z_0} + a_0 + a_1(z - z_0) + \cdots, \tag{1}$$

at $z = z_0$, which are regular in the remainder of D and satisfy

$$\limsup_{z \to C} |F(z)| \leqq 1. \tag{2}$$

The family Ω is compact. Thus the real number

$$\sigma(\zeta_0) = \max_{F \in \Omega} |F(\zeta_0)| \tag{3}$$

is defined for every $\zeta_0 \neq z_0$ of D.

Since the function $F(z) \equiv 1$ is in the class Ω, it is clear that $\sigma(\zeta_0) \geqq 1$ in D. The question raised by Robinson is whether or not there exists in D a nonempty set A on which $\sigma(\zeta_0) = 1$. He has solved the problem for the case of an annulus. If the annulus under consideration consists of the points z with $0 < r < |z| < 1$, and if $r < z_0 < 1$, then he has shown [3] that for $-1 < \zeta_0 < -r$ we have $\sigma(\zeta_0) = 1$, while $\sigma(\zeta_0) > 1$ in the remainder of D. Hence the set A consists here of the segment $-1 < \zeta_0 < -r$. From this result Robinson obtains a highly elegant treatment of questions of the Schwarz lemma type for an annulus.

We shall proceed in the opposite direction. We shall use results obtained in the author's thesis [2] for bounded functions to extend Robinson's theorem to domains of connectivity $n \geqq 3$.

It follows from the methods in the paper cited that there is a unique extremal function $F_0(z; \zeta_0) \in \Omega$ with

$$F_0(\zeta_0; \zeta_0) = \sigma(\zeta_0). \tag{4}$$

If $G(z; \zeta)$ denotes the Green's function of D, we have

Presented to the Society, September 10, 1948; received by the editors May 25, 1948, and, in revised form, August 19, 1948.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$\log |F_0(z;\zeta_0)| = G(z;z_0) - \sum_{\nu=1}^{n-1} G(z;z_\nu), \tag{5}$$

where the points $z_1, \cdots, z_{n-1}$ depend on ζ_0, and may lie on C. When $\sigma(\zeta_0)=1$, one of the z_ν lies at z_0 and we have the extremal function

$$F_0(z;\zeta_0) \equiv 1. \tag{6}$$

The proof of these statements follows so closely the material of Chapter VII of [2] that we omit it here.

The function $F_0(z;\zeta_0)$ depends continuously on the parameter ζ_0 in the sense of uniform convergence in every closed subdomain of D not containing z_0. Also, $\sigma(\zeta_0)$ is a continuous function of ζ_0 for $\zeta_0 \neq z_0$ in D, and $z_1, \cdots, z_{n-1}$ depend continuously on ζ_0. For if

$$\lim_{m\to\infty} \zeta_m = \zeta_0 \neq z_0, \tag{7}$$

then

$$\begin{aligned} \sigma(\zeta_0) &= F_0(\zeta_0;\zeta_0) \\ &= \lim_{m\to\infty} |F_0(\zeta_m;\zeta_0)| \\ &\leqq \liminf_{m\to\infty} F_0(\zeta_m;\zeta_m) \\ &= \liminf_{m\to\infty} \sigma(\zeta_m). \end{aligned} \tag{8}$$

On the other hand, since Ω is normal and compact, we can pick a subsequence ζ_μ^* of ζ_m and a function $F \in \Omega$ so that

$$\lim_{\mu\to\infty} \sigma(\zeta_\mu^*) = \limsup_{m\to\infty} \sigma(\zeta_m), \tag{9}$$

$$\lim_{\mu\to\infty} F_0(z;\zeta_\mu^*) = F(z), \tag{10}$$

$$\begin{aligned} \sigma(\zeta_0) &\geqq |F(\zeta_0)| \\ &= \lim_{\mu\to\infty} |F_0(\zeta_0;\zeta_\mu^*)| \\ &= \lim_{\mu\to\infty} \lim_{\nu\to\infty} |F_0(\zeta_\nu^*;\zeta_\mu^*)| \\ &= \lim_{\mu\to\infty} F_0(\zeta_\mu^*;\zeta_\mu^*) = \limsup_{m\to\infty} \sigma(\zeta_m). \end{aligned} \tag{11}$$

Therefore $\sigma(\zeta_m) \to \sigma(\zeta_0)$ as $m\to\infty$. If $F_0(z;\zeta_m)$ does not approach $F_0(z;\zeta_0)$ as $m\to\infty$, then there exist subsequences ζ_μ^* and ζ_μ^{**} of ζ_m such that

$$\lim_{\mu\to\infty} F_0(z;\zeta_\mu^*) = F_1(z), \tag{12}$$

$$\lim_{\mu\to\infty} F_0(z;\zeta_\mu^{**}) = F_2(z), \tag{13}$$

where F_1 and F_2 are extremal functions in Ω satisfying

$$F_1(\zeta_0) = F_2(\zeta_0) = \sigma(\zeta_0), \qquad F_1 \not\equiv F_2. \tag{14}$$

But the extremal function $F_0(z;\zeta_0)$ is unique. Hence we arrive at a contradiction, and it follows that $F_0(z;\zeta_m)$ tends to $F_0(z;\zeta_0)$ as m tends to infinity.

We now choose any curve Γ in $D-A$, with its end points both lying at z_0, which cannot be deformed to a point in $D-A$ in such a way that its end points remain fixed. In topological language, we choose Γ to represent an element of the fundamental group of $D-A$ which is not the identity. The assumption that such a curve Γ exists will lead us to a contradiction.

Since Γ lies entirely inside $D-A$, there exists an $\epsilon>0$ such that $\sigma(\zeta_0)\geqq 1+\epsilon$ for $\zeta_0\in\Gamma$. For clearly $\sigma(\zeta_0)\to\infty$ as $\zeta_0\to z_0$, and $\sigma(\zeta_0)$ is continuous and greater than 1 on every closed arc of Γ which does not contain z_0.

Let $\zeta(t)$ be the parametric representation of Γ, $0\leqq t\leqq 1$. We suppose without loss of generality that $\zeta(t)\neq z_0$ for $0<t<1$, while $\zeta(0)=\zeta(1)=z_0$. Hence

$$\bar{\sigma}(t) = \sigma(\zeta(t)) \tag{15}$$

is continuous for $0<t<1$, and $F_0(z;\zeta(t))$ is continuous in its dependence on t for $0<t<1$. We denote by Λ_t the inverse image of the ray $\bar{\sigma}(t)\leqq w\leqq\infty$ under the mapping $w=F_0(z;\zeta(t))$.

Since Γ cannot be shrunk to a point in $D-A$, there is a point u not in $D-A$ such that the order

$$p = \operatorname{ord}(\Gamma) = \int_0^1 d\arg\{\zeta(t)-u\} \tag{16}$$

of Γ about u is not zero. We denote by $\Gamma(t_0)$ the curve which consists of Λ_{t_0} plus the arc of Γ corresponding to values of t in the interval $t_0<t\leqq 1$. Now one end point of Λ_{t_0} is at z_0 and the other is at $\zeta(t_0)$. Hence $\Gamma(t_0)$ is a curve whose order $p(t_0)$ about u is well defined.

We maintain that $p(t_0)=p$ for small values of t_0, say for $0<t_0<\delta$. For given any $M>1$, there is a δ sufficiently small so that if $0<t_0<\delta$ we have $\bar{\sigma}(t_0)>M$. Now for any $F\in\Omega$ we have $\log|F(z)|\leqq G(z;z_0)$. Hence if $0<t_0<\delta$, then Λ_{t_0} lies in a neighborhood $G(z;z_0)>\log M$ of z_0, and such a neighborhood becomes arbitrarily small for sufficiently large M. Therefore it follows that $p(t_0)=p$ for $0<t_0<\delta$.

Similarly, we find there is a $\Delta>0$ such that $p(t_0)=0$ for $1-\Delta<t_0<1$.

Now $p \neq 0$. Therefore $p(t_0)$ cannot be continuous in the interval

$$\delta/2 \leqq t_0 \leqq 1 - \Delta/2, \tag{17}$$

since $p(t_0)$ is an integral-valued function.

The residue of $F_0(z; \zeta(t))$ at $z = z_0$ is bounded away from zero in the interval (17), since $\bar{\sigma}(t) \geqq 1+\epsilon$ there and $F_0(z; \zeta(t))$ depends continuously on t. Also, $F_0(z; \zeta(t))$ assumes each value w with $|w| > 1$ precisely once in D, and hence its inverse function is uniquely defined for $|w| > 1$. Indeed, the inverse function,

$$z - z_0 = H(w; \zeta(t)) = \frac{\beta_t}{w} + \frac{b_2}{w^2} + \frac{b_3}{w^3} + \cdots, \qquad |w| > 1, \tag{18}$$

is schlicht for $|w| > 1$ and its distortion β_t at infinity is bounded away from zero and infinity, since $\beta_t = \alpha_{F_0}$, $F_0 = F_0(z; \zeta(t))$.

It follows that Δ_t depends continuously on t in the sense of uniform convergence of its parametric representation. Therefore $p(t_0)$ is continuous in the interval (17), and we arrive at a contradiction.

We have proved the following theorem.

THEOREM. *Let* Ω *be the class of functions* $F(z)$ *in* D *which satisfy* (2) *and have at most one simple pole* (1) *at* $z_0 \in D$, *and let* $\sigma(\zeta_0)$ *be defined by* (3). *If* $A \subset D$ *is the set of points* $\zeta_0 \in D$ *such that* $\sigma(\zeta_0) = 1$, *then the region* $D - A$ *is simply-connected.*

It is of interest to consider, following Robinson, the question of how many poles we must allow the functions in our class to have in order that the set A be vacuous. To be precise, we consider the class $\Omega(z_1, \cdots, z_m)$ of functions $F(z)$ satisfying (2) and regular in D except for possible simple poles at $z_1, \cdots, z_m$. If several of the z_j coincide, we allow the poles in question to have correspondingly increased multiplicities. We define the function $\sigma(\zeta_0)$ by (3) with the new class $\Omega = \Omega(z_1, \cdots, z_m)$, and we denote by $A(z_1, \cdots, z_m)$ the set of points $\zeta_0 \in D$ with $\sigma(\zeta_0) = 1$.

If $m \geqq n$, the set $A(z_1, \cdots, z_m)$ is seen to be vacuous by the results of the papers [1, 2]. Indeed, the variation of zeros introduced there shows that the extremal function $F_0 \in \Omega(z_1, \cdots, z_m)$ with $F_0(\zeta_0) = \sigma(\zeta_0)$ satisfies an identity

$$\log |F_0(z)| = \sum_{j=1}^{m} G(z; z_j) - \sum_{j=1}^{n-1} G(z; \zeta_j), \tag{19}$$

where the points $\zeta_1, \cdots, \zeta_{n-1}$ depend on ζ_0. Since the number of zeros ζ_j of the extremal function is less than the number of poles z_j, the extremal function is not constant, and we have $\sigma(\zeta_0)>1$.

The same type of argument can be used to show that, in general, if $m=n-1\geqq 2$, then $A(z_1, \cdots, z_m)$ is vacuous. However, in every domain of connectivity $n\geqq 2$ there exist points $z_1, \cdots, z_{n-1}$ such that the set $A(z_1, \cdots, z_{n-1})$ for the class $\Omega(z_1, \cdots, z_{n-1})$ is not empty. For let $z_1^*, \cdots, z_{n-1}^*$ be the critical points of the Green's function, $G(z; \zeta_0)$, of D with normalization point at $\zeta_0\in D$. We maintain that ζ_0 is in the set $A(z_1^*, \cdots, z_{n-1}^*)$.

Indeed, let $F\in\Omega(z_1^*, \cdots, z_{n-1}^*)$, and set

$$p'(z) = \frac{\partial G(z;\zeta_0)}{\partial x} - i\frac{\partial G(z;\zeta_0)}{\partial y}, \qquad z = x+iy. \tag{20}$$

Then[3]

$$|F(\zeta_0)| = \left|-\frac{1}{2\pi i}\oint_C F(z)p'(z)dz\right| \leqq \frac{1}{2\pi}\oint_C |dp(z)| = 1. \tag{21}$$

The technique of contour integration which we use here is precisely the same as that developed in the paper [2].

We introduce the harmonic functions $\omega_i(z)$ known as the harmonic measures of the C_i in D, $i=1, \cdots, n$. The function $\omega_i(z)$ is harmonic in D and assumes the boundary values 1 on C_i and the boundary values 0 on the remainder of C. We see that any function

$$h(z) = G(z;\zeta_0) + \sum_{i=1}^{n-1}\lambda_i\omega_i(z) \tag{22}$$

can be used in the role of $G(z;\zeta_0)$ in the argument of the preceding paragraph, provided the normal derivative of $h(z)$ is non-negative on C.

We now return to the consideration of $\Omega=\Omega(z_0)$. Suppose that z_0 is a critical point of $h(z)$. Then we can apply the method embodied in the relation (21) to functions $F\in\Omega$ in order to show that $\zeta_0\in A$. If $n\geqq 3$, it will in general be possible to choose the parameters $\lambda_1^*, \cdots, \lambda_{n-1}^*$ in such a way that

$$h^*(z) = G(z;\zeta_0^*) + \sum_{i=1}^{n-1}\lambda_i^*\omega_i(z) \tag{23}$$

has a critical point at z_0 and has a non-negative normal derivative on

---

[3] Here and in (25) we apply Cauchy's theorem to contour integrals over C of the function F. This is permissible, since F has limits in angle almost everywhere on C. These limits can be used in the contour integrals (21) and (25), as is seen from the Lebesgue convergence theorem.

C, for all values of ζ_0^* in some small neighborhood of ζ_0. Indeed, this is possible if all $n-1$ of the critical points of (22) lie in D and the vectors

$$\left[\frac{\partial \omega_i(z)}{\partial x} - i\frac{\partial \omega_i(z)}{\partial y}\right]\Big|_{z=z_0}, \qquad i = 1, \cdots, n-1, \tag{24}$$

do not lie on a line through the origin. Thus we shall have

$$\begin{aligned} |F(\zeta_0^*)| &= \left| -\frac{1}{2\pi i}\oint_C \left[\frac{\partial h^*}{\partial x} - i\frac{\partial h^*}{\partial y}\right] F(z)dz \right| \\ &\leqq \frac{1}{2\pi}\oint_C \frac{\partial h^*}{\partial \nu} ds = 1, \end{aligned} \tag{25}$$

where ν represents the inner normal to C.

Therefore we can conclude that if $n \geqq 3$ the set $A = A(z_0)$ has, in general, interior points. There will be special cases in which this is not so, because our conditions upon the parameters $\lambda_1^*, \cdots, \lambda_{n-1}^*$ in order that $h^*(z)$ have a critical point at z_0 and a non-negative normal derivative on C can degenerate. This happens, for example, when D is an annulus $r < |z| < 1$ with slits along the segment $-1 < z < -r$, and $z_0 > 0$. For this case of symmetry one sees from Robinson's original theorem for circular rings [3] that the set A lies on the real axis.

It is clear that further progress in the investigation of the present set of problems can be achieved by finding precise information concerning the relative locations of the singularities and critical points of the harmonic functions (23) with non-negative normal derivative on C.

BIBLIOGRAPHY

1. L. Ahlfors, *Bounded analytic functions*, Duke Math. J. vol. 14 (1947) pp. 1–11.
2. P. R. Garabedian, *Schwarz's lemma and the Szegö kernel function*, Thesis, Harvard University, 1948.
3. R. M. Robinson, *Analytic functions in circular rings*, Duke Math. J. vol. 10 (1943) pp. 341–354.

HARVARD UNIVERSITY

SOME GENERALIZATIONS OF HELLY'S THEOREM ON CONVEX SETS

ALFRED HORN

Helly's well known theorem on convex sets states that if F is a family of closed bounded convex sets in n-dimensional space such that every $n+1$ members of F have a point in common, then there exists a point which belongs to all members of F. The interest of this theorem lies in the role played by the dimensionality of the containing space. If we are given that every k members of F have a point in common, where $k<n+1$, the conclusion is false. In this paper we shall show that significant conclusions can still be drawn in this case.

Consider the case where the sets lie in the plane and every two have a point in common. If we project the members of F on a line L, we obtain a family of convex subsets of L to which the hypotheses of Helly's theorem apply. It is an immediate consequence that in any direction there exists a line which intersects all members of F. Much less obvious is the statement that through *every point* there passes a line which intersects all members of F. This is the type of result which will be proved in §4 below. Instead of parallel projection, we shall project the members of F centrally on a circle (or a sphere, when the dimension of the containing space is greater than two). Then we shall apply, instead of Helly's theorem, a theorem proved below concerning "convex" sets (in the sense described in §1) lying on a circle (or sphere). Such sets (in a more restricted sense) were considered in a paper by C. V. Robinson [2].[1] In that paper the author was interested in conditions on a family of "strictly convex" sets in order that they all have a point in common. Our interest however is in proving a theorem for an arbitrary family of "convex" subsets of a sphere with the only restriction being that every k members of the family have a point in common. This is carried out in §3.

I wish to express my thanks to my colleague F. A. Valentine, with whom I had helpful discussions. A special case of Theorem 3 was proved in our joint paper [1].

1. Definitions and notation. We denote n-dimensional Euclidean space by E_n, and if $x=(x_1, \cdots, x_n)$ is a point of E_n, we write

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

Presented to the Society, November 27, 1948; received by the editors July 8, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

A subspace of E_n, that is, a hyperplane through the origin, is a subset of E_n which is closed under addition and scalar multiplication.

The surface of the unit sphere in E_n, that is, the set $E_x(x \in E_n, |x| = 1)$, will be denoted by S_n. A great k-sphere S_k is the intersection of S_n with a k-dimensional subspace of E_n. It follows that a great 0-sphere is the empty set, and a great 1-sphere is a pair of diametrically opposite points of S_n. A great 2-sphere is also called a great circle.

A major arc is a closed arc of a great circle which is greater than a semi-circle (this includes as a special case the whole great circle). A minor arc is a closed arc of a great circle which is smaller than a semi-circle. A semi-circle is neither a major nor a minor arc. If x and y are linearly independent points of S_n, the minor arc joining x and y is the set

$$E_z(z = \alpha x + \beta y / |\alpha x + \beta y|, \alpha \geqq 0, \beta \geqq 0, \alpha + \beta > 0).$$

The symbol $[x, y]$ is reserved for the minor arc joining x and y.

We shall use the term convex for subsets of S_n in a sense which is wider than that usually understood.

Definition 1. A subset A of S_n is convex if whenever x and y are linearly independent points of A, then either the minor or the major arc joining x and y lies entirely in A.

It follows that any set consisting of just two diametrically opposite points is convex, and that any subset of S_1 is convex.

Definition 2. A subset A of S_n is strictly convex if whenever x and y are linearly independent points of A, then the minor arc $[x, y]$ lies in A.

If A and B are sets, we say that A intersects B if $A \cdot B \neq 0$.

2. **Convex sets which are not strictly convex.** An immediate consequence of definitions 1 and 2 is the following lemma.

Lemma 1. *If A and B are strictly convex subsets of S_n, then $A \cdot B$ is strictly convex. If A is a convex subset of S_n and S_k is a great k-sphere, then $A \cdot S_k$ is convex.*

Lemma 2. *Let A be a closed convex subset of S_3 which contains a major arc M which is a proper subset of a great circle S_2, and suppose $A - S_2 \neq 0$. Then there exists a point z in $A - S_2$ such that every great circle through z intersects A in a major arc.*

Proof. It will suffice to find an interior point z of A such that $-z \in A$. Let x and y be the end points of M. If $b \in A - S_2$, all the minor arcs joining b and points interior to $[-x, -y]$ lie in A, since no point interior to $[x, y]$ is in A. Hence $A-S_1$ contains an interior point w. If $-w$ is not in A, all the minor arcs joining w and points of M lie entirely in A. Letting H denote the hemisphere bounded by S_1 and containing w, it follows that A contains H minus the spherical triangle x, y, w. Now there exists a point p interior to $[y, w]$ such that $[p, x]$ is not in A, for A is closed and $[x, y]$ is not in A. Since A is convex, the major arc joining p and x lies in A. Clearly, this major arc contains a point $z\in A\cdot H$ with the required properties.

THEOREM 1. *If A is a closed convex subset of S_n which is not strictly convex, and which is contained in no great $n-1$-sphere, then A contains at least one of every pair of diametrically opposite points of S_n.*

PROOF. We proceed by induction on n. The cases $n=1$ and $n=2$ are obvious. When $n\geq 3$, A contains a major arc M which is a proper subset of a great circle S_1. Let $n=3$, and let z be as described in Lemma 2. If $x\in S_3-A$, the opposite point $-x$ must lie in A because the great circle through z and x intersects A in a major arc. Now suppose the theorem holds for $n<m$, $m>3$. Let z_1 and z_2 be the end points of M. By Lemma 2, if $p\in A-S_1$, we can find a point z in the great 3-sphere S_3 through S_1 and p such that every great circle on S_3 through z intersects A in a major arc. Let $z_3, \cdots, z_m$ be such that $z_1, \cdots, z_m$ are linearly independent points of A, and such that every great circle joining z_i to a point of S_1 intersects A in a major arc, $3\leq i\leq m$. Let y be in the interior of $[z_1, z_2]$. Observe y is not in A. Now let x be any point of S_m. We have

$$x = \alpha_1 y + \alpha_2 z_2 + \cdots + \alpha_m z_m.$$

If $\alpha_3=0$, x lies on the great $m-1$-sphere containing M, $z_4, \cdots, z_m$. By the induction hypothesis, it follows that either x or $-x$ is in A. If $\alpha_3\neq 0$, we have

$$x = \alpha_1 y + \beta w + \alpha_4 z_4 + \cdots + \alpha_m z_m,$$

where $\beta w=\alpha_2 z_2+\alpha_3 z_3$, $\beta=|\alpha_2 z_2+\alpha_3 z_3|$. Note that either w or $-w$ is in A. Clearly, the points $y, w, z_4, \cdots, z_m$ are linearly independent. The great $m-1$-sphere through $y, w, z_4, \cdots, z_m$ intersects A in a closed convex set B by Lemma 1. Also B contains a major arc which is a proper subset of the great circle through z_4 and y, and B contains $m-1$ linearly independent points $-y, z_4, \cdots, z_m, \pm w$. Hence by the induction hypothesis, either x or $-x$ is in B, and therefore in A.

3. The main theorem.

THEOREM 2. *Let F be a family of closed convex subsets of S_n which is* *such that every k members of F have a point in common,* $1 \leqq k \leqq n$. *Then given any great* $n-k$*-sphere* S_{n-k}, *there is a great* $n-k+1$*-sphere containing* S_{n-k} *and intersecting every member of F.*

As a noteworthy special case, we have the following theorem.

THEOREM 3. *Let F be a family of closed convex subsets of* S_n *which is such that every n members of F have a point in common. Then there exists a point x such that every member of F contains either x or* $-x$.

PROOF. If we denote the statement of Theorem 2 by $T(n, k)$, then the statement of Theorem 3 is $T(n, n)$. We are going to prove $T(n, k)$ by induction on n. The statement $T(1, 1)$ is trivially true. The rest of the proof is divided into three parts.

Part 1. If $1 \leqq k \leqq n$, $T(k, k)$ implies $T(n, k)$.

PROOF. Let E_{n-k} be the subspace generated by S_{n-k} and let E_k be the subspace of points orthogonal to E_{n-k}. We may suppose that no member of F intersects S_{n-k}, for otherwise it would intersect every great $n-k+1$-sphere containing S_{n-k}. If $x \in E_n$, let $P(x)$ denote the projection of x on E. By our assumption, if $x \in A \in F$, then $P(x) \neq 0$. Let $A' = E_z(z = P(x)/|P(x)|, x \in A)$, and $F' = E_B(B = A', A \in F)$. It is easily seen that A' is a convex subset of $S_k = S_n \cdot E_k$, whenever $A \in F$. In fact, if z and z' are linearly independent points of A', that is, $z = P(x)/|P(x)|$, $z' = P(x')/|P(x')|$, where x and x' are linearly independent points of A', then whenever $\alpha^2 + \alpha'^2 > 0$

$$\frac{\alpha z + \alpha' z'}{|\alpha z + \alpha' z'|} = \frac{P(\beta x + \beta' x')}{|P(\beta x + \beta' x')|} = \frac{P(\beta x + \beta' x'/|\beta x + \beta' x'|)}{|P(\beta x + \beta' x'/|\beta x + \beta' x'|)|},$$

where $\beta = \alpha/|P(x)|$, $\beta' = \alpha'/|P(x')|$. Hence, whenever w is on the minor (or major) arc joining z and z', it is the image of a point on the minor (or major) arc joining x and x'. Moreover A' is closed whenever A is closed. Furthermore every k members of F' have a point in common. By $T(k, k)$ there is a point w such that every member of F' contains w or $-w$. But the statement $\pm w \in A'$ is equivalent to $P(x)/|P(x)| = \pm w$ for some $x \in A$, that is, $P(x)$ is a multiple of w. Thus every member A of F contains a point in the subspace E_{n-k+1} generated by E_{n-k} and w, and the great $n-k+1$-sphere $S_n \cdot E_{n-k+1}$ intersects every member of F. To complete the proof of Theorem 2, we need only show:

Part 2. If $n > 1$, $T(n-1, n-1)$ implies $T(n, n)$.

PROOF. Let F be a family satisfying the hypotheses of $T(n, n)$, and suppose $T(n, n)$ is false. Then to each $x \in S_n$, we can assign a member $A(x)$ of F such that neither x nor $-x$ belongs to $A(x)$. Since $A(x)$ is closed, we can find a number $\delta(x)$ such that neither z nor $-z$ belongs to $A(x)$ whenever $|z-x|<\delta(x)$. The sets $E_z(|z-x|<\delta(x),\ z\in S_n,\ x\in S_n)$ cover S_n, and we may select a finite covering of S_n from these. Let the corresponding members of F be $A(x_1), \cdots, A(x_p)$. Then no point x is such that all of the sets $A(x_i)$ contain either x or $-x$. If we can prove $T(n, n)$ for a finite family F, we shall have arrived at a contradiction. Accordingly, we henceforth assume F is finite.

Next we prove that $T(n, n)$ is true if one of the members, say B, of F lies entirely on some great $n-1$-sphere S_{n-1}. Let $A'=A\cdot S_{n-1}$ for each A in F, and let F' be the family of sets A', $A\in F$. Every $n-1$ members of F' have a point in common, since they have a point in common with B. Also by Lemma 1, every member of F' is a closed convex subset of S_{n-1}. An application of $T(n-1, n-1)$ completes the proof.

Accordingly, we assume that no member of the finite family F lies entirely on a great $n-1$-sphere. By Theorem 1, we may assume that every member of F is strictly convex.

To prove $T(n, n)$ for a finite family of strictly convex closed sets, we proceed by induction on the number of members of F. $T(n, n)$ being trivially true whenever F has at most n members, assume $T(n, n)$ holds whenever F has fewer than r members, where $r>n$. Before completing the proof of part 2, we need the following result.

Part 3. Suppose F has r members $A_1, \cdots, A_r$, and suppose there is a point x such that $x\in A_1\cdot \cdots \cdot A_s$, and $-x\in A_{s+1}\cdot \cdots \cdot A_{s+t}$, where $s>0$, $t\geq 0$, and $s+t\leq r-1$. Then $T(n-1, n-1)$ implies there exists a point y such that either

(a) every member of F contains either y or $-y$, or

(b) y belongs to $s+1$ members of F, or

(c) y belongs to s members of F and $-y$ belongs to $t+1$ other members of F.

Proof. Every $n-1$ members of the family $\{A_1\cdot A_r, \cdots, A_{r-1}\cdot A_r\}$ have a point in common. By Lemma 1, each of these sets is strictly convex. Since $T(n-1, n-1)$ implies $T(n, n-1)$, there exists a great circle S_2 containing x and $-x$ which intersects all of these sets. Suppose neither x nor $-x$ belongs to A_r. If $A_r\cdot S_2$ consists of a pair of diametrically opposite points y and $-y$, then $\pm y\in A_i\cdot A_r\cdot S_2\subset A_i$, $i=1, \cdots, r$, and we have (a). Otherwise, $A_r\cdot S_2$ is a connected subset of S_2 which is in the interior of one of the semi-circles bounded by x and $-x$. Moreover, $A_i\cdot S_2$ is connected, $i=1, \cdots, s$, since $A_i\cdot S_2$ contains a point in $A_i\cdot A_r\cdot S_2$ as well as x. Choose a point x_i in each of the sets $A_i\cdot A_r\cdot S_2$, and let y be the nearest of the points $x_1, \cdots, x_s$ to x. Then y must belong to $A_1\cdot \cdots \cdot A_s\cdot A_r$, since the minor arc joining

x_i and x must contain y. This is case (b).

We now return to the proof of part 2. Suppose F has r members. By the induction hypothesis, there is a point x such that $x \in A_1 \cdot \cdots \cdot A_s$, while $-x \in A_{s+1} \cdot \cdots \cdot A_{r-1}$, where $1 \leqq s \leqq r-1$. If $T(n, n)$ is false, by part 3 there is a point y common to $s+1$ members of F. Applying part 3 again, we can find a point z such that (b) or (c) hold with s replaced by $s+1$ and $t \geqq 0$. If z is not common to $s+2$ members of F, by applying part 3 repeatedly we can eventually find a point common to $s+2$ members of F. Repeating the process, we finally arrive at a point common to $r-1$ members of F. Another application of part 3 completes the proof.

4. Convex sets in the ordinary sense.

In this section we use the term convex in the ordinary sense unless otherwise stated. A hyperplane in E_n is a translation of a subspace of E_n. We define a -1-dimensional hyperplane as the empty set.

THEOREM 4. *Let F be a family of bounded, closed convex sets in E_n which is such that every k members of F have a point in common, $1 \leqq k \leqq n+1$. Then through each $n-k$-dimensional hyperplane there passes an $n-k+1$-dimensional hyperplane which intersects every member of F.*

PROOF. The case $k=n+1$ is Helly's theorem. Suppose $1 \leqq k \leqq n$, and let H_{n-k} be an $n-k$-dimensional hyperplane. By a change of coordinates, we may suppose that H_{n-k} passes through the origin. We may also assume that no member of F intersects H_{n-k}. If $A \in F$, let A' be the set $E_y(y = x/|x|, x \in A)$. Clearly, A' is a closed subset of S_n. If y_1 and y_2 are linearly independent points of A', that is, $y_i = x_i/|x_i|$, $x_i \in A$, $i = 1, 2$, and if $\alpha_1 \geqq 0$, $\alpha_2 \geqq 0$, $\alpha_1 + \alpha_2 > 0$, then

$$\frac{\alpha_1 y_1 + \alpha_2 y_2}{|\alpha_1 y_1 + \alpha_2 y_2|} = \frac{\lambda x_1 + (1-\lambda)x_2}{|\lambda x_1 + (1-\lambda)x_2|}$$

where

$$\lambda = \frac{\alpha_1/|x_1|}{\alpha_1/|x_1| + \alpha_2/|x_2|} \geqq 0.$$

Hence A' is a strictly convex subset of S_n (in the sense of Definition 2). Let F' be the family of sets A', $A \in F$. F' satisfies the hypotheses of Theorem 2, and we conclude that through $S_{n-k} = H_{n-k} \cdot S_n$ there passes a great $n-k+1$-sphere S_{n-k+1} intersecting all members of F'. Let H_{n-k+1} be the subspace generated by S_{n-k+1}. The statement $S_{n-k+1} \cdot A' \neq 0$ implies there is a point x in A such that $x/|x| \in S_{n-k+1}$

and hence $x \in H_{n-k+1}$. Thus H_{n-k+1} intersects every member of F. This completes the proof.

As shown in the proof of Theorem 2 (see part 1) the cases $k \leqq n$ of Theorem 4 can be made to depend on the case $k=n$. It does not appear possible to derive the case in n from the Helly case $k=n+1$.

Upon learning of Theorem 4, S. Karlin and L. S. Shapley [3] gave a short proof using Helly's Theorem and the non-existence of a non-vanishing tangential vector field on a sphere of any dimension such that the vectors at antipodal points are equal. Their method does not yield Theorem 2.

REFERENCES

1. Alfred Horn and F. A. Valentine, *Some properties of L sets in the plane*, Duke Math. J. vol. 16 (1949) pp. 131–140.

2. C. V. Robinson, *Spherical theorems of Helly type and congruence indices of spherical caps*, Amer. J. Math. vol. 64 (1942) pp. 260–272.

3. S. Karlin and L. S. Shapley, *Some applications of a theorem on convex functions*, Project Rand, publication P-74, April, 1949.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES

A THEOREM IN FINITE PROJECTIVE GEOMETRY

CHUNG-TAO YANG

It is known[1] that if F is a Galois field of order p^m and S_n^m a finite projective n-space over F, then each line of S_n^m passes through p^m+1 points and in S_n^m appear $(N_{n,0}^m N_{n,1}^m \cdots N_{n,t}^m)/(N_{t,0}^m N_{t,1}^m \cdots N_{t,t}^m)$ t-spaces S_t^m, where

$$N_{i,j}^m \equiv p^{mi} + p^{m(i-1)} + \cdots + p^{mj}.$$

In case F possesses a subfield F' of order p^r, there exists in S_n^m at least one n-subspace S_n^r, that is, a finite projective n-space, on which the points contained in a line are p^r+1 in number. The converse is also true.

The object of this note is to prove the following theorem.

In order to divide an S_n^m into several S_n^r such that one and only one S_n^r contains a given point, it is necessary and sufficient that r is a divisor of m and that m/r is relatively prime to $n+1$.

We first prove the necessity of the condition.

From the above remark, m is evidently divisible by r. By hypothesis every point of S_n^m is contained in one and only one S_n^r; we infer that $N_{n,0}^r = (p^{r(n+1)}-1)/(p^r-1)$ is a divisor of $N_{n,0}^m = (p^{m(n+1)}-1)/(p^m-1)$. Hence $(m/r, n+1)=1$ is a consequence of the following lemma.

LEMMA. *Let α, β, and $a>1$ be three natural integers;*

$$(a-1)(a^{\alpha\beta}-1)/(a^\alpha-1)(a^\beta-1)$$

is an integer if and only if $(\alpha, \beta)=1$.

To prove this we note that $(\alpha, \beta)=1$ implies $(a^\alpha-1; \alpha^\beta-1)=a-1$, and both $a^\alpha-1$ and $a^\beta-1$ are divisors of $a^{\alpha\beta}-1$, so that

$$(a-1)(a^{\alpha\beta}-1)/(a^\alpha-1)(a^\beta-1)$$

is an integer.

Conversely, suppose that $(a-1)(a^{\alpha\beta}-1)/(a^\alpha-1)(a^\beta-1)$ is an integer. If on the contrary $(\alpha, \beta)>1$, on denoting a prime factor of (α, β) by q so that $\alpha=\gamma q$ and $\beta=\delta q$, we have

$$\frac{(a-1)(a^{\alpha\beta}-1)}{(a^\alpha-1)(a^\beta-1)} = \frac{(a-1)(a^{\gamma q(\delta q-1)} + a^{\gamma q(\delta q-2)} + \cdots + a^{\gamma q} + 1)}{a^{\delta q}-1}.$$

Received by editors August 2, 1948.

[1] O. Veblen and W. H. Bussey, *Finite projective geometries*, Trans. Amer. Math. Soc. vol. 7 (1906) p. 244.

As $a^{\gamma q(\delta q-i)} \equiv a^{\gamma q(\delta-k)}$ (mod $a^{\delta q}-1$) if $i \equiv k$ (mod δ) and $\delta > k$, we obtain that

$$\frac{q(a-1)(a^{\gamma q(\delta-1)}+a^{\gamma q(\delta-2)}+\cdots+a^{\gamma q}+1)}{a^{\delta q}-1}=\frac{q(a-1)(a^{\gamma\delta q}-1)}{(a^{\gamma q}-1)(a^{\delta q}-1)}$$

is an integer. Consequently, $q(a-1)(a^{\gamma\delta q}-1) \geqq (a^{\gamma q}-1)(a^{\delta q}-1)$, namely,

$$qa^{\gamma\delta q+1}+a^{\gamma q}+a^{\delta q}+q \geqq a^{\gamma\delta q^2}+qa^{\gamma\delta q}+qa+1.$$

On the other hand we easily derive

$$qa^{\gamma\delta q} \geqq 2a^{\gamma\delta q} \geqq a^{\gamma q}+a^{\delta q},$$
$$qa+1>q,$$

and

$$a^{\gamma\delta q^2}=a^{\gamma\delta q^2-\gamma\delta q-1}a^{\gamma\delta q+1} \geqq qa^{\gamma\delta q+1},$$

which contradicts the inequality just obtained. Hence the lemma follows.

We now prove the sufficiency of the condition stated in the theorem. Consider an irreducible polynomial in $F[x]$:

$$x^{n+1}-a_0x^n-a_1x^{n-1}-\cdots-a_{n-1}x-a_n;$$

then the projective collineation in S_n^m, which carries the point with coordinates $(x_0, x_1, \cdots, x_n)$ into the point with coordinates $(x_0', x_1', \cdots, x_n')$ given by

$$\begin{aligned}\rho x_0' &= a_0x_0+x_1, \quad \rho x_1' = a_1x_0+x_2, \cdots, \\ \rho x_{n-1}' &= a_{n-1}x_0+x_n, \quad \rho x_n' = a_nx_0,\end{aligned} \tag{1}$$

is fixed point free. It may be shown[3] that the collineation (1) is of order $N_{n,0}^m$, when and only when a root of

$$x^{n+1}-a_0x^n-a_1x^{n-1}-\cdots-a_{n-1}x-a_n=0 \tag{2}$$

in the algebraic extension of F with degree $n+1$, namely F^*, is of an order divisible by $N_{n,0}^m$.

Since r is a divisor of m, there exists in F a subfield F' of order p^r. Let

$$x^{n+1}-b_0x^n-b_1x^{n-1}-\cdots-b_{n-1}x-b_n$$

be any irreducible polynomial in $F'[x]$. A reference to $(m/r, n+1)=1$

---

[3] J. Singer, *A theorem in finite projective geometry and some applications to number theory*, Trans. Amer. Math. Soc. vol. 43 (1938) p. 379.

shows that it is also irreducible in $F[x]$. Hence the projective collineation

$$\rho x_0' = b_0x_0 + x_1,\ \rho x_1' = b_1x_0 + x_2, \cdots,\\ \rho x_{n-1}' = b_{n-1}x_0 + x_n,\ \rho x_n' = b_nx_0 \tag{3}$$

is fixed point free and admits a fixed S_n^r determined by the points $(1, 0, 0, \cdots, 0)$, $(0, 1, 0, \cdots, 0)$, $\cdots$, $(0, 0, \cdots, 0, 1)$.

Construct an $S_n^{m(n+1)}$ over F^* to contain the given S_n^m, and consider the projective collineations (1) and (3) on $S_n^{m(n+1)}$; then they are collineations with exactly $n+1$ independent fixed points. If ξ is a root of (2) in F^*, then the fixed points of (1) are $(y_0^{p^{im}}, y_1^{p^{im}}, \cdots, y_n^{p^{im}})$, $i=0, 1, \cdots, n$, where $(y_0, y_1, \cdots, y_n)$ is determined by

$$\xi y_0 = a_0y_0 + y_1,\ \xi y_1 = a_1y_0 + y_2, \cdots,\\ \xi y_{n-1} = a_{n-1}y_0 + y_n,\ \xi y_n = a_ny_0.$$

Similarly we get the fixed points of (3), $(z_0^{p^{im}}, z_1^{p^{im}}, \cdots, z_n^{p^{im}})$, $i=0, 1, \cdots, n$, where $(z_0, z_1, \cdots, z_n)$ is determined by

$$\eta z_0 = b_0z_0 + z_1,\ \eta z_1 = b_1z_0 + z_2, \cdots, \eta z_{n-1} = b_{n-1}z_0 + z_n,\ \eta z_n = b_nz_0,$$

with a root η of the equation

$$x^{n+1} - b_0x^n - b_1x^{n-1} - \cdots - b_{n-1}x - b_n = 0.$$

The projective collineation on $S_n^{m(n+1)}$, which leaves invariant $(y_0^{p^{im}}, y_1^{p^{im}}, \cdots, y_n^{p^{im}})$, $i=0, 1, \cdots, n$, and the given S_n^m, is completely determined by a point A on the S_n^m and its image B. Since B may be selected in at most $N_{n,0}^m$ ways so far as A is given and (1), being of order $N_{n,0}^m$, meets such a condition, we infer that all these projective collineations form a cyclic group of order $N_{n,0}^m$.

There exists $(n+1)^2$ elements c_{ik} in F, $i, k=0, 1, \cdots, n$, such that

$$\begin{aligned} y_0 &= c_{00}z_0 + c_{01}z_1 + \cdots + c_{0n}z_n,\\ y_1 &= c_{10}z_0 + c_{11}z_1 + \cdots + c_{1n}z_n,\\ &\cdots\cdots\cdots\cdots\cdots\cdots,\\ y_n &= c_{n0}z_1 + c_{n1}z_2 + \cdots + c_{nn}z_n. \end{aligned}$$

The transform T of (1) by

$$\begin{aligned} x_0' &= c_{00}x_0 + c_{01}x_1 + \cdots + c_{0n}x_n,\\ x_1' &= c_{10}x_0 + c_{11}x_1 + \cdots + c_{1n}x_n,\\ &\cdots\cdots\cdots\cdots\cdots\cdots,\\ x_n' &= c_{n0}x_0 + c_{n1}x_1 + \cdots + c_{nn}x_n \end{aligned}$$

is a projective collineation of order $N^m_{n,0}$ and leaves invariant $(z_0^{p^{im}}, z_1^{p^{im}}, \cdots, z_n^{p^{im}})$, $i=0, 1, \cdots, n$, and the given S^m_n. Therefore (3) is a power of T. Since a fixed S^r_n of (3) has been obtained, on denoting it by R^r_n we have that

$$T^i(R^r_n), \qquad i = 1, 2, \cdots, N^m_{n,0}/N^r_{n,0},$$

are the fixed S^r_n of (3), where $T^i(R^r_n)$ represents the image of R^r_n effected by T^i. These $N^m_{n,0}/N^r_{n,0}$ fixed S^r_n evidently satisfy the condition of the theorem. Thus we have completed the proof.

NATIONAL UNIVERSITY OF CHEKIANG

SOME CONSEQUENCES OF A WELL KNOWN THEOREM ON CONICS

R. A. ROSENBAUM AND JOSEPH ROSENBAUM

Graustein [4, p. 296][1] proves the following theorem:

THEOREM I. *If three point conics have a common chord, and the three conics are taken in pairs and the common chord of each pair which is opposite to the given common chord is drawn, the three resulting lines are concurrent.*

He remarks that several well known theorems, including those of Pascal and the existence of the radical center of 3 non-coaxal circles, are obtainable as special cases of the above. The following result also follows directly from Theorem I:

COROLLARY 1. *The joins of the intersections of the opposite sides of a complete quadrangle with a conic passing through two vertices of the quadrangle are concurrent.*

This corollary furnishes a simple proof of Ex. 155, p. 307 of Baker [1]: Let A, B, C, O be 4 points of a conic; let a line meet BC, CA, AB respectively in L, M, N; and OL, OM, ON meet the conic again in P, Q, R respectively. Then AP, BQ, CR meet in a point, lying on the line LMN.

It does not seem to have been noted that the following theorem may be obtained directly from Theorem I.

Received by the editors August 28, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

THEOREM 1. *The intersections of each of the conics of a pencil with a fixed conic $\mathcal{C}$ passing through 2 of the base points of the pencil are pairs of points of an involution on $\mathcal{C}$.*

PROOF. Suppose that the base points of the pencil of conics are A, B, C, D; that $\mathcal{C}$ passes through A and B; and that $\mathcal{C}$ intersects any 3 conics Σ_1, Σ_2, Σ_3 of the pencil in P_1, Q_1; P_2, Q_2; and P_3, Q_3. Then, by Theorem I, the lines P_1Q_1, P_2Q_2, CD are concurrent; so also are the lines P_1Q_1, P_3Q_3, CD. Hence the lines P_iQ_i all pass through the same point, for all points of intersection P_i, Q_i of $\mathcal{C}$ with an arbitrary conic, Σ_i, of the pencil. But this last is just the condition that P_i, Q_i are pairs of an involution on $\mathcal{C}$.

This result might also have been obtained more laboriously (for the proof of Theorem I itself is easy) by the method which Baker [2, pp. 134–138] uses for general involutions.

The involution theorem of Desargues follows from Theorem 1 as a special case by letting $\mathcal{C}$ consist of the line AB and a line not through any of the points A, B, C, D.

Theorem I leads to a poristic result in the following manner: Let Σ, Σ' be two conics through A, B, C, D. Choose A_1 on Σ, and draw AA_1 to meet Σ' in α_1. Let α_1B meet Σ in A_2; AA_2 meet Σ' in α_2, and so on. Starting with A_1 on Σ, we thus set up a sequence of points, A_i, on Σ: AA_i meets Σ' in α_i, and α_iB meets Σ in A_{i+1}. Then we have the following theorem.

THEOREM 2. *If A_n coincides with A_1 for some position of A_1, then A_n will coincide with A_1 for all positions of A_1.*

PROOF. Choose B_1 on Σ and let the intersection of BB_1 with Σ' be called β_1. The lines AA_1 and BB_1 constitute a conic through A and B; hence, by Theorem I, A_1B_1, $\alpha_1\beta_1$, and CD are collinear, at O, say. If the intersection of β_1A with Σ is called B_2, then, by the same argument, A_2B_2 passes through O. (We have, then, two sets of lines, A_iB_i and $\alpha_i\beta_i$, all concurrent at O). Now, if, for some A_1, A_n coincides with A_1, B_n will coincide with B_1, for all B_1. Hence, in turn, A_n will coincide with A_1, for all A_1.

The condition for periodicity when the two conics are circles may be easily obtained:

THEOREM 3. *Let S, S' be two circles, centered at O, O', intersecting at A and B. The sequence $\{A_i\}$ on S is obtained as follows: A_iA intersects S' at α_i; α_iB intersects S at A_{i+1}. Then a necessary and sufficient condition that A_n shall coincide with A_1 for all starting points, A_1, on S is that angle $OAO' = m\pi/n$ where m and n have no common factor.*

PROOF. In triangle $A_iB\alpha_i$, each of the angles A_i and α_i is constant, because they are inscribed angles subtending the fixed arcs AB and BA of circles S and S'. Hence, the third angle, B, is also constant. Now when A_i has the particular position where A_iA is perpendicular to AB, then BA_i and $B\alpha_i$ are diameters; so that for this position angle $A_iB\alpha_i$ is angle OBO'. Hence angle $A_iB\alpha_i$ equals angle OBO' for all positions of A_i. Since angle $A_iB\alpha_i$ is the same as angle ABA_{i+1}, and since angle A_iBA_{i+1} equals 1/2 arc A_iA_{i+1}, it follows that arc A_iA_{i+1} equals twice angle OBO', for all positions of A_i.

The condition for A_n to coincide with A_1 is obviously that $\sum_1^n$ arc $A_iA_{i+1}=2m\pi$, or that angle OAO' (=angle OBO') $=m\pi/n$.

This result is related to Steiner's porism on a ring of circles tangent to two fixed circles and to the neighboring circles of the ring. (See, for example, Coolidge [3, p. 30] or Johnson [5, p. 115].)

COROLLARY 2. *Given two circles S_1, S_2 with their line of centers intersecting S_1 in P_1Q_1 and S_2 in P_2Q_2. Let the circles constructed on P_1Q_2 and P_2Q_1 as diameters be Σ_1, Σ_2. Then, if either of the pairs (S_1, S_2), (Σ_1, Σ_2) is a Steiner pair, the other is a pair of the type of Theorem* 3, *and conversely.*

REFERENCES

1. H. F. Baker, *Introduction to plane geometry.*
2. ———, *Principles of geometry*, vol. II.
3. J. L. Coolidge, *Treatise on the circle and the sphere.*
4. W. C. Graustein, *Introduction to higher geometry.*
5. R. A. Johnson, *Modern geometry.*

REED COLLEGE AND
THE MILFORD SCHOOL

ON THE FREQUENCY OF PAIRS OF SQUARE-FREE NUMBERS WITH A GIVEN DIFFERENCE

L. MIRSKY

If k is a positive integer, then the function

$$f(x) = f(x, k) = \sum_{n \leq x} | \mu(n)\mu(n+k) |$$

enumerates the number of pairs of square-free integers with fixed difference k such that the smaller of the two does not exceed x. The purpose of the present note is to establish the following result.

THEOREM. *As $x \to \infty$ we have*

$$f(x) = \prod_{p} \left(1 - \frac{2}{p^2}\right) \prod_{p^2 | k} \left(1 + \frac{1}{p^2 - 2}\right) x + O(x^{2/3} \log^{4/3} x),$$

where the O-constant may depend upon k.

In a previous publication[1] I considered the more general sum

$$F(x) = \sum_{n \leq x} \mu_r(n + k_1) \cdots \mu_r(n + k_s),$$

where $k_1, \cdots, k_s$ are distinct integers, r is an integer greater than 1, and $\mu_r(n)$ is defined as 0 or 1 according as n is or is not divisible by the rth power of a prime. I showed that, for $x \to \infty$,

$$F(x) = Ax + O(x^{2/(r+1)+\epsilon}), \tag{1}$$

where A is a constant which can be expressed as an infinite series or else as a product ranging over primes. The asymptotic formula (1) generalized and sharpened an earlier estimate due to Pillai.[2] The present note furnishes a slight improvement on (1) for the case $r=2$, $s=2$. The factor x^ϵ in (1) arose from the expression $\max_{\nu \leq x} d(\nu)$, and could not, therefore, be replaced by a power of $\log x$ by the method previously used.

Our notation is as follows. The letters x, y denote positive numbers; all other small letters denote positive integers unless otherwise stated, and p is reserved for primes.

The O-notation refers to the passage $x \to \infty$, and O-constants de-

---

Received by the editors July 8, 1948.

[1] L. Mirsky, *Note on an asymptotic formula connected with r-free integers*, Quart. J. Math. Oxford Ser. vol. 18 (1947) pp. 178–182.

[2] S. S. Pillai, *On sets of square-free integers*, J. Indian Math. Soc. N.S. vol. 2 (1936) pp. 116–118.

pend at most upon k.

(a, b) denotes the highest common factor of a and b.

$d(n)$ denotes the number of divisors of n.

We write

$$N_1(x, a, b, k) = \sum_{au-bv=k,\, bv\leqq x} 1; \qquad N_2(x, a, b, k) = \sum_{au^2-bv^2=k,\, bv^2\leqq x} 1.$$

We begin with two preliminary estimates.

LEMMA 1. *If* $(a, b)\nmid k$, *then* $N_1(x, a, b, k)=0$.
If $(a, b)\mid k$, *then* $N_1(x, a, b, k)=x(a, b)/ab+O(1)$.

This result is effectively case $s=2$ of Lemma 2 of my paper referred to above. The proof is extremely simple and may be left to the reader.

LEMMA 2. $N_2(x, a, b, k)=O(\log x)$.

For the case when ab is not a square it was shown by Estermann[3] that

$$N_2(x, a, b, k) \leqq 2d(k)\{\log (x+k)+1\}.$$

If, on the other hand, ab is a square, the required result follows trivially. For write $a=l^2t$, $b=m^2t$. If $t\nmid k$, then $N_2(x, a,b , k)=0$, while if $t\mid k$, then

$$\begin{aligned} N_2(x, a, b, k) &= N_2(xt^{-1}, l^2, m^2, kt^{-1}) \\ &\leqq \sum_{l^2u^2-m^2v^2=kt^{-1}} 1 \leqq \sum_{r^2-s^2=kt^{-1}} 1 = O(1). \end{aligned}$$

We now come to the proof of the theorem. Since

$$|\mu(n)| = \sum_{m^2\mid n} \mu(m)$$

we have

$$f(x) = \sum_{a^2c-b^2d=k,\, b^2d\leqq x} \mu(a)\mu(b) = \sum\nolimits_1 + \sum\nolimits_2, \tag{2}$$

say, where $ab\leqq y$ in $\sum_1$ and $ab>y$ in $\sum_2$. Here y denotes a function of x to be fixed later.

Writing

$$K = \sum_{(a,b)^2\mid k} \mu(a)\mu(b)\frac{(a, b)^2}{a^2b^2}$$

---

[3] T. Estermann, *Einige Sätze über quadratfreie Zahlen*, Math. Ann. vol. 105 (1931) pp. 653–662, Hilfssatz 2 (Anhang).

we have, by Lemma 1,

$$\begin{aligned}\textstyle\sum_1 &= \sum_{ab\leqq y} \mu(a)\mu(b)N_1(x, a^2, b^2, k)\\ &= x \sum_{ab\leqq y,\,(a^2,b^2)\mid k} \mu(a)\mu(b)\frac{(a, b)^2}{a^2b^2} + O(y\log y)\\ &= Kx + O\left(x\sum_{ab>y}\frac{1}{a^2b^2}\right) + O(y\log y)\\ &= Kx + O(xy^{-1}\log y) + O(y\log y). \qquad (3)\end{aligned}$$

Again, by Lemma 2,

$$\begin{aligned}\left|\textstyle\sum_2\right| &\leqq \sum_{a^2c-b^2d=k,\,b^2d\leqq x,\,ab>y} 1 \leqq \sum_{a^2c-b^2d=k,\,b^2d\leqq x,\,cd<x(x+k)y^{-2}} 1\\ &= \sum_{cd<x(x+k)y^{-2}} N_2(x, c, d, k) = O(x^2y^{-2}\log^2 x). \qquad (4)\end{aligned}$$

Putting $y=x^{2/3}\log^{1/3}x$ we obtain, by (2), (3), and (4),

$$f(x) = Kx + O(x^{2/3}\log^{4/3}x).$$

Finally, it is clear that $K=\prod_p \chi_p$, where

$$\chi_p = \sum_{u,v=0,1;\,(p^u,p^v)^2\mid k} \mu(p^u)\mu(p^v)\frac{(p^u, p^v)^2}{p^{2u+2v}} = \begin{cases}1 - 1/p^2 & \text{if } p^2\mid k,\\ 1 - 2/p^2 & \text{if } p^2\nmid k.\end{cases}$$

The theorem now follows at once.

It may be worth mentioning that the method of the present note also enables us to investigate the sums

$$\sum_{n\leqq x}\left|\mu(q_1n + k_1)\mu(q_2n + k_2)\right|,$$

and

$$\sum_{n\leqq x}\left|\mu((q_1n + k_1)(q_2n + k_2))\right|$$

(where k_1, k_2, q_1, q_2 are given integers), and to obtain asymptotic formulae for these sums with error terms of the form $O(x^{2/3}\log^{4/3}x)$.

Another result which can be established by an argument analogous to that used above is as follows.

Let $t_1(n)=O(\log^{\alpha_1}n)$, $t_2(n)=O(\log^{\alpha_2}n)$, *where* α_1, α_2 *are any given real numbers, and let*

$$T_1(n) = \sum_{m^2\mid n} t_1(m), \qquad T_2(n) = \sum_{m^2\mid n} t_2(m).$$

Then

$$\sum_{n\leqq x} T_1(n)T_2(n+k) = Cx + O(x^{2/3}\log^{\alpha_1+\alpha_2+4/3}x),$$

where

$$C = \sum_{(a,b)^2\mid k} t_1(a)t_2(b)\frac{(a,b)^2}{a^2b^2}\,.$$

The theorem we have proved in detail is the special case $t_1(n) = t_2(n) = \mu(n)$ of this result.

Added in proof (September 1949). In a recent paper (Quart. J. Math. Oxford Ser. vol. 20 (1949) pp. 65–79) F. V. Atkinson and Lord Cherwell obtained a generalization of Lemma 2 in which squares are replaced by rth powers. This new estimate enables me to extend the result of the present paper to numbers not divisible by rth powers, and to prove that, for any $r\geqq 2$,

$$\sum_{n\leqq x}\mu_r(n)\mu_r(n+k) = \prod_p\left(1-\frac{2}{p^r}\right)\prod_{p^r\mid k}\left(\frac{p^r-1}{p^r-2}\right)x$$
$$+ O(x^{2/(r+1)}(\log x)^{(r+2)/(r+1)}).$$

University of Sheffield

THE IRREGULARITY OF AN ALGEBRAIC SURFACE AND A THEOREM ON REGULAR SURFACES

H. T. MUHLY[1]

1. Introduction. In a recent paper [3][2] O. Zariski and the writer have shown that the arithmetic genus of a field Σ of algebraic functions of two variables can be invariantly defined with the aid of the Hilbert characteristic function associated with a pair of models of Σ. The first objective of this note is to show that the irregularity of Σ can be defined in a similar manner. The second objective is to obtain more general results on the existence of integral bases for regular surfaces than were obtained in [2].

2. The irregularity. We consider a field Σ of algebraic functions of two independent variables over a ground field k. The field k is assumed to be of characteristic zero and to be maximally algebraic in Σ. We use the notations and definitions of [3]. In particular, if U and V are normal models of Σ we let $\{A_m\}$ ($\{B_n\}$) denote the system of curves cut out on U (V) by the hypersurfaces of order m (n) of its ambient space. The dimension increased by one of the complete system $|A_m+B_n|$ (which system is regarded as lying on the join W of U and V) is denoted by $r(m, n)$. The transformation $T: U\rightarrow V$ is said to be proper [3, definition 2] if $T(A)$ is normal for a generic $A\in\{A_1\}$.

LEMMA 1. *If the transformation* $T: U\rightarrow V$ *is proper then there exists an integer* n_0 *such that for* $n\geqq n_0$ *and* $m\geqq i$ *the complete system* $|A_m+B_n|$ *cuts a complete series on the generic curve* A_i *of the system* $|A_i|$, *where* i *is an arbitrary positive integer.*

PROOF. Let A be a nonsingular irreducible hyperplane section of U such that $T(A)$ is normal. (Such hyperplane sections exist in view of the Bertini-Zariski theorems [7] and [9] and the fact that T is proper.) If $\bar{r}(m, n)$ is the r-function associated with the pair $(A, T(A))$ in the sense of [3] (article 2), then by formula 4.1 of [3], there exists an integer n_0 such that when $n\geqq n_0$ the function $r(m, n)$ satisfies the addition formula

$$r(m, n) = r(m-1, n) + \bar{r}(m, n). \tag{2.1}$$

Presented to the Society, September 10, 1948; received by the editors July 20, 1948.

[1] Research paper done under Office of Naval Research contract N6ori-71, Task Order XVIII (NR-043 054) at the University of Illinois and Harvard University. The ideas expressed in this paper represent the personal views of the author and are not necessarily those of the Office of Naval Research.

[2] Numbers in brackets refer to the bibliography.

If π is the genus of A, if μ and ν are respectively the orders of A and $T(A)$, and if $n\nu > 2\pi - 2$, then the function $\bar{r}(m, n)$ is given by the formula $\bar{r}(m, n) = m\mu + n\nu - \pi + 1$, so that by repeated application of (2.1),

$$r(m, n) = r(m - i, n) + i(m\mu + n\nu - \pi + 1) - 2^{-1}i(i - 1)\mu. \tag{2.2}$$

If μ_i and π_i are the degree and genus of the system $|A_i|$ the well known formulas of Noether yield

$$\begin{aligned}\mu_i &= i^2\mu,\\ \pi_i &= i\pi + 2^{-1}i(i - 1)\mu - i + 1,\end{aligned} \tag{2.3}$$

so that (2.2) becomes

$$r(m, n) = r(m - i, n) + i(m\mu + n\nu) - \pi_i + 1. \tag{2.4}$$

Let A_i be a generic[3] member of $|A_i|$ and observe that $|A_m + B_n|$ cuts a series of order $(m\mu + n\nu)i$ on A_i. If necessary we increase n_0 so that $n_0\nu > 2\pi - 2$. It then follows from the second of formulas (2.3) that if $m \geqq i$ and $n \geqq n_0$ the inequality, $(m\mu + n\nu)i > 2\pi_i - 2$, holds so that the series cut on A_i by $|A_m + B_n|$ is non-special. If d is its deficiency, then the dimension of this series is $(m\mu + n\nu)i - \pi_i - d$, and since the residual system of $|A_m + B_n|$ with respect to A_i is the complete system $|A_{m-i} + B_n|$,

$$r(m, n) = r(m - i, n) + (m\mu + n\nu)i - \pi_i - d + 1. \tag{2.5}$$

Equations (2.4) and (2.5) together imply that $d = 0$, q.e.d.

It is well known that if C_1 and C_2 are generic members of a linear system $|C|$ of curves on an algebraic surface then the characteristic series cut by $|C|$ on C_i $(i = 1, 2)$ will have the same deficiency. This deficiency is therefore a character of the system $|C|$. We quote the following lemma proved in [4] (page 70).

LEMMA 2. *If the system* $|C + D|$ *cuts out a complete series on a generic curve* $C \in |C|$, *then the deficiency of the characteristic series of* $|C|$ *is not greater than that of* $|D|$.

If we let $\alpha(m)$ and $\beta(n)$ denote the deficiencies of the characteristic series of the complete systems $|A_m|$ and $|B_n|$ respectively then Lemmas 1 and 2 together yield the following theorem.

THEOREM 1. *If the transformation* $T: U \rightarrow V$ *is proper then there exists*

---

[3] The term "generic" is here used only to signify that A_i is irreducible and has genus π_i, where π_i is the genus of $|A_i|$. Unless otherwise specified the term will be used in this sense throughout the text.

an integer n_0 such that $\alpha(m) \leqq \beta(n)$ when $n \geqq n_0$ and m is arbitrary.

PROOF. By Lemma 1 there is an integer n_0 such that $|A_m+B_n|$ cuts a complete series on the generic curve of $|A_m|$ when $n \geqq n_0$ and m is arbitrary. By Lemma 2 it follows that $\alpha(m) \leqq \beta(n)$, q.e.d.

COROLLARY 1. *If both $T: U \rightarrow V$ and $T^{-1}: V \rightarrow U$ are proper then there exist integers m_0 and n_0 such that $\alpha(m) = \beta(n)$ if $m \geqq m_0$ and $n \geqq n_0$.*

It follows that the deficiency $\alpha(m)$ of the characteristic series of the complete system determined by the m-fold of the system of hyperplane sections of any normal model U of Σ is independent of m if m is sufficiently large. Indeed, if V is any model of Σ in regular correspondence with U (so that both $T: U \rightarrow V$ and $T^{-1}: V \rightarrow U$ are proper [3]) and if m_0 and n_0 are the integers determined by the pair (U, V) in the sense of Corollary 1, then $\alpha(m) = \beta(n_0)$ for all $m \geqq m_0$. Moreover, $\alpha(m_0)$ is the maximum value of $\alpha(m)$ since $\alpha(m) \leqq \beta(n_0) = \alpha(m_0)$ for all m. This maximum value assumed by the function $\alpha(m)$ is therefore a character of the model U. We denote it by $\delta(U)$. The non-negative numerical character $\delta(U)$ is a ***relative invariant*** of U, for if U and V are in regular correspondence then $\delta(U) = \alpha(m_0) = \beta(n_0) = \delta(V)$.

COROLLARY 2. *If U and V are normal models of Σ such that*[4] *$U < V$ then $\delta(U) \leqq \delta(V)$.*

PROOF. If U and V are normal and if $U < V$ then $T: U \rightarrow V$ is proper. Hence by Theorem 1 there exists an integer n_0 such that $\delta(U) \leqq \beta(n_0) \leqq \delta(V)$, q.e.d.

COROLLARY 3. *If U and V are nonsingular models of Σ then $\delta(U) = \delta(V)$.*

PROOF. It is shown in [8] that there exist models U_1 and V_1 such that the correspondence $T_1: U_1 \rightarrow V_1$ is regular and such that U_1 (V_1) is obtained from U (V) by a sequence of quadratic transformations with simple centers. Since such quadratic transformations and their inverses are proper [3, Lemma 3] and since T_1 and T_1^{-1} are proper Corollary 1 implies that $\delta(U) = \delta(U_1) = \delta(V_1) = \delta(V)$, q.e.d.

Since the character $\delta(U)$ has the same value for all nonsingular models of Σ we can regard it as a character of Σ. It is this character which we define to be the irregularity q of the field Σ, and Σ is said to be regular or irregular according as $q=0$, or $q>0$. The use of the term "irregularity" for this character of Σ is justified by the well known work of Castelnuovo (see [4, chap. IV]) who has obtained the above results by different methods.

---

[4] The notation $U < V$ is used to indicate that the birational transformation T^{-1}: $V \rightarrow U$ has no fundamental points on V; or equivalently, the local ring $Q(P')$ contains the local ring $Q(P)$, when $P(\subset U)$ and $P'(\subset V)$ are a pair of corresponding points in the birational correspondence T.

COROLLARY 4. *If U is any normal model of Σ then $\delta(U)\leqq q$.*

PROOF. There exists a nonsingular model V such that $U<V$ (see [6]). Hence $\delta(U)\leqq\delta(V)=q$, q.e.d.

3. **Regular models.** Let U be a normal model of Σ, and let $\{A_m\}$ be the system cut out on U by the hypersurfaces of order m of its ambient space.

LEMMA 3. *There exists an integer n_0 such that if $n\geqq n_0$ the complete system $|rA_n|$ cuts out a complete series on the generic curve of $|A_n|$ for any integer $r\geqq 2$.*

PROOF. We regard U as being in regular birational correspondence with itself under the identity correspondence and we identify the systems $\{A_m\}$ and $\{B_m\}$ of the preceding article. By Lemma 1 there exists an integer n_0 such that $|A_{m+n}|$ cuts a complete series on a generic $A_m\in|A_m|$ when $n\geqq n_0$. It follows that $|A_{n+ns}|$ cuts a complete series on a generic A_n if $n\geqq n_0$ and s is an arbitrary positive integer. Hence $|A_{rn}|(=|rA_n|)$ cuts a complete series on A_n if $n\geqq n_0$ and $r\geqq 2$, q.e.d.

THEOREM 2. *If U is a normal model of Σ such that the relative invariant $\delta(U)$ is zero, then there exists an integer h_0 such that if U_h is any derived arithmetically normal model of U belonging to a character of homogeneity $h\geqq h_0$, then the generic hyperplane section of U_h is arithmetically normal. In fact, any irreducible nonsingular hyperplane section of U_h is an arithmetically normal curve.*

PROOF. Let h_0 be the integer n_0 determined in Lemma 3 and let h be a character of homogeneity[5] of U such that $h\geqq h_0$. If $\{A_n\}$ denotes as usual the system cut out on U by the hypersurfaces of order n of its ambient space, and if U_h is the derived arithmetically normal model of U belonging to the character h, then the hypersurfaces of order r in the ambient space of U_h will cut out the complete system $|A_{hr}|$ on U_h, $r=1, 2, \cdots$ [1]. Since $h\geqq h_0$, it follows that $|A_{hr}|$ will cut out a complete series on the generic curve of $|A_h|$ if $r\geqq 2$. If necessary, we increase h_0 so that for $h \geqq h_0$ the deficiency of the characteristic series of $|A_h|$ will equal $\delta(U)$. Then since $\delta(U)$ is zero, the system $|A_h|$ will cut a complete series on a generic curve of $|A_h|$. It therefore follows that $|A_{hr}|$ cuts a complete series on a generic curve $A_h \in |A_h|$ for $r=1, 2, \cdots$, so that A_h is arithmetically normal [1].

---

[5] For the definition and properties of characters of homogeneity see [5, articles 20 and 21].

To show that any nonsingular irreducible hyperplane section of U_h is arithmetically normal we observe that the term "generic" is used to signify an irreducible member A_h of $|A_h|$ of genus π_h as was pointed out in footnote 3. In view of the fact that $|A_h|$ is cut out on U_h by the hyperplanes of its ambient space, it is a straightforward matter to show that any two irreducible nonsingular hyperplane sections of U_h have the same genus.[6] It follows that any such member of $|A_h|$ is generic in the sense in which we have used the term, q.e.d.

Models of Σ which satisfy the hypothesis of Theorem 2 (that is, normal models U such that $\delta(U)=0$) will be called *regular* models of Σ. If Σ is a regular field, then since $0 \leqq \delta(U) \leqq q$, it follows that every normal model of Σ is a regular model. Whether or not irregular fields possess regular models is an open question.

4. **Integral bases.** Let $\mathfrak{o} = k[x_0, x_1, \cdots, x_n]$ be the integrally closed ring of homogeneous coordinates along an arithmetically normal model W of Σ. A triple (y_0, y_1, y_2) of elements of $\mathfrak{o}$ will be called an *admissible* set of independent variables for $\mathfrak{o}$ if (a) y_i is homogeneous of degree one, and (b) $\mathfrak{o}$ is integrally dependent on $k[y_0, y_1, y_2]$. The elements y_0, y_1, y_2 will then be algebraically independent over k, and the quotients y_1/y_0, y_2/y_0 will form a transcendence base for Σ/k. If $\nu = [\Sigma: k(y_1/y_0, y_2/y_0)]$, and if there exist ν elements $\omega_1, \omega_2, \cdots, \omega_\nu$ in $\mathfrak{o}$ which are linearly independent over $k[y_0, y_1, y_2]$ and form a modular base for $\mathfrak{o}$ over $k[y_0, y_1, y_2]$, then the set (y_0, y_1, y_2) will be called a *regular set* of independent variables for $\mathfrak{o}$. The elements $\omega_1, \omega_2, \cdots, \omega_\nu$ are said to be an independent integral base for $\mathfrak{o}$ over $k[y_0, y_1, y_2]$. The ring $\mathfrak{o}$ will be called a *regular ring* if every admissible set of independent variables is regular. It is evident that if the set (y_0, y_1, y_2) is regular then the set (x_0, x_1, x_2), $x_i = \sum a_{ij} y_j$, $a_{ij} \in k$, $|a_{ij}| \neq 0$, is also regular.

---

[6] Let $\mathfrak{o}$ be the ring of homogeneous coordinates on U_h and let $\chi(n)$ be the number of independent homogeneous elements of degree n in $\mathfrak{o}$. If A_h is an irreducible hyperplane section of U_h of order ν and genus p, then the fact that the prime ideal of A_h in $\mathfrak{o}$ is a principal ideal, together with the fact that A_h is nonsingular, implies that $\chi(n) = 2^{-1}n(n-1)\nu + (\nu - p + 1)n + c$, where c is a constant. (See [2, article 6] for details.) Since the function $\chi(n)$ is independent of the curve A_h used to compute it, the fact that all irreducible nonsingular members of $|A_h|$ have the same genus is established.

Let U be a regular model of Σ and let U_h be a derived arithmetically normal model of U such that every irreducible nonsingular hyperplane section of U_h is arithmetically normal. Let $\mathfrak{o}$ be the ring of homogeneous coordinates on U_h and let (y_0, y_1, y_2) be an admissible set of independent variables for $\mathfrak{o}$. Since $\mathfrak{o}$ depends integrally on $k[y_0, y_1, y_2]$, it follows that the ideal $\Sigma\mathfrak{o}y_i$ is irrelevent so that the net of curves, $c_0y_0+c_1y_1+c_2y_2=0$, $c_i \in k$, has no base points on U_h. Since U_h has only a finite number of singularities it follows [9] that the generic curve of this net is nonsingular. Since the quotients y_1/y_0 and y_2/y_0 form a transcendence base for Σ/k, the general curve of the net is irreducible [7]. Hence, after applying a linear transformation to the quantities y_i if necessary, we can assume that (a) the ideals $\mathfrak{o}y_i$ are prime, (b) the curves $y_i=0$, $i=0, 1, 2$, intersect pair by pair at simple points of U_h, and (c) the curves $y_i=0$ are arithmetically normal. It is shown in [2, article 8] that these conditions are sufficient to insure the existence of an independent integral base for $\mathfrak{o}$ over $k[y_0, y_1, y_2]$.[7] Hence every admissible set of independent variables in $\mathfrak{o}$ is regular, and $\mathfrak{o}$ is a regular ring. We can therefore assert the following theorem.

THEOREM 3. *If U is a regular model of Σ then the ring of homogeneous coordinates along a derived arithmetically normal model of U belonging to a sufficiently high character of homogeniety is a regular ring.*

5. **Regular fields.** Let Σ be a regular field and let ξ_1, ξ_2 be an arbitrary transcendence base for Σ/k. The question has been raised by O. Zariski as to whether or not the integral closure $\mathfrak{T}$ in Σ of the ring $k[\xi_1, \xi_2]$ always has an independent integral base over $k[\xi_1, \xi_2]$. Although we cannot answer this question, the following theorem throws some light on the problem.

THEOREM 4. *If (ξ_1, ξ_2) is a transcendence base for the regular field Σ/k, then there exist integers h such that the integral closure in Σ of the ring $R_h=k[\xi_1^h, \xi_2^h]$ has an independent integral base over R_h.*

PROOF. For any h the integral closure in Σ of R_h coincides with the integral closure $\mathfrak{T}$ of R_1. The ring $\mathfrak{T}$ is a finite integral domain, so that there exist elements $\xi_3, \xi_4, \cdots, \xi_n$ in $\mathfrak{T}$ such that $\mathfrak{T}=k[\xi_3, \xi_4, \cdots, \xi_n]$. Let η_0 be a transcendental over Σ and let $\eta_i=\eta_0\xi_i$, $i=1, 2$. If

$$(5.1)\quad \xi_i^\nu + a_{i1}(\xi_1, \xi_2)\xi_i^{\nu-1} + \cdots + a_{i\nu}(\xi_1, \xi_2) = 0, \qquad i = 3, 4, \cdots, n,$$

---

[7] Condition (b) above is somewhat weaker than the corresponding condition (b) given in [2, article 8]. However, the stronger form was used only as a matter of convenience to simplify the details of the proof, as an examination of the proof will show.

is the equation of integral dependence for ξ_i over R_1 and if m is an integer which is not less than the greatest of the numbers $j^{-1}\cdot\deg a_{ij}$, $i=3, 4, \cdots, n; j=1, 2, \cdots, \nu$, then on multiplying (5.1) by $\eta_0^{m\nu}$ we find that $\eta_0^m\cdot\xi_i$ depends integrally on $k[\eta_0, \eta_1, \eta_2]$. By the transitivity of integral dependence it follows that $\eta_0^m\cdot\xi_i$ depends integrally on $k[\eta_0^m, \eta_1^m, \eta_2^m]$. We let $y_i=\eta_i^m$, $i=0, 1, 2; y_j=\eta_0^m\cdot\xi_j$, $j=3, 4, \cdots, n$. The quantities $y_0, y_1, \cdots, y_n$ can be regarded as the coordinates of the general point of a model W of Σ. Moreover, the ring $k[y]$ of homogeneous coordinates on W depends integrally on $k[y_0, y_1, y_2]$.

Let ρ be a character of homogeneity of W and let U be the derived normal model of W belonging to the character ρ. Since Σ is regular, the model U is regular and hence possesses a derived normal model U_σ which has a regular ring $\mathfrak{o}=k[x_0, x_1, \cdots, x_s]$ of homogeneous coordinates. We put $g=\rho\cdot\sigma$ and observe that g is a character of homogeneity of W and that U_σ is a derived normal model W_g of W belonging to the character g.

The quantities y_0^g, y_1^g, y_2^g are homogeneous of degree g when the degree is measured with respect to W, but they are of degree one when measured with respect to W_g. These elements are in $\mathfrak{o}$, and it is not difficult to see that every element of $\mathfrak{o}$ depends integrally on $k[y_0^g, y_1^g, y_2^g]$. In fact, every element of $\mathfrak{o}$ depends integrally on $k[y]$, so that $\mathfrak{o}$ is integral over $k[y_0, y_1, y_2]$. Since this latter ring is integral over $k[y_0^g, y_1^g, y_2^g]$, it follows that (y_0^g, y_1^g, y_2^g) is an admissible set of independent variables for $\mathfrak{o}$. Since $\mathfrak{o}$ is a regular ring, the set (y_0^g, y_1^g, y_2^g) is a regular set of independent variables.

After applying a nonsingular linear transformation with coefficients in k we can assume that $x_i=y_i^g$, $i=0, 1, 2$. Then $x_i/x_0=\xi_i^{gm}$, $i=1, 2$. If $h=gm$, then the ring $\mathfrak{o}_0=k[x_1/x_0, x_2/x_0, \cdots, x_s/x_0]$ depends integrally on R_h, and since $\mathfrak{o}_0$ is integrally closed, $\mathfrak{o}_0=\mathfrak{T}$. By [2, Theorem 2.1] the fact that $\mathfrak{o}$ has an independent modular base over $k[x_0, x_1, x_2]$ consisting of $\mu=[\Sigma: k(x_1/x_0, x_2/x_0)]$ elements implies that $\mathfrak{o}_0$ has an independent integral base over $k[x_1/x_0, x_2/x_0]$, that is, $\mathfrak{T}$ has an independent integral base over R_h, q.e.d.

BIBLIOGRAPHY

1. H. T. Muhly, *A remark on normal varieties*, Ann. of Math. vol. 42 (1941).

2. ———, *Integral bases and a characterization of regular surfaces*, Trans. Amer. Math. Soc. vol. 54 (1943).

3. H. T. Muhly and O. Zariski, *Hilbert's characteristic function and the arithmetic genus of an algebraic variety*, to appear in Trans. Amer. Math. Soc.

4. O. Zariski, *Algebraic surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3, 1935.

5. ———, *Some results in the arithmetic theory of algebraic varieties*, Amer. J. Math. vol. 61 (1939).

6. ———, *The reduction of the singularities of an algebraic surface*, Ann. of Math. vol. 40 (1939).

7. ———, *Pencils on an algebraic variety and a new proof of a theorem of Bertini*, Trans. Amer. Math. Soc. vol. 50 (1941).

8. ———, *A simplified proof for the resolution of singularities of an algebraic surface*, Ann. of Math. vol. 43 (1942).

9. ———, *The theorem of Bertini on the variable singular points of a linear system of varieties*, Trans. Amer. Math. Soc. vol. 56 (1944).

UNIVERSITY OF ILLINOIS

ON THE EUCLIDEAN ALGORITHM IN QUADRATIC NUMBER FIELDS

H. CHATLAND

1. **Introduction.** Let m be a square-free rational integer. The field $R(m^{1/2})$ is said to be Euclidean or that the Euclidean algorithm exists in $R(m^{1/2})$ if for integers α, $\beta \neq 0 \subset R(m^{1/2})$ there exists an integer $\gamma \subset R(m^{1/2})$ such that

$$|N(\alpha - \beta\gamma)| < |N(\beta)|.$$

The problem of determining in what fields $R(m^{1/2})$ the algorithm exists has been worked out except for m equal to a prime of the form $24n+1$ and greater than 97. In this paper it is shown that the Euclidean algorithm does not exist for $m=24n+1>97$ except possibly for $m=193$, 241, 313, 337, 457, and 601. The problem is not settled in these six cases.

2. **Previous results.** In order that a field be Euclidean the class number must be 1. However, this condition is not sufficient for, as Dedekind pointed out [1][1], the field $R(-19^{1/2})$ has class number 1 but is not Euclidean. L. E. Dickson [2] showed that for m negative the Euclidean algorithm exists only if $m=-1$, -2, -3, -7, and -11. For m positive, the algorithm has been shown to exist for the following values of m:

(1) 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73, 97.

Except for the last two values in (1) the proofs have been obtained by O. Perron [3], A. Oppenheim [4], R. Remak [5], N. Hofreiter [6], and A. Berg [7]. It was pointed out by I. Schur [4, p. 351] that the algorithm does not exist for $m=47$. A. Oppenheim [4] proved that for $m=23$ and $m=53$ the algorithm does not exist. N. Hofreiter [8] proved non-existence for $m\equiv 14 \pmod{24}$ and [6] for $m=77$ and $m\equiv 21 \pmod{24}$, $m>21$. E. Berg [7] and J. F. Keston [9] proved non-existence for $m\not\equiv 1 \pmod 4$ except for the values listed in (1). Also, apart from (1) H. Behrbohm and L. Rédei [10] showed that the algorithm can exist only in the following three cases.

I. $m = p \equiv 13 \pmod{24}$,

Presented to the Society, June 19, 1948; received by the editors May 11, 1948, and in revised form, July 23, 1948.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

II. $m = p \equiv 1 \pmod 8$,

III. $m = pq$ with $p \equiv q \equiv 3 \pmod 8$ or $p \equiv q \equiv 7 \pmod 8$

where p and q are primes.

For sufficiently large m, P. Erdös and Ch. Ko [11] proved that the algorithm cannot exist in cases I and II. H. Heilbronn [12] proved a similar result for case III. L. Schuster [13] showed that except for $m=33$ and 57 in case III the algorithm exists at most for $m\equiv 1 \pmod{24}$. A. Brauer [14] proved that the algorithm cannot exist in case I for $p>109$. There remained then in case I only the values $m=61$ and $m=109$. L. Rédei [15] proved the non-existence of the algorithm for these two values. By an entirely different method L. K. Hua and W. T. Sheh [17] proved that the algorithm does not exist for $m=61$. L. Rédei [18] obtained the result in case III that if the algorithm exists then $m=3q$. This result coupled with that of Schuster [13] completes case III in which the algorithm exists only for $m=21$, 33, 57. There remain those values of $m=p\equiv 1 \pmod 8$. For $p=73$, 97, L. Rédei [15] proved the existence of the algorithm and for p of the form $24n+17$ and greater than 41 the non-existence of the algorithm. The case $p=24n+17$ and $p>41$ was also treated by L. Hua and S. Min [16] who left in doubt however $p=89$, 113, and 137.

3. **Present results.** Recently H. Danveport [19] proved that the Euclidean algorithm does not exist for quadratic fields whose discriminants exceed $(128)^2$. In their paper, mentioned above, Erdös and Ko prove the following theorem:

THEOREM. *For a prime p of the form $4n+1$, the Euclidean algorithm cannot exist in $R(p^{1/2})$, if p can be written in the form*

$$p = q_1m_1 + q_2m_2, \tag{2}$$

where m_1, m_2, q_1, q_2 are all positive and quadratic non-residues (mod p), *and where the q_i are odd primes which divide q_im_i to an odd power for $i=1, 2$.*

In this paper a representation of the form (2) is given for each prime of the form $24n+1$ greater than 97 and less than $(128)^2$ except for $p=193$, 241, 313, 337, 457, and 601. In the case of these last six it can be shown that no such representation exists. Hence for them no conclusion can be drawn, concerning the existence of the algorithm, by this method.

Representations of primes $p=24n+1$ in the form $p=q_1m_1+q_2m_2$

$409=14\times 19+11\times 13$
$433=7\times 29+10\times 23$
$577=14\times 13+5\times 79$
$673=17\times 15+19\times 22$
$769=23\times 21+13\times 22$
$937=5\times 7+41\times 22$
$1009=11\times 52+19\times 23$
$1033=11\times 13+89\times 10$
$1129=17\times 19+31\times 26$
$1153=17\times 14+61\times 15$
$1201=17\times 26+23\times 33$
$1249=19\times 44+7\times 59$
$1297=5\times 154+17\times 31$
$1321=17\times 39+47\times 14$
$1489=17\times 39+59\times 14$
$1609=19\times 14+17\times 79$
$1657=7\times 45+61\times 22$
$1753=7\times 165+13\times 46$
$1777=5\times 21+19\times 88$
$1801=19\times 78+11\times 29$
$1873=23\times 60+17\times 29$
$1993=7\times 5+89\times 22$
$2017=5\times 26+17\times 111$
$2089=7\times 190+11\times 69$
$2113=5\times 277+7\times 104$
$2137=13\times 129+5\times 92$
$2161=23\times 70+19\times 29$
$2281=7\times 51+13\times 148$
$2377=5\times 62+13\times 159$
$2473=5\times 31+19\times 122$
$2521=11\times 53+17\times 114$
$2593=5\times 275+29\times 42$
$2617=7\times 122+41\times 43$
$2689=13\times 46+17\times 123$
$2713=11\times 153+5\times 206$
$2833=7\times 15+11\times 248$
$2857=5\times 138+11\times 197$
$2953=5\times 42+13\times 211$
$3001=7\times 65+19\times 134$
$3049=11\times 94+13\times 155$
$3121=7\times 220+17\times 93$
$3169=7\times 86+17\times 151$
$3217=5\times 87+13\times 214$
$3313=13\times 45+11\times 248$
$3361=11\times 68+13\times 201$
$3433=7\times 10+19\times 177$
$3457=5\times 42+17\times 191$
$3529=13\times 57+17\times 164$
$3673=5\times 13+11\times 328$
$3697=5\times 92+13\times 249$
$3769=17\times 14+11\times 321$
$3793=19\times 60+7\times 379$
$3889=11\times 38+13\times 267$
$4057=29\times 20+19\times 183$
$4129=7\times 29+13\times 302$
$4153=13\times 102+11\times 257$
$4177=11\times 45+7\times 526$
$4201=11\times 38+13\times 291$
$4273=5\times 58+7\times 569$
$4297=5\times 19+11\times 382$
$4441=13\times 70+11\times 321$
$4513=19\times 15+7\times 604$
$4561=17\times 110+13\times 207$
$4657=5\times 46+19\times 233$
$4729=11\times 93+17\times 218$
$4801=23\times 21+17\times 254$
$4969=7\times 152+11\times 355$
$4993=5\times 57+11\times 428$
$5113=5\times 93+7\times 664$
$5209=11\times 306+19\times 97$
$5233=5\times 17+11\times 468$
$5281=7\times 38+17\times 295$
$5449=7\times 95+13\times 368$
$5521=7\times 89+31\times 158$
$5569=19\times 68+13\times 329$
$5641=7\times 172+29\times 153$
$5689=17\times 132+13\times 265$
$5737=37\times 20+19\times 263$
$5857=5\times 78+7\times 781$
$5881=31\times 13+11\times 498$
$5953=5\times 7+11\times 538$
$6073=5\times 29+19\times 312$

$6121 = 7\times 87+13\times 424$
$6217 = 17\times 15+11\times 542$
$6257 = 5\times 56+43\times 139$
$6337 = 5\times 61+13\times 464$
$6361 = 17\times 21+19\times 316$
$6481 = 13\times 342+11\times 185$
$6529 = 11\times 141+19\times 262$
$6553 = 11\times 280+23\times 151$
$6577 = 5\times 69+19\times 328$
$6673 = 5\times 308+29\times 177$
$6793 = 7\times 235+11\times 468$
$6841 = 29\times 67+31\times 158$
$6961 = 7\times 104+23\times 271$
$7057 = 13\times 80+11\times 547$
$7129 = 7\times 374+13\times 347$
$7177 = 5\times 38+17\times 411$
$7297 = 7\times 15+29\times 248$
$7321 = 7\times 312+11\times 467$
$7369 = 13\times 201+29\times 164$
$7393 = 23\times 29+19\times 354$
$7417 = 5\times 17+13\times 564$
$7489 = 37\times 70+23\times 213$
$7537 = 5\times 51+11\times 662$
$7561 = 13\times 73+29\times 228$
$7681 = 13\times 102+31\times 205$
$7753 = 5\times 416+31\times 183$
$7873 = 13\times 10+29\times 267$
$7993 = 5\times 88+7\times 1079$
$8017 = 19\times 90+17\times 371$
$8089 = 17\times 295+29\times 106$
$8161 = 7\times 29+23\times 346$
$8209 = 13\times 14+23\times 349$
$8233 = 17\times 165+23\times 236$
$8329 = 31\times 219+11\times 140$
$8353 = 5\times 17+13\times 636$
$8377 = 13\times 135+11\times 602$
$8521 = 13\times 31+11\times 738$
$8641 = 7\times 136+11\times 699$
$8689 = 29\times 195+37\times 82$
$8713 = 5\times 404+23\times 291$
$8737 = 37\times 30+29\times 263$
$8761 = 19\times 71+17\times 436$
$8929 = 13\times 114+11\times 677$
$9001 = 13\times 42+19\times 445$
$9049 = 11\times 53+17\times 498$
$9241 = 17\times 286+29\times 151$
$9337 = 5\times 203+19\times 438$
$9433 = 13\times 80+11\times 763$
$9601 = 13\times 174+41\times 179$
$9649 = 41\times 212+11\times 87$
$9697 = 11\times 30+17\times 551$
$9721 = 11\times 774+17\times 71$
$9769 = 17\times 390+43\times 73$
$9817 = 5\times 42+13\times 739$
$10009 = 11\times 105+19\times 466$
$10177 = 5\times 7+11\times 922$
$10273 = 5\times 238+31\times 293$
$10321 = 7\times 41+29\times 346$
$10369 = 13\times 207+11\times 698$
$10513 = 5\times 28+11\times 943$
$10657 = 29\times 353+7\times 60$
$10729 = 41\times 53+23\times 372$
$10753 = 5\times 351+11\times 818$
$10993 = 5\times 156+7\times 1459$
$11113 = 5\times 611+17\times 474$
$11161 = 7\times 66+13\times 823$
$11257 = 17\times 190+23\times 349$
$11329 = 11\times 140+13\times 753$
$11353 = 7\times 19+17\times 660$
$11497 = 7\times 60+11\times 1007$
$11593 = 11\times 40+19\times 587$
$11617 = 13\times 245+17\times 496$
$11689 = 7\times 264+13\times 757$
$11833 = 7\times 20+11\times 1063$
$11953 = 5\times 22+13\times 911$
$12049 = 19\times 156+23\times 395$
$12073 = 7\times 15+11\times 1088$
$12097 = 11\times 120+13\times 829$
$12241 = 13\times 126+23\times 461$
$12289 = 19\times 165+23\times 398$
$12409 = 13\times 63+19\times 610$
$12433 = 13\times 160+17\times 609$
$12457 = 19\times 330+23\times 269$
$12553 = 11\times 230+13\times 771$

$12577 = 5\times 126+13\times 919$
$12601 = 29\times 132+31\times 283$
$12721 = 17\times 52+19\times 623$
$12841 = 23\times 86+17\times 639$
$12889 = 11\times 195+17\times 632$
$13009 = 7\times 23+11\times 1168$
$13033 = 5\times 102+7\times 1789$
$13177 = 7\times 80+11\times 1147$
$13249 = 7\times 13+17\times 774$
$13297 = 5\times 153+13\times 964$
$13417 = 11\times 40+19\times 683$
$13441 = 29\times 110+17\times 603$
$13513 = 5\times 126+13\times 991$
$13537 = 5\times 63+11\times 1202$
$13633 = 5\times 569+31\times 348$
$13681 = 11\times 69+13\times 994$
$13729 = 17\times 328+31\times 263$
$13873 = 11\times 78+19\times 685$
$13921 = 11\times 118+13\times 971$
$14281 = 13\times 124+41\times 309$
$14401 = 11\times 145+19\times 674$
$14449 = 11\times 62+13\times 1059$

$14593 = 11\times 1123+5\times 448$
$14713 = 11\times 399+29\times 356$
$14737 = 11\times 80+31\times 447$
$14929 = 11\times 42+17\times 851$
$15073 = 5\times 366+41\times 323$
$15121 = 11\times 129+31\times 442$
$15193 = 17\times 250+31\times 353$
$15217 = 7\times 117+23\times 626$
$15241 = 11\times 78+19\times 757$
$15289 = 17\times 469+31\times 236$
$15313 = 5\times 281+19\times 732$
$15361 = 7\times 305+17\times 778$
$15601 = 17\times 56+19\times 771$
$15649 = 11\times 492+29\times 353$
$15817 = 5\times 577+53\times 244$
$15889 = 7\times 374+23\times 577$
$15913 = 11\times 70+19\times 797$
$15937 = 7\times 165+19\times 778$
$16033 = 7\times 15+11\times 1448$
$16057 = 7\times 60+19\times 823$
$16249 = 11\times 680+37\times 237$
$16273 = 5\times 322+31\times 473$

BIBLIOGRAPHY

1. P. G. L. Dirichlet, *Vorlesungen über Zahlentheorie*, ed. by R. Dirichlet, 4th ed., Braunschweig, 1894, p. 451.

2. L. E. Dickson, *Algebren und ihre Zahlentheorie*, Zürich and Leipzig, 1927, pp. 150–151.

3. O. Perron, *Quadratische Zahlkörper mit Euklidischen Algorithmus*, Math. Ann. vol. 107 (1932) pp. 489–495.

4. A. Oppenheim, *Quadratic fields with and without Euclid's algorithm*, Math. Ann. vol. 109 (1934) pp. 349–352.

5. R. Remak, *Über den Euklidischen Algorithmus in reellquadratischen Zahlkörpern*, Jber. Deutschen Math. Verein. vol. 44 (1934) pp. 238–250.

6. N. Hofreiter, *Quadratische Körper mit und ohne Euklidischen Algorithmus*, Monatshefte für Mathematik und Physik vol. 42 (1935) pp. 397–400.

7. E. Berg, *Über die Existenz eines Euklidischen Algorithmus in quadratischen Zahlkörpern*, Kungl. Fysiografiska Sällakapets i Lund Förhandlingar vol. 5 (1935) No. 5.

8. N. Hofreiter, *Quadratische Zahlkörper ohne Euklidischen Algorithmus*, Math. Ann. vol. 110 (1935) pp. 195–196.

9. J. F. Keston, *Existence of a Euclidean algorithm in quadratic fields*, Thesis, Yale University, 1935; cf. Bull. Amer. Math. Soc. vol. 41 (1935) p. 186.

10. H. Behrbohm and L. Rédei, *Dern Euklidische Algorithmus in quadratischen Körpern*, J. Reine Angew. Math. vol. 174 (1935) pp. 192–205.

11. P. Erdös and Ch. Ko, *Note on the Euclidean algorithm*, J. London Math. Soc. vol. 13 (1938) pp. 3–8.

12. H. Heilbronn, *On Euclid's algorithm in real quadratic fields*, Proc. Cambridge Philos. Soc. vol. 34 (1938) pp. 521–526.

13. L. Schuster, *Reelquadratische Zahlkorper ohne Euklidischen Algorithmus*, Monatshefte für Mathematik und Physik vol. 47 (1938) pp. 117–127.

14. A. Brauer, *On the non-existence of the Euclidean algorithm in certain quadratic number fields*, Amer. J. Math. vol. 62 (1940) pp. 697–716.

15. L. Rédei, *Zur Frage des Euklidischen Algorithmus in quadratischen Zahlkorpern*, Math. Ann. vol. 118 (1942) pp. 588–608.

16. L. K. Hua and S. H. Min, *On the distribution of quadratic non-residues and the Euclidean algorithm in real quadratic fields*. II, Trans. Amer. Math. Soc. vol. 56 (1944) pp. 547–569.

17. L. K. Hua and W. T. Sheh, *On the lack of an Euclidean algorithm in* $R(61^{1/2})$, Amer. J. Math. vol. 67 (1945) pp. 209–211.

18. L. Rédei, *Über den Euklidischen Algorithmus in reellquadratischen Zahlkörpern*, Matematiskai és Fizikai Lapok vol. 47 (1940) pp. 78–90.

19. H. Davenport, *Indefinite binary quadratic forms, and Euclid's algorithm in real quadratic fields*, To appear in Proc. London Math. Soc.

OHIO STATE UNIVERSITY

TRANSCENDENCE OF FACTORIAL SERIES WITH PERIODIC COEFFICENTS

VERNE E. DIETRICH AND ARTHUR ROSENTHAL

It is well known that every real number α can be represented by a factorial series

$$\alpha = \frac{a_1}{1!} + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!} + \cdots, \tag{1}$$

where the a_n $(n=1, 2, \cdots)$ are integers and, moreover, $0 \leqq a_n < n$ (for $n=2, 3, \cdots$). This representation is unique for the irrational numbers α, while every rational α can be represented either with almost all[1] $a_n=0$ or with almost all $a_n=n-1$.

The representation (1) was discussed and the aforesaid properties were proved by M. Stéphanos [1].[2] But an even more general type of series had already been studied by G. Cantor [2] (not known to M. Stéphanos). These series have later been called "Cantor series" (cf. [3]).

In this note we consider the case in which the factorial series (1) has periodic coefficients a_n and we prove the following theorem:

THEOREM 1. *Every number α represented by a factorial series* (1) *with periodic coefficients is transcendental* (*except for the trivial case where almost all a_n are zero*).

The above condition $0 \leqq a_n < n$ (for $n=2, 3, \cdots$) is not used at all in the following proof. Moreover, the coefficients a_n need not necessarily be integers; the a_n can be *any algebraic numbers*. Then Theorem 1 and its proof still hold.

We generalize Theorem 1 further:

THEOREM 2. *If the power series*[3]

$$\phi(z) = \sum_{n=1}^{\infty} \frac{a_n}{n!} z^n \tag{2}$$

has algebraic coefficients a_n (*not almost all of them being zero*) *which form a periodic sequence, then $\phi(z)$ is a transcendental number for every*

Presented to the Society, November 26, 1948; received by the editors August 12, 1948.

[1] The expression "almost all" is used in the sense of "all but a finite number."

[2] Numbers in brackets refer to the bibliography at the end.

[3] Under the conditions of Theorem 2, $\phi(z)$ is an entire function.

algebraic $z(\neq 0)$.

For $z=1$ Theorem 2 furnishes Theorem 1. Hence we prove Theorem 2.

The general case in which the period starts with a_s $(s\geqq 1)$ can immediately be reduced to the case in which the period begins with a_1. One has only to subtract and add $s-1$ terms, that is, algebraic numbers. Therefore it suffices to assume that the period starts at a_1.

Let m be the "length" of the period, that is, the number of coefficients a_n belonging to the period, and let ω be a primitive mth root of unity. Then

$$e^{\omega^k\cdot z} = 1 + \frac{\omega^k\cdot z}{1!} + \frac{\omega^{k2}\cdot z^2}{2!} + \cdots$$
$$+ \frac{\omega^{k(m-1)}\cdot z^{(m-1)}}{(m-1)!} + \frac{\omega^{km}\cdot z^m}{m!} + \cdots .$$

Thus, since the sum of the rth powers of all the mth roots of unity is zero if r is not divisible by m, we obtain:

$$\sum_{k=1}^{m} e^{\omega^k\cdot z} = m + 0 + \cdots + 0 + m\frac{z^m}{m!}$$
$$+ 0 + \cdots + 0 + m\frac{z^{2m}}{(2m)!} + \cdots$$

and hence

$$\frac{a_m}{m}\sum_{k=1}^{m} e^{\omega^k\cdot z} = a_m + 0 + \cdots + 0 + a_m\frac{z^m}{m!}$$
$$+ 0 + \cdots + 0 + a_m\frac{z^{2m}}{(2m)!} + 0 + \cdots .$$

Similarly for $r=1, 2, \cdots, m-1$ we have

$$\frac{a_{m-r}}{m}\sum_{k=1}^{m} \omega^{kr}e^{\omega^k\cdot z} = 0 + \cdots + 0 + a_{m-r}\frac{z^{m-r}}{(m-r)!}$$
$$+ 0 + \cdots + 0 + a_{m-r}\frac{z^{2m-r}}{(2m-r)!} + 0 + \cdots .$$

By summing over $r=0, 1, 2, \cdots, m-1$, we obtain

$$\sum_{k=1}^{m} e^{\omega^k\cdot z}\left(\frac{1}{m}\sum_{r=0}^{m-1} a_{m-r}\omega^{kr}\right) = \phi(z) + a_m$$

or, if we set $(1/m)\sum_{r=0}^{m-1} a_{m-r}\omega^{kr} = A_k$ $(k=1, 2, \cdots, m)$, we have

$$\sum_{k=1}^{m} A_k e^{\omega^k \cdot z} - [\phi(z) + a_m]\, e^0 = 0. \tag{3}$$

It is impossible that all coefficients A_k $(k=1, 2, \cdots, m)$ vanish; for then the equation of $(m-1)$st degree

$$\sum_{r=0}^{m-1} a_{m-r} z^r = 0$$

would have m different roots ω^k $(k=1, 2, \cdots, m)$, while the case that all $a_n = 0$ $(n=1, 2, \cdots, m)$ has been excluded. But now, since for an algebraic $z \neq 0$ the numbers $\omega^k \cdot z$ $(k=1, 2, \cdots, m)$ are algebraic, different, and nonzero and since the A_k $(k=1, 2, \cdots, m)$ are also algebraic numbers and at least one of them is not zero, it follows from (3) by Lindemann's general theorem ([4, 5], cf. also [6]) that $\phi(z) + a_m$ and hence also $\phi(z)$ is a transcendental number for every algebraic $z \neq 0$.

BIBLIOGRAPHY

1. M. Stéphanos, *Sur une propriété remarquable des nombres incommensurables*, Bull. Soc. Math. France vol. 7 (1879) pp. 81–83.

2. G. Cantor, *Ueber die einfachen Zahlensysteme*, Zeitschrift fur Mathematik und Physik vol. 14 (1869) pp. 121–128.

3. O. Perron, *Irrationalzahlen*, Göschens Lehrbücherei, vol. 1, Berlin, 1st ed., 1921, 2d ed., 1939, 3d ed., 1947, §33.

4. F. Lindemann, *Ueber die Zahl π*, Math. Ann. vol. 20 (1882) pp. 213–225.

5. ———, *Ueber die Ludolph'sche Zahl*, Sitzungsberichte der Akademie der Wiss. Berlin (1882) pp. 679–682.

6. K. Weierstrass, *Zu Lindemann's Abhandlung: "Ueber die Ludolph'sche Zahl"*, Sitzungsberichte der Akademie der Wissenschaften, Berlin (1885) pp. 1067–1086.

PURDUE UNIVERSITY

A NONHOMOGENEOUS MINIMAL SET

E. E. FLOYD

1. Introduction. In this note we consider the following question: *does there exist a compact minimal set which is of dimension* 0 *at some of its points and of positive dimension at others?* We answer the question in the affirmative by constructing a compact plane set X and a homeomorphism T of X onto X such that X is minimal with respect to T (that is, contains no proper closed subset Y with $T(Y) \subset Y$) and such that X possesses the desired property. As a result, there exist nonhomogeneous minimal sets.

An outline of the procedure is as follows. A compact, totally disconnected subset A of the x-axis in the plane and a homeomorphism f of A onto A are defined so that A is minimal with respect to f. Two real functions b_0 and b_1 are then defined on A with $0 \leqq b_0(x) \leqq b_1(x) \leqq 1$. We then let X be the set of all points $(x, tb_1(x)+(1-t)b_0(x))$ for $x \in A$ and $0 \leqq t \leqq 1$, thus in effect erecting a vertical interval or a point over each $x \in A$. Then T is defined so as to send the point determined by x and t into the point determined by $f(x)$ and t.

2. The example.

DEFINITIONS. Let A_i denote the set of integers $1, \cdots, 3^i$ and let π_{i+1} be the map from A_{i+1} to A_i defined by $\pi_{i+1}(p) = p \bmod 3^i$ for $p \in A_{i+1}$. Let A designate the limit space of the sequence (A_i, π_{i+1}) [1].[1] Then A is a compact totally disconnected metric space. Let f_i be the map from A_i onto A_i defined by $f_i(p) = (p+1) \bmod 3^i$; then $\pi_{i+1} f_{i+1} = f_i \pi_{i+1}$. It follows that the map defined by $f(x) = f_i((x_i))$ for $x = (x_i) \in A$ is a homeomorphism of A onto A. Moreover A is minimal with respect to f for if $x = (x_i) \in A$ and $y = (y_i) \in A$, then $f^{y_n - x_n}(x)$ has its first n coordinates equal to those of y.

Let $x = (x_i) \in A$; the points of A_{i+1} mapping onto x_i under π_{i+1} are $x_i + \alpha \cdot 3^i$, $\alpha = 0, 1, 2$. Define α_i by $x_{i+1} = x_i + \alpha_i \cdot 3^i$. We call the subsequence $\beta_1, \beta_2, \cdots$ of $\alpha_1, \alpha_2, \cdots$ consisting of all $\alpha_i \neq 1$ the *associated sequence* for x, and define several functions of x:

Let $a(x)$ *be the number of elements in the associated sequence for* x ($a(x)$ is either a non-negative integer or ∞).

Let $b_0(x) = 0$ *if* $a(x) = 0$, $b_0(x) = (1/2) \sum_{j=1}^{a(x)} \beta_j / 2^j$ *if* $a(x) > 0$.

Let $b_1(x) = b_0(x) + \sum_{j > a(x)} 1/2^j = b_0(x) + 1/2^{a(x)}$ if $a(x) < \infty$, *and let*

Presented to the Society, September 10, 1948; received by the editors July 20, 1948.

[1] Numbers enclosed in brackets refer to the bibliography at the end of the paper.

$b_1(x) = b_0(x)$ if $a(x) = \infty$.

Let $b_t(x) = tb_1(x) + (1-t)b_0(x)$ for $0 \leqq t \leqq 1$.

Since A is a totally disconnected compact metric space, it may be embedded in the x-axis of the plane. Let X be the set of all points $(x, b_t(x))$, $x \in A$ and $0 \leqq t \leqq 1$. Define a transformation T of X into X by $T((x, b_t(x))) = (f(x), b_t(f(x)))$.

THEOREM. *The space X is compact and of dimension 0 at some points and of dimension 1 at the others. Furthermore T is a homeomorphism of X onto X and X is minimal with respect to T.*

PROOF. The proof will be divided into four parts.

(a) *X is compact.* We state the following, omitting proof: the function $b_0(x)$ is lower semi-continuous on A and $b_1(x)$ is upper semi-continuous on A. Let $(x_j, b_{t_j}(x_j))$ be an infinite sequence of points in X. Some subsequence, which we suppose the same as the original, converges to a point $z = (x, y)$ of the plane. Since A is compact we have $x \in A$. For $y \in A$ let $V(y) = \bigcup_{0 \leqq t \leqq 1} (y, b_t(y))$. By the semi-continuity statements we have $\limsup V(x_i) \subset V(x)$. But $(x_i, b_t(x_i)) \in V(x_i)$ and hence $z \in V(x) \subset X$.

(b) *X is of dimension 0 at some of its points and dimension 1 at the others.* Since A is totally disconnected, the components of X are the sets $V(x)$, $x \in A$. If $a(x) = \infty$, then $V(x)$ is a single point and the dimension of X at x is 0. If $a(x)$ is finite then $V(x)$ is a line segment and the dimension of X at x is greater than or equal to 1. This dimension is exactly 1 since X is the subset of a product of a space of dimension 0 and a space of dimension 1.

(c) *T is a homeomorphism of X onto X.* Let $x = (x_i) \in A$. Suppose for some m $x_{m+1} \neq 3^{m+1}$; let m be the first such integer. We say that x is of *type* 1 if $m = 0$; if $m > 0$ we say that x is of *type* 2 or 3 according as α_m is 0 or 1. Then:

(i) If x is of type 1, $a(f(x)) = a(x)$, $b_t(f(x)) = b_t(x)$.

(ii) If x is of type 2, $a(f(x)) = a(x) - 1$; $b_t(f(x)) = 2b_t(x) - 2 + 1/2^{m-2}$.

(iii) If x is of type 3, $a(f(x)) = a(x) + 1$, $b_t(f(x)) = (1/2)b_t(x) - 1/2 + 1/2^{m-1}$.

Of the above statements we prove (ii) as typical. We then have $x = (x_i)$ where $x_i = 3^i$ for $i \leqq m$, $x_{m+1} = 3^m$, $x_i \neq 3^i$ for $i \geqq m+1$. The associated sequence is then $2, \cdots, 2, 0, \beta_{m+1}, \cdots$ led by $(m-1)$ 2's. Then

$$(*) \quad b_0(x) = \sum_{j=1}^{m-1} 1/2^j + \sum_{j \geqq m+1} \beta_j/2^{j+1} = 1 - 1/2^{m-1} + \sum_{j \geqq m+1} \beta_j/2^{j+1}.$$

Also $f(x) = (y_i)$ where $y_j = 1$ for $i \leqq m$, $y_{m+1} = 1 + 3^m$, $y_i = 1 + x_i$ for

$i \geqq m+1$. The associated sequence of $f(x)$ is then $0, \cdots, 0, \beta_{m+1}, \cdots$ led by $(m-1)$ 0's and hence $b_0(f(x)) = \sum_{j \geqq m+1} \beta_j/2^j$. Solving this with (*) we obtain $b_0(f(x)) = 2b_0(x) - 2 + 1/2^{m-2}$. Also, by inspecting the associated sequences, it follows that $a(f(x)) = a(x) - 1$. These formulas imply the result for $b_t(f(x))$.

Inspection of these formulas together with consideration of the one exceptional point shows that $a(f(x))$ is infinite if and only if $a(x)$ is infinite. Then $V(f(x))$ is a point if and only if $V(x)$ is a point. Since f is 1–1 and onto, T is then 1–1 and onto X.

The continuity of T at $(x, b_t(x))$ follows from the formulas in case x is one of the types; for the exceptional point continuity may be checked directly.

(d) *X is minimal with respect to T.* Consider the point $x = (1, 1, \cdots) \in A$ and the point $z = (x, 0) \in X$. We show that z is an almost periodic point[2] under T and that $\bigcup_{n=-\infty}^{+\infty} T^n(z)$ is dense in X. It will then follow that X is minimal under T. To show z almost periodic, we notice that $V(x) = z$; that is, X is of dimension 0 at z. Then there exist arbitrarily small neighborhoods U of x in A and V of z in X such that $(y, b_t(y)) \in V$ if and only if $y \in U$. But then almost periodicity of x under f implies the almost periodicity of z under T.

Let $w = (y, b_t(y)) \in X$ where $y = (y_i) \in A$. We must now define a sequence (k_n) of integers with $T^{k_n}(z) \rightarrow w$. Since $b_0(y) \leqq b_t(y) \leqq b_1(y)$, it follows from the definitions that the associated sequence $(\beta_j | j = 1, \cdots, a(y))$ may be extended by addition of elements to obtain a sequence $(\beta_j | j = 1, \cdots, \infty)$, each β_j either 0 or 2, such that $b_t(y) = \sum_{j=1}^{\infty} \beta_j/2^{j+1}$. Define α_i by $y_{i+1} = y_i + \alpha_i \cdot 3^i$. Let $n > 0$; let m be the number of elements in $\alpha_1, \cdots, \alpha_{n-1}$ which are not 1. Let $y'_j = y_j$ for $j \leqq n$, $y'_{n+j+1} = y'_{n+j} + \beta_{m+j+1} \cdot 3^{n+j}$ for $0 \leqq j \leqq n-1$, and $y'_j = y'_{2n}$ for $j \geqq 2n$. By construction, $y^{(n)} = (y'_j)$ is a point of A. Since from some point on the coordinates of $y^{(n)}$ are identical, then $k_n = y'_{2n} - 1$ has the property that $f^{k_n}(x) = y^{(n)}$. The associated sequence of $y^{(n)}$ is $\beta_1, \beta_2, \cdots, \beta_m, \cdots, \beta_{m+n}, 0, \cdots$ and hence $b_0(y^{(n)}) = \sum_{j=1}^{m+n} \beta_j/2^{j+1}$. Then $y^{(n)} \rightarrow y$ and $b_0(y^{(n)}) \rightarrow b_t(y)$ as n increases. Then $T^{k_n}(z) = (y^{(n)}, b_0(y^{(n)}))$ which converges to w as n increases. This completes the proof.

REMARK. Let P be a topological space and let g be a homeomorphism from P onto itself. Then a point $p \in P$ is said to be regularly almost periodic if and only if for each neighborhood V of p there exists a positive integer k such that the closure of $\bigcup_{n=-\infty}^{+\infty} g^{nk}(p)$

---

[2] The point z is said to be almost periodic under T if to $\epsilon > 0$ there corresponds a positive integer N such that in every set of N consecutive integers appears an integer n with $\rho(z, T^n(z)) < \epsilon$.

is contained in V. An example has been given by Garcia and Hedlund [2] of a minimal set some of whose points are regularly almost periodic while others are not. The example of the preceding theorem also has this property. It is easily verified that each point of A is regularly almost periodic with respect to f; it follows that points of X at which the dimension is 0 are also regularly almost periodic. All points of X, however, are not regularly almost periodic, for otherwise a theorem of Garcia and Hedlund [2] would give us that T has equicontinuous powers, which is false.

BIBLIOGRAPHY

1. H. Freudenthal, *Entwicklungen von Räumen und ihren Gruppen*, Compositio Math. vol. 4 (1937) pp. 145–234.

2. M. Garcia and G. A. Hedlund, *The structure of minimal sets*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 954–964.

UNIVERSITY OF VIRGINIA

AN INEQUALITY RELATED TO THE ISOPERIMETRIC INEQUALITY

L. H. LOOMIS AND H. WHITNEY

In this note we shall prove the following theorem.

THEOREM 1. *Let m be the measure of an open subset O of Euclidean n-space, and let $m_1, \cdots, m_n$ be the $(n-1)$-dimensional measures of the projections of O on the coordinate hyperplanes. Then*

$$m^{n-1} \leqq m_1 m_2 \cdots m_n. \tag{1}$$

Note that for n-dimensional intervals with faces parallel to the coordinate hyperplanes, (1) holds with the equality sign.

With any reasonable definition of the $(n-1)$-dimensional measure s of the boundary of O, $s \geqq 2m_i$ for each i, so that (1) gives

$$m^{n-1} \leqq s^n/2^n; \tag{2}$$

this is the isoperimetric inequality, without the best constant. Since the proof of the isoperimetric inequality with the best constant is difficult,[1] and since its applications do not necessarily require the best constant, our elementary proof of the theorem may be of some interest.

We first reduce the problem to a combinatorial one, in the following theorem.

THEOREM 2. *Let S be a set of cubes from a cubical subdivision of n-space; let S_i be the set of $(n-1)$-cubes obtained by projecting the cubes of S onto the ith coordinate hyperplane. Let N and N_i be the numbers of cubes in S and S_i respectively. Then*

$$N^{n-1} \leqq N_1 N_2 \cdots N_n. \tag{3}$$

Assuming Theorem 2, we prove Theorem 1 as follows. Given $\epsilon > 0$, choose a cubical subdivision of n-space into cubes of side δ, with δ so small that if S is the set of cubes interior to O forming the set $\bar{S}$, $\mu(O-\bar{S}) < \epsilon$ (μ = measure). Then

$$[\mu(\bar{S})]^{n-1} = N^{n-1}\delta^{n(n-1)} \leqq (N_1\delta^{n-1}) \cdots (N_n\delta^{n-1}) \leqq m_1 \cdots m_n,$$

and since ϵ is arbitrary, (1) follows.

Received by the editors July 31, 1948.

[1] See E. Schmidt, *Über das isoperimetrische Problem in Raum von n Dimensionen*, Math. Zeit. vol. 44 (1939) pp. 689–788.

Proof of Theorem 2.[3] If $n=2$, the theorem is clear; we shall use induction on n. Each cube of S projects into an interval on the first coordinate axis; let $I_1, \cdots, I_k$ be the intervals thus obtained. Let T_i be the set of cubes projecting onto I_i, and let T_{ij} be the set of $(n-1)$-cubes obtained by projecting the cubes of T_i into the jth coordinate hyperplane $(j=2, \cdots, n)$. Let a_i and a_{ij} be the numbers of cubes in T_i and T_{ij} respectively. Clearly

$$\sum_{i=1}^{k} a_i = N, \qquad a_i \leqq N_1 \ (i = 1, \cdots, k), \tag{4}$$

$$\sum_{i=1}^{k} a_{ij} = N_j \qquad (j = 2, \cdots, n). \tag{5}$$

Also, by induction,

$$a_i^{n-2} \leqq a_{i2} \cdots a_{in} \qquad (i = 1, \cdots, k). \tag{6}$$

From (6) and the second part of (4) we obtain

$$a_i^{n-1} \leqq N_1 a_{i2} \cdots a_{in} \qquad (i = 1, \cdots, k).$$

Now using successively the first part of (4), the above inequality, Hölder's inequality, and (5), we see that

$$\begin{aligned} N = \sum_{i=1}^{k} a_i &\leqq \sum_{i=1}^{k} N_1^{1/(n-1)} \prod_{j=2}^{n} a_{ij}^{1/(n-1)} \\ &\leqq N_1^{1/(n-1)} \prod_{j=2}^{k} \left(\sum_{i=1}^{m} a_{ij} \right)^{1/(n-1)} = \prod_{j=1}^{n} N_j^{1/(n-1)}, \end{aligned}$$

as required.

Harvard University

---

[3] The authors are indebted to M. R. Demers for a simplification in the proof of this theorem.

SOME ELEMENTARY TOPOLOGICAL PROPERTIES OF ESSENTIAL MAXIMAL MODEL CONTINUA

W. R. SCOTT[1]

1. **Introduction.** Let T: $z=t(w)$, $w\in D$, be a bounded continuous transformation from a bounded connected open set D in the w-plane into the z-plane. Radó and Reichelderfer [3, p. 263][2] have defined essential maximal model continua $\gamma(z)$ for such transformations. If A is a set in the z-plane, let $E(A)$ be the point set sum of the e.m.m.c. $\gamma(z)$ for $z\in A$. Thus $E(A)$ is a subset of D. We are primarily interested in the topological properties of $E(A)$. Since most of the reasoning is valid for more general spaces, it is carried out first for these spaces and then in §6 the results are applied to the above situation. These results are also applicable to the set $E^+(A)$ and $E^-(A)$ defined by Reichelderfer in [4].

2. **Preliminaries.** Let T: $y=t(x)$, $x\in X$, be a single-valued continuous transformation from a normal Hausdorff space X into a topological space Y (these terms are used in the sense of [1, chap. 1]). Suppose that for each set $A\subset Y$ there has been defined a set $E(A)\subset X$ such that

$$E(A) = \sum_{y\in A} E(y), \tag{1}$$

$$E(y) = \sum_{\gamma\in S(y)} \gamma, \tag{2}$$

where $S(y)$ is a class of disjoint nonempty closed connected sets $\gamma\subset X$ such that $\gamma\subset T^{-1}(y)$ and having the following property:

P. If $\gamma_0\in S(y_0)$ and if G is any open set such that $\gamma_0\subset G$, then there exists a neighborhood $N(y_0)$ such that if $y\in N(y_0)$ then there exists a $\gamma\in S(y)$ such that $\gamma\subset G$.

Let $f(y)$ be the number of $\gamma\in S(y)$. Thus $f(y)=0, 1, 2, \cdots$, or $+\infty$, no distinction being made between infinite cardinals.

3. **Closed sets** A.

THEOREM 1. *Let T, A, $E(A)$, and $f(y)$ be given as in §2. If A is closed and if $f(y)$, $y\in A$, is upper semi-continuous and bounded on A, then $E(A)$ is closed.*

Received by the editors May 27, 1948.

[1] This paper was written under the joint sponsorship of the U. S. Navy, Office of Naval Research, and the Ohio State University Research Foundation.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

PROOF. Deny the theorem. Then there exists a point $x_0 \in C(E(A)) - E(A)$, where $C(E(A))$ is the closure of $E(A)$. Since A is closed and since $E(A) \subset T^{-1}(A)$ which is closed, we have $t(x_0) = y_0 \in A$. Let $f(y_0) = k$. Then there exist disjoint $\gamma_i \in S(y_0)$, $i = 1, \cdots, k$, and $x_0 \cdot \sum \gamma_i = 0$. Since each γ_i is closed, and since X is a normal Hausdorff space, there exist disjoint open sets $G_0, \cdots, G_k$, such that $x_0 \in G_0$, $\gamma_i \subset G_i$, $i = 1, \cdots, k$. By property P, there exist neighborhoods $N_i(y_0)$ such that if $y \in N_i(y_0)$ then there exists a $\gamma \in S(y)$ such that $\gamma \subset G_i$. Since $f(y)$ is upper semi-continuous on A, there is a neighborhood $N'(y_0)$ such that if $y \in N'(y_0) \cdot A$, then $f(y) \leqq k$. Let

$$N(y_0) = N'(y_0) \cdot N_1(y_0) \cdots N_k(y_0).$$

Then if $y \in N(y_0) \cdot A$, all of the $\gamma \in S(y)$ lie in $\sum_{i=1}^{k} G_i$. But since $x_0 \in G_0$ and $G_0 \cdot \sum_{i=1}^{k} G_i = 0$, this implies that $x_0 \notin C(E(A))$. This is a contradiction. Thus $E(A)$ is closed.

It will be noted that the fact that each γ is connected was not used in the proof of this theorem.

4. Connected sets A.

LEMMA 1. *Let T, A, and $E(A)$ be given as in §2. Let $E(A) = E_1 + E_2$ where $E_1 \cdot C(E_2) + C(E_1) \cdot E_2 = 0$. Let $f_i(y)$, $y \in A$, be the number of $\gamma \in S(y)$ such that $\gamma \subset E_i$. Then $f_i(y)$ is lower semi-continuous on A.*

PROOF. Note that since a $\gamma_0 \in S(y_0)$ is connected, either $\gamma_0 \subset E_i$ or or $\gamma_0 \cdot E_i = 0$. Hence $f_1(y) + f_2(y) = f(y)$, $y \in A$. It will be sufficient to show that $f_1(y)$ is lower semi-continuous on A. Let $f_1(y_0) \geqq n$, $y_0 \in A$. If $n = 0$, then $f_1(y) \geqq n$, $y \in A$. If $n > 0$, there exist $\gamma_j \subset E_1 \cdot E(y_0)$, $j = 1, \cdots, n$. Hence $\gamma_1, \cdots, \gamma_n$, and $C(E_2)$ are disjoint closed sets. Therefore there exist disjoint open sets $G_1, \cdots, G_n$ such that $C(E_2) \cdot \sum G_j = 0$ and $\gamma_j \subset G_j$. By property P there exists a neighborhood $N(y_0)$ such that if $y \in N(y_0)$ then there exists a $\gamma_j \subset G_j \cdot E(y)$. If, in addition, $y \in A$, then $f_1(y) \geqq n$. But this proves that $f_1(y)$ is lower semi-continuous on A.

In particular, if $E_2 = 0$, we have that $f(y)$, $y \in A$, is lower semi-continuous on A. Thus, if $A = Y$, this shows that $f(y)$ is lower semi-continuous on Y.

LEMMA 2. *Let A be a connected subspace of Y. Let $g_i(y)$, $y \in A$, be a (finite) integral-valued, lower semi-continuous function, $i = 1, \cdots, n$. Let $g(y) = \sum g_i(y)$, $y \in A$, be upper semi-continuous. Then $g_i(y)$ is constant on A for each i.*

PROOF. The function $g^*(y) = \sum_{j \neq i} g_j(y)$ is lower semi-continuous.

Hence $g_i(y)$ is upper semi-continuous, being the difference of an upper semi-continuous function $g(y)$ and a lower semi-continuous function $g^*(y)$. Since $g_i(y)$ is also lower semi-continuous, it is continuous. Since A is connected, $g_i(y)$ is constant on A.

THEOREM 2. *Let T, A, $E(A)$, and $f(y)$ be given as in §2. Let A be connected, and suppose that $f(y)$ is upper semi-continuous and $f(y) \leqq k < +\infty$, $y \in A$. Then $E(A)$ has at most k components. Let these components be $E_1, \cdots, E_r$, $r \leqq k$. Let $f_i(y)$, $y \in A$, be the number of $\gamma \in S(y)$ such that $\gamma \subset E_i$. Then $f_i(y)$ is constant for each i.*

PROOF. If $E(A)=0$, the theorem is trivial. If $E(A) \neq 0$, let $E_1, \cdots, E_r$ be such that $E_i \neq 0$, $E_i \cdot C(E_j)=0$ if $i \neq j$, and $\sum E_i = E(A)$. Let $f_i(y)$, $y \in A$, be defined as in the theorem. Clearly $f_i(y)$ is integral-valued and $\not\equiv 0$. Moreover, by Lemma 1, $f_i(y)$ is lower semi-continuous on A. Since $f(y) = \sum f_i(y)$ is upper semi-continuous and bounded on A, Lemma 2 is applicable, and $f_i(y) \equiv k_i \geqq 1$, $i=1, \cdots, r$. But since

$$k \geqq f(y) = \sum_{i=1}^{r} f_i(y) \geqq \sum_{i=1}^{r} 1 = r,$$

we have $r \leqq k$. But if $E(A)$ had more than k components, then there would exist $E_1, \cdots, E_{k+1}$ such that $E_i \neq 0$, $E_i \cdot C(E_j)=0$ if $i \neq j$, and $\sum E_i = E(A)$. Hence $E(A)$ has at most k components and the theorem is true.

5. Open connected sets A.

THEOREM 3. *Let T: $y=t(x)$, $x \in X$, be a continuous transformation from a connected, locally connected, locally compact, separable metric space X into a topological space Y. Let A be a connected open set in Y, let $E(A)$ be given as in §2, and suppose that $f(y) \not\equiv 0$ is upper semi-continuous and bounded on A. Let $y_1 \in A$, $y_2 \in A$. Then there exist points $x_1 \in E(y_1)$, $x_2 \in E(y_2)$, and an arc C from x_1 to x_2 such that $T(C) \subset A$.*

PROOF. Let $E_1(A)$ be any component of $E(A)$. Let $f_1(y)$, $y \in A$, be the number of $\gamma \in S(y)$ such that $\gamma \subset E_1(A)$. Then by Theorem 2, $f_1(y) \equiv k_1 > 0$, $y \in A$. Hence there exist an $x_1 \in E(y_1) \cdot E_1(A)$ and an $x_2 \in E(y_2) \cdot E_1(A)$. Let G be the component of $T^{-1}(A)$ which contains $E_1(A)$. Then [2, Theorem I, 14.1] G is open and connected. Hence [2, Theorems I, 14.3 and II, 5.2] G is arcwise connected. Therefore there exists an arc $C \subset G$ joining x_1 and x_2. Clearly $T(C) \subset A$.

6. Plane transformations. Let T: $z=t(w)$, $w \in D$, be a bounded continuous transformation from a bounded connected open set D in the w-plane into the z-plane. Radó and Reichelderfer [3, p. 263] have defined essential maximal model continua (e.m.m.c.) γ for such transformations. Let $S(z)$ be the class of (disjoint) e.m.m.c. of z. If $\gamma \in S(z)$ then $\gamma \subset T^{-1}(z)$ and $S(z)$ has property P [3, Lemmas 2.8 and 2.16]. Define $E(A)$ by equations (1) and (2). The function $f(z)$ becomes the essential multiplicity function $\kappa(z)$ [3, p. 263]. The conditions in Theorems 1, 2, and 3 are satisfied, and we have the following result.

COROLLARY. *Let T: $z = t(w)$, $w \in D$, be a bounded continuous transformation from a bounded connected open set D in the w-plane into the z-plane. Let κ (z) be upper semi-continuous on A with $\kappa(z) \leqq k < +\infty$, $z \in A$. Then if A is closed, $E(A)$ is closed; if A is connected, $E(A)$ has at most k components.*

Theorem 3 of course yields a corollary also.

Reichelderfer[3] [4] has recently defined a class $S^+(z)$ for such plane transformations as follows. $S^+(z_0)$ consists of all e.m.m.c. γ_0 of z_0 such that if G is any open set containing γ_0 then there exists a finitely-connected Jordan region $R \subset G$ containing γ_0 in its interior and such that $\mu(z_0, T, R) > 0$ (see [3, p. 263] for notation). Reichelderfer [4] shows that $S^+(z)$ has property P. Define $E^+(A)$ by equations (1) and (2). The function $f(z)$ is then denoted by $\kappa^+(z)$. Then Theorems 1, 2, and 3 apply in this case.

Similar definitions and conclusions hold for $E^-(A)$ (see [4]).

7. **Examples.** The discussion will be restricted to bounded continuous plane transformations T: $z = t(w)$, $w \in D$, where D is an open simply-connected set, and to the functions $\kappa(z)$ and $E(z)$ mentioned in §6. The following examples shed further light on Theorems 1, 2, and 3. We have let $w = u + iv$, $z = x + iy$, and have used whichever notation is more convenient.

I. In Theorem 1, if $f(z)$ is merely assumed to be bounded, then $E(A)$ is not necessarily closed. For let D be the disc $|w| < 2$, and let

$$z = \begin{cases} w, & |w| \leqq 1 \\ w/|w|, & 1 < |w| < 2. \end{cases}$$

Let A be the closed disc $|z| \leqq 1$. Then $\kappa(z) \leqq 1$, $z \in A$, and $E(A)$ is the set $|w| < 1$ which is not closed in D.

II. In Theorem 1, if $\kappa(z) \equiv +\infty$, $z \in A$, then $E(A)$ is not necessarily closed. Sierpinski [5] has constructed a continuous function $x = g^*(u)$, $0 \leqq u \leqq 1$, such that $\kappa(x, g^*) = +\infty$ for $0 < x < 1$ except at a countable set of points. A slight modification of his function yields a continuous function $x=g(u)$, $0\leqq u\leqq 1$, such that $\kappa(x, g)\equiv +\infty$, $0<x<1$, and such that $g(1)=1$. To construct the required example, let D be the open rectangle: $0<u<3$, $0<v<1$. Let T be defined by the relations

---

[3] The author had the privilege of reading this paper in manuscript.

$$x=\begin{cases} g(u), & 0<u\leqq 1,\\ (3-u)/2, & 1<u\leqq 2,\\ 1/2, & 2<u<3,\end{cases}$$
$$y=v, \qquad 0<v<1.$$

Let A be the line segment: $1/2\leqq x\leqq 3/4$, $y=1/2$. Then $\kappa(z)\equiv +\infty$, $z\in A$, and $E(A)$ is not closed because it contains the segment $3/2<u<2$, $v=1/2$, but not the point $u=2$, $v=1/2$.

III. In Theorem 1, if A is open and $\kappa(z)\equiv k<+\infty$, $z\in A$, it does not follow that $E(A)$ is open. For let D be the disc $|w|<2$, and let

$$z=t(w)=\begin{cases}(w-1)w, & 1\leqq |w|<2,\\ |w|-1, & 0\leqq |w|<1.\end{cases}$$

Let A be the disc $|z|<2$. Then it follows that $\kappa(z)\equiv 1$, $z\in A$, and $E(A)$ is the set of points w such that $1\leqq |w|<2$. Hence $E(A)$ is not open.

IV. In Theorem 2, $E(A)$ may be connected, thus having only 1 component, even though $\kappa(z)\equiv k<+\infty$, $z\in A$. For let $z=w^k$, $|w|<2$. Let A be the unit circle $|z|=1$. Then $\kappa(z)\equiv k$, $z\in A$. But $E(A)$ is the unit circle $|w|=1$, and is therefore connected.

V. Theorem 3 does not hold for line segments A. For let D be the open rectangle: $-2<u<2$, $-2<v<2$. Let S be the point set consisting of the line segment $u=0$, $-1\leqq v\leqq 1$, and the curve $v=\sin 2\pi/u$, $0<u\leqq 1$. Let S' be the point set consisting of S and the two line segments $-2<u<0$, $v=0$, and $1<u<2$, $v=0$. Let $d((u, v), S')$ be the distance from the point (u, v) to the set S'. Define the transformation T as follows:

$$x=\begin{cases} u, & -2<u<0,\\ 0, & 0\leqq u\leqq 1,\\ u-1, & 1<u<2,\end{cases}$$
$$y=\begin{cases} d((u, v), S'), & (u, v)\text{ above } S',\\ 0, & (u, v)\in S',\\ -d((u, v), S') & (u, v)\text{ below } S'.\end{cases}$$

T is clearly continuous. Let A be the line segment: $-1\leqq x\leqq 1/2$, $y=0$. Then $\kappa(z)\equiv 1$, $z\in A$, and $T^{-1}(A)=E(A)$ consists of the line segments $-1 \leqq u < 0$, $v=0$, and $1 < u \leqq 3/2$, $v=0$, and the set S. Hence if $z_1 = (-1, 0)$ and $z_2 = (1/2, 0)$ there is no arc $w_1 w_2 \subset D$ such that $t(w_1) = z_1$, $t(w_2) = z_2$, $T(\text{arc } w_1 w_2) \subset A$.

VI. There remains the question as to whether, in the case of plane transformations, we could require $C \subset E(A)$ in Theorem 3. This would be impossible in the general case, as may be shown by an example similar to the one described in V.

BIBLIOGRAPHY

1. S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, 1942.

2. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, 1942.

3. T. Radó and P. V. Reichelderfer, *A theory of absolutely continuous transformations in the plane*, Trans. Amer. Math. Soc. vol. 49 (1941) pp. 258–307.

4. P. V. Reichelderfer, *Law of transformation for essential generalized jacobians*, Duke Math. J. vol. 16 (1949) pp. 73–83.

5. W. Sierpinski, *Sur une fonction continue dans un intervalle qui prend chaque valeur de cet intervalle $\aleph_0$ fois*, Publications Mathématiques Univerzitet Belgrade vols. 6 and 7 (1938) pp. 84–85

OHIO STATE UNIVERSITY

A NEW PROOF OF E. CARTAN'S THEOREM ON THE TOPOLOGY OF SEMI-SIMPLE GROUPS

GEORGE DANIEL MOSTOW[1]

1. **Introduction.** E. Cartan has proved[2] that a connected semi-simple Lie group is topologically the direct product of a compact subgroup and a Euclidean space. Cartan first proved this theorem in 1927 by a reduction to special cases, and not until 1929 did he free his proof from the consideration of special cases. As a result Cartan's proof is diffused among several journals. Moreover, Cartan employs in an essential way the theory of symmetric Riemannian spaces and makes use of a result whose proof seems to be lacking (Ann. École Norm. loc. cit. p. 367, §20).

In this paper there will be given a more direct proof which eliminates the use of symmetric Riemannian spaces. The author wishes to acknowledge his debt to Professor C. Chevalley who suggested in an oral communication Lemma 1.4 below and to whom a proof of Theorem 1, essentially the same as the one given here, was known.

2. **Definitions and preliminaries.** Let $\mathcal{G}$ be a semi-simple Lie algebra over a field K of characteristic zero. Let ad g denote the linear transformation $x \rightarrow [g, x]$, where $x, g \in \mathcal{G}$. By a Cartan subalgebra is meant a subalgebra $\mathcal{H}$ of $\mathcal{G}$ maximal with respect to properties (1) $\mathcal{H}$ is abelian, that is, $[\mathcal{H}, \mathcal{H}] = 0$; and (2) $H \in \mathcal{H}$ implies ad H is semi-simple, that is, its minimal equation has no repeated factor. It is a theorem that a Cartan subalgebra is a maximal abelian subalgebra (but not conversely).

Let $\mathcal{G}$ be a semi-simple Lie algebra, that is, $\mathcal{G}$ contains no abelian ideal, and let $\mathcal{H}$ be any Cartan subalgebra. Assume that the base field K is algebraically closed. A nonzero linear function α defined on $\mathcal{H}$ is called a "root" if and only if there exists an X in $\mathcal{G}$ such that $[H, X] = \alpha(H)X$ for all $H \in \mathcal{H}$. Any nonzero Y in $\mathcal{G}$ with this property is said to "belong to α."

The following is known:[3]

R1. If X_α, X_β belong to α, β respectively, then $[X_\alpha, X_\beta]$ belongs

---

Presented to the Society, September 10, 1948; received by the editors September 2, 1948.

[1] O.N.R. Fellowship holder, 1948–1949.

[2] Ann. École Norm. vol. 44 (1927) p. 367, cf. Bull. Soc. Math. France vol. 55 (1927) pp. 122–125, J. Math. Pures Appl. (9) vol. 8 (1929) pp. 24–27.

[3] Cf. H. Weyl, *Continuous groups*, vol. 2, Institute for Advanced Study Notes, 1935, pp. 69–87.

to $\alpha+\beta$ or is zero according as $\alpha+\beta$ is a root or not.

R2. The linear subspace of elements of $\mathcal{G}$ belonging to a root is one-dimensional.

Notation. "X_α" denotes an element belonging to the root α.

Clearly any collection $\{X_\alpha\}$ with distinct subscripts are linearly independent over K mod $\mathcal{H}$ and $\{\mathcal{H}, X_\alpha | \text{all roots } \alpha\}$ spans $\mathcal{G}$.

Suppose α, β are roots. Define

$$k(\alpha, \beta) = \max l(\beta - \nu\alpha \text{ is a root, } 0 \leqq \nu \leqq l;\ \nu, l \text{ integers}),$$

$$k'(\alpha, \beta) = \max l(\beta + \nu\alpha \text{ is a root, } 0 \leqq \nu \leqq l;\ \nu, l \text{ integers}).$$

Define $B(X, Y) = \text{Trace}(\text{ad } X \text{ ad } Y)$. B is called the "fundamental bilinear form."

R3. $B(X, Y)$ is nondegenerate on $\mathcal{G}$ and on $\mathcal{H}$. If $\alpha+\beta\neq 0$, $B(X_\alpha, X_\beta)=0$. $B(H, X_\alpha)=0$, if $H\in\mathcal{H}$, for all roots α.

Define H_α as the element of $\mathcal{H}$ such that $B(H, H_\alpha)=\alpha(H)$ for all $H\in\mathcal{H}$.

R4.

$$\beta(H_\alpha) = \frac{k(\alpha, \beta) - k'(\alpha, \beta)}{2}\alpha(H_\alpha),$$

$$\frac{4}{\alpha(H_\alpha)} = \sum_\beta [k(\alpha, \beta) - k'(\alpha, \beta)]^2,$$

$$[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha})H_\alpha.$$

For each root α select an X_α. Define $C_{\alpha\beta}$ by the relations

$$[X_\alpha, X_\beta] = C_{\alpha,\beta}X_{\alpha+\beta} \text{ if } \alpha+\beta \text{ is a root,}$$

$$C_{\alpha,\beta} = 0 \text{ if } \alpha+\beta \text{ is not a root.}$$

R5. $C_{\alpha,\beta}C_{-\alpha,-\beta} = -2^{-1}k'(\alpha, \beta)[k(\alpha, \beta)+1]\alpha(H_\alpha)$.

R6. $\{X_\alpha | \text{all roots } \alpha\}$ may be so chosen that $[X_\alpha, X_{-\alpha}]=H_\alpha$ and $C_{\alpha,\beta} = -C_{-\alpha,-\beta}$. Here $B(X_\alpha, X_{-\alpha})=1$ and $C_{\alpha,\beta}$ is real if K is the complex numbers.

3. **An algebraic theorem.** We shall now deal exclusively with Lie algebras over the real and complex number fields R and C respectively.

Let $\mathcal{G}$ be a Lie algebra over C. By a real form of $\mathcal{G}$ is meant a real linear subspace which (1) is closed under the formation of brackets; (2) spans $\mathcal{G}$; and (3) has a base which is linearly independent over C. If $\mathcal{G}$ is a Lie algebra over R, by the complexification of $\mathcal{G}$ is meant the augmented linear space over C obtained by augmenting R to C.

Let $\mathcal{G}$ be a Lie algebra over C and let $\mathcal{G}_R$ be a real form. Consider the real-linear transformation θ of $\mathcal{G}$ (considered as a linear space over R):

$$\theta: \ X + (-1)^{1/2}Y \to X - (-1)^{1/2}Y, \quad \text{where} \quad X, Y \in \mathcal{G}_R.$$

DEFINITION. θ is called the "conjugation of $\mathcal{G}$ with respect to $\mathcal{G}_R$."

Note that $\mathcal{G}_R$ is the set of elements which are invariant under θ.

Note too that $\theta([X+(-1)^{1/2}Y, U+(-1)^{1/2}V]) = \theta(([X, U] -[Y, V])+(-1)^{1/2}([X, V]+[Y, U])) = [X, U]-[Y, V]-(-1)^{1/2} \cdot([X, V)+[Y, V]) = [X-(-1)^{1/2}Y, U-(-1)^{1/2}V]$ if $X, Y, U, V \in \mathcal{G}_R$. Thus $\theta([X, Y) = [\theta(X), \theta(Y)]$ if $X, Y \in \mathcal{G}$.

DEFINITION. A compact form of a complex Lie algebra is a real form on which the fundamental bilinear form is negative definite.

This section is devoted to the proof of the following theorem.

THEOREM 1. *Let $\mathcal{G}$ be a semi-simple Lie algebra over R, let $\mathcal{G}_C$ denote its complexification, and let θ denote the conjugation of $\mathcal{G}_C$ with respect to $\mathcal{G}$. Then θ leaves invariant a compact real form $\mathcal{G}_K$.*

Notation. "X'" denotes $\theta(X)$, where $X \in \mathcal{G}_C$.

Clearly $\mathcal{G}_C$ is a semi-simple Lie algebra. Assume that for each root α an X_α is so chosen as to make $\{X_\alpha | \text{all roots } \alpha\}$ satisfy the condition in R6 above.

LEMMA 1.1. *Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$ and let $\mathcal{H}_C$ denote the complexification of $\mathcal{H}$. $\mathcal{H}_C$ is a Cartan subalgebra of $\mathcal{G}_C$. Moreover to each root α there corresponds a root α' such that*

(1) *$\alpha'(H) = \bar{\alpha}(H)$ for $H \in \mathcal{H}$,*

(2) *$[H, X_\alpha'] = \alpha'(H)X_\alpha'$ for $H \in \mathcal{H}_C$,*

(3) *if $\alpha, \beta, \alpha+\beta$ are roots, then $\alpha'+\beta'$ is a root and $(\alpha+\beta)' = \alpha'+\beta'$.*

PROOF. Inasmuch as $\{\text{ad } H | H \in \mathcal{H}_C\}$ is a commutative set of linear transformations and is spanned by semi-simple transformations, and inasmuch as C is algebraically closed, all the elements of $\{\text{ad } H | H \in \mathcal{H}_C\}$ are semi-simple.

Suppose $X+(-1)^{1/2}Y$, where $X, Y \in \mathcal{G}$, is an element of a Cartan subalgebra of $\mathcal{G}_C$ which includes $\mathcal{H}_C$. Then $[H, X] = [H, Y] = 0$ for $H \in \mathcal{H}$. $\mathcal{H}$ being a maximal abelian subalgebra, X and Y must be in $\mathcal{H}$ so that $X+(-1)^{1/2}Y \in \mathcal{H}_C$. It follows that $\mathcal{H}_C$ is a Cartan subalgebra of $\mathcal{G}_C$.

From

$$\theta([X, Y]) = [X', Y']$$

and

$$\theta(\alpha X) = \bar{\alpha}X' \qquad \text{if } X, Y \in \mathcal{G}_C$$

we infer that

$$[H, X'_\alpha] = \bar{\alpha}(H)X'_\alpha, \qquad \text{if } H \in \mathcal{H}.$$

Lemma 1.1 follows from this.

LEMMA 1.2. $C_{\alpha,\beta} = \pm C_{\alpha',\beta'}$.

PROOF. $-C^2_{\alpha,\beta} = -2^{-1}k'(\alpha, \beta)[k(\alpha, \beta)+1]\alpha(H_\alpha)$. From Lemma 1.1 it follows directly that

$$k(\alpha, \beta) = k(\alpha', \beta'), \; k'(\alpha, \beta) = k'(\alpha', \beta').$$

Since

$$\frac{4}{\alpha(H_\alpha)} = \sum_\beta [k(\alpha, \beta) - k'(\alpha, \beta)]^2$$

it follows that $C_{\alpha,\beta} = \pm C_{\alpha',\beta'}$.

LEMMA 1.3. $H_{\alpha'} = H'_\alpha$.

PROOF. Since the fundamental bilinear form B is real on $\mathcal{G}$, $B(X+(-1)^{1/2}Y, \; U+(-1)^{1/2}V)$ is the complex conjugate of $B(X-(-1)^{1/2}Y, \; U-(-1)^{1/2}V)$ if $XY \in \mathcal{G}$, that is, $B(Z', W')$ is the complex conjugate of $B(Z, W)$ if $Z, M \in \mathcal{G}_C$. Hence $B(H, H'_\alpha)$ is the complex conjugate of $B(H, H_\alpha)$ is the complex conjugate of $\alpha(H) = \alpha'(H)$ if $H \in \mathcal{H}$. Consequently $B(H, H'_\alpha) = \alpha'(H)$ for all $H \in \mathcal{H}_C$ and thus by definition $H_{\alpha'} = H'_\alpha$.

Let u_α be defined by the relation

$$X'_\alpha = u_\alpha X_{\alpha'} \qquad \text{(all roots } \alpha).$$

Since $C_{\alpha,\beta}$ are real, $u_\alpha u_\beta = \pm u_{\alpha+\beta}$ if $\alpha, \beta, \alpha+\beta$ are roots. Also $\bar{u}_\alpha u_{\alpha'} = 1$, $u_\alpha u_{-\alpha} = 1$.

LEMMA 1.4. *The choice of* $\{X_\alpha | \text{all roots } \alpha\}$ *can be modified without changing the constants of structure* $C_{\alpha,\beta}$ *so as to make* $|u_\alpha| = 1$ *for all roots* α.

PROOF.[4] Let $r_\alpha = |u_\alpha|^{-1/2}$ and let $Y_\alpha = r_\alpha X_\alpha$ for all roots α. Then $r_\alpha r_\beta = r_{\alpha+\beta}$ and $r_\alpha r_{-\alpha} = 1$ if $\alpha, \beta, \alpha+\beta$ are roots. It follows immediately that if we replace the $\{X_\alpha\}$ by $\{Y_\alpha\}$, then the constants of structure are unchanged. Moreover

$$\theta(Y_\alpha) = v_\alpha Y_{\alpha'}$$

where $|v_\alpha| = |r_\alpha u_\alpha r_{\alpha'}^{-1}| = |u_\alpha|^{-1/2} \cdot |u_\alpha| \cdot |u_{\alpha'}|^{1/2} = 1$. The lemma is thus proved.

---

[4] The proof given here is a simplification, pointed out by Dr. Harish-Chandra, of the author's proof.

We assume now that $\mathcal{G}_c$ has a base $\{H_{\alpha_i}, X_\alpha | \text{all roots } \alpha\}$ such that $|u_\alpha| = 1$. Since $u_\alpha u_{-\alpha} = 1$, it follows that $u_{-\alpha} = \bar{u}_\alpha$.

Proof of Theorem 1. Let $\mathcal{H}_0$ denote the real linear space spanned by $\{H_\alpha | \text{all roots } \alpha\}$ and let $\mathcal{G}_K$ denote the real linear spaced spanned by $\{(-1)^{1/2}\mathcal{H}_0, z_\alpha X_\alpha - \bar{z}_\alpha X_{-\alpha} | \text{all roots } \alpha, z_\alpha \text{ assuming all complex values}\}$. Since $C_{\alpha,\beta}$ is real and $C_{-\alpha,-\beta} = -C_{\alpha,\beta}$, it is readily seen that $\mathcal{G}_K$ is closed under formation of brackets. Moreover $\{H_1, \cdots, H_l, X_\alpha - X_{-\alpha}, (-1)^{1/2}(X_\alpha + X_{-\alpha}) | \text{all pairs of roots } \alpha \text{ and } -\alpha; H_1, \cdots, H_l$ base for $(-1)^{1/2}\mathcal{H}_0\}$ is a base for $\mathcal{G}_c$ with respect to complex numbers. It follows that $\mathcal{G}_K$ is a real form of $\mathcal{G}_c$. If $Z \in \mathcal{G}_K$, then $Z = (-1)^{1/2}H + \sum_\alpha (z_\alpha X_\alpha - \bar{z}_\alpha X_{-\alpha})$, where $H \in \mathcal{H}_0$. Thus

$$\begin{aligned}\theta(Z) &= -(-1)^{1/2}H' + \sum_\alpha (\bar{z}_\alpha u_\alpha X_{\alpha'} - z_\alpha u_{-\alpha} X_{-\alpha'}) \\ &= -(-1)^{1/2}H' + \sum_\alpha (\bar{z}_\alpha u_\alpha X_{\alpha'} - z_\alpha \bar{u}_\alpha X_{-\alpha'})\end{aligned}$$

where $H' \in \mathcal{H}_0$. Hence $\mathcal{G}_k$ is invariant under the conjugation θ. Finally, $\mathcal{G}_K$ is a compact real form. For if $Z \in \mathcal{G}_K$, then

$$Z = (-1)^{1/2}H + \sum_\alpha (z_\alpha X_\alpha - \bar{z}_\alpha X_{-\alpha}), \qquad H \in \mathcal{H}_0.$$

By R3, $B(Z, Z) = -B(H, H) - \sum_\alpha z_\alpha \bar{z}_\alpha$. But

$$B(H, H) = \text{Tr } (\text{ad } H)(\text{ad } H) = \sum_\alpha \alpha^2(H) \geqq 0$$

with inequality holding only if $H = 0$, since all roots assume real values on $\mathcal{H}_0$ (cf. R4). Hence $B(Z, Z) < 0$ if $Z \neq 0$. Thus $\mathcal{G}_K$ is a compact real form invariant under the conjugation θ.

4. **The theorem.** Let G be a connected real semi-simple Lie group, $\mathcal{G}$ its Lie algebra, and let ad G denote its adjoint group, that is, the Lie group of autmorphisms of $\mathcal{G}$ induced by the inner automorphisms of G. Let $\mathcal{G}_C$ denote the complexification of $\mathcal{G}$, θ the conjugation of $\mathcal{G}_C$ with respect to $\mathcal{G}$, $\mathcal{G}_K$ a real compact form of $\mathcal{G}_C$ which is invariant under θ. ad $\mathcal{G}_C$, being a subalgebra of the Lie algebra of all linear transformations of $\mathcal{G}_C$, determines a Lie subgroup G_C^* of the Lie group of all nonsingular linear transformations on $\mathcal{G}_C$. To $\mathcal{G}$ there corresponds a real linear subspace $\mathcal{G}^* \subset \text{ad } \mathcal{G}_C$ under the correspondence:

$$X \in \mathcal{G} \leftrightarrow \text{ad } X \in \text{ad } \mathcal{G}_C.$$

To $\mathcal{G}_K$ there corresponds, under the correspondence

$$X \in \mathcal{G}_K \leftrightarrow \text{ad } X \in \text{ad } \mathcal{G}_C,$$

a real linear subspace $\mathcal{G}_K^* \subset \text{ad } \mathcal{G}_C$. Regarding ad $\mathcal{G}_C$ as a Lie algebra over the reals (that is, regard ad $\mathcal{G}_C$ as a linear space over the reals; since the bracket operation satisfies the identities of a Lie algebra, ad $\mathcal{G}_C$ is a real Lie algebra) $\mathcal{G}^*$ and $\mathcal{G}_K^*$ are then subalgebras and to them correspond subgroups G^* and G_K^* of G_C^*. Clearly $G^* \cong \text{ad } G$ topologically as well as algebraically.

LEMMA 2.1. *$\mathcal{G} = \mathcal{G} \cap \mathcal{G}_K + \mathcal{G} \cap (-1)^{1/2}\mathcal{G}_K$ (directly), that is, $\mathcal{G}$ has a base $\{X_1, \cdots, X_P, (-1)^{1/2}X_{P+1}, \cdots, (-1)^{1/2}X_r\}$ with $X_i \in \mathcal{G}_K$ $(i=1, 2, \cdots, r)$. Moreover, there is base $\mathcal{C}$ for $\mathcal{G}_C$ relative to which the matrix of* ad X *is real skew-symmetric for all $X \in \mathcal{G}_K$.*

PROOF. Since the real form $\mathcal{G}_K$ is invariant under θ and since θ^2 is the identity, there is a base $\{X_1, \cdots, X_p, X_{p+1}, \cdots, X_r)$ of $\mathcal{G}_K$ such that $\theta(X_i) = X_i$ $(i=1, \cdots, p)$ and $\theta(X_j) = -X_j$ $(j=p+1, \cdots, r)$. Inasmuch as $\theta((-1)^{1/2}X_j) = (-1)^{1/2}X_j$ $(j=p+1, \cdots, r)$, one infers that $(-1)^{1/2}X_j \in \mathcal{G}$ $(j=p+1, \cdots, r)$. Clearly $\{X_1, \cdots, X_P; (-1)^{1/2}X_{P+1}, \cdots, (-1)^{1/2}X_r\}$ is a base for $\mathcal{G}$, that is, $\mathcal{G} = \mathcal{G} \cap \mathcal{G}_K + \mathcal{G} \cap (-1)^{1/2}\mathcal{G}_K$ (directly).

The fundamental bilinear form being negative definite on the compact real form $\mathcal{G}_K$, a base $\mathcal{C} = \{Z_1, \cdots, Z_r\}$ can be selected for $\mathcal{G}_C$ such that $Z_i \in \mathcal{G}_K$ $(i=1, \cdots, r)$ and such that $B(X, Y) = -\sum_{i=1}^{r} x_i y_i$ if $X = \sum_{i=1}^{r} x_i Z_i$, $Y = \sum_{i=1}^{r} y_i Z_i$. By the Jacobian identities, ad $[X, Y] = \text{ad } Y \text{ ad } X - \text{ad } X \text{ ad } Y$. Hence

$$\begin{aligned} B(\text{ad } X(Y), Z) + B(Y, \text{ad } X(Z)) &= \text{Trace ad } [X, Y] \text{ ad } Z + \text{Trace ad } Y \text{ ad } [X, Z] \\ &= \text{Trace } (\text{ad } X)(\text{ad } Y \text{ ad } Z) - \text{Trace } (\text{ad } Y \text{ ad } Z)(\text{ad } X) \\ &= 0. \end{aligned}$$

It follows that, relative to $\mathcal{C}$, the matrix of ad X is real skew-symmetric if $X \in \mathcal{G}_K$.

DEFINITION. Let $\mathcal{V}$ be a linear space over a field K and let $\mathcal{E}$ denote the group of nonsingular linear transformations on $\mathcal{V}$. Let $\mathcal{C}$ denote a base for $\mathcal{V}$ and T_M the matrix of $T \in \mathcal{E}$ relative to $\mathcal{C}$. If $H \subset \mathcal{E}$, $\{T_M | T \in H\}$ is denoted by H_M. A subgroup G of $\mathcal{E}$ is called "algebraic relative to $\mathcal{C}$" if G_M is the intersection of an algebraic subvariety of the space of matrices of all linear transformations of $\mathcal{V}$ with $\mathcal{E}_M$.

It is to be noted that if a group is algebraic relative to one base for $\mathcal{V}$, then it is algebraic relative to any other base. Hence one can use simply the term "algebraic."

In our usage of algebraic groups, the field K will always be the reals.

DEFINITION. Let $\mathcal{G}$ be a Lie algebra over the complex numbers and $\mathcal{G}_0$ a real form of $\mathcal{G}$. The set of all automorphisms of $\mathcal{G}$ is called "aut $\mathcal{G}$." The subset of aut $\mathcal{G}$ keeping $\mathcal{G}_0$ invariant is called "aut $\mathcal{G}_0$."

Let T be a complex-linear transformation of $\mathcal{G}_C$. If $\mathcal{G}_C$ is regarded as a linear space over the reals, then the transformation T is a fortiori real-linear.

LEMMA 2.2. *Let $\mathcal{G}_C$ be regarded as a linear space over the reals. Then* aut $\mathcal{G}_C$, aut $\mathcal{G}$, *and* aut $\mathcal{G}_K$ *are algebraic groups.*

PROOF. Let $\mathfrak{D}=\{X_1, \cdots, X_r, (-1)^{1/2}X_1, \cdots, (-1)^{1/2}X_r\}$ be a base for $\mathcal{G}_C$ considered as a linear space over the reals. Let X_{r+k} denote $(-1)^{1/2}X_k$ $(k=1, \cdots, r)$. Let ι denote the real-linear transformation determined by the mapping

$$X_k \rightarrow (-1)^{1/2}X_k,$$
$$(-1)^{1/2}X_k \rightarrow -X_k \qquad (k=1, \cdots, r).$$

Let T_M denote the matrix of T relative to $\mathfrak{D}$ if T is a real-linear transformation of $\mathcal{G}_C$. Finally, let the real numbers C_{ij}^k be defined by $[X_i, X_j]=\sum_{k=1}^{2r} C_{ij}^k X_k$ $(i, j, k=1, \cdots, 2r)$.

Clearly $T\in$aut $\mathcal{G}_C$ if and only if det $T\neq 0$ and

(1) $T\iota=\iota T$,

(2) $[TX_i, TX_j]=T[X_i, X_j]$ $(i, j=1, \cdots, r)$.

The first condition states that

$$T_M\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}=\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}T_M,$$

where E is the $r\times r$ identity matrix. Condition (2) states that

$$\sum_{\alpha,\beta=1}^{2r} C_{\alpha\beta}^k a_i^\alpha a_j^\beta = \sum_{\gamma=1}^{2r} C_{ij}^\gamma a_\gamma^k \qquad (i, j = 1, \cdots r),$$

where $(a_p^q)=T_M$ $(p, q=1, \cdots, 2r)$. Thus aut $\mathcal{G}_C$ is algebraic.

To prove that aut $\mathcal{G}$ is algebraic, assume $\{X_1, \cdots, X_r\}$ chosen so as to be a subset of $\mathcal{G}$. Then, clearly, $T\in$aut $\mathcal{G}$ if and only if det $T\neq 0$ and

(1) $T\iota=\iota T$,

(2) $[TX_i, TX_j]=T[X_i, X_j]$ $(i, j=1, \cdots, r)$,

(3) $a_i^j=0$ if $j>r$.

Thus aut $\mathcal{G}$ is algebraic. By a similar argument aut $\mathcal{G}_K$ may be proved to be algebraic.

LEMMA 2.3. *G^*, G_K^*, G_C^* are the components of the identity of* aut $\mathcal{G}$, aut $\mathcal{G}_K$, aut $\mathcal{G}_C$, *respectively.*

Proof. $\mathcal{G}$, $\mathcal{G}^*$, $\mathcal{G}_C^*$ are all semi-simple Lie algebras. The lemma now follows easily from the observations:

(1) If $\exp tX$ is an automorphism of $\mathcal{G}_C$ (or $\mathcal{G}$ or $\mathcal{G}_K$) for all real t, then X is a derivation of $\mathcal{G}_C$ (or $\mathcal{G}$ or $\mathcal{G}_K$).

(2) All the derivations of a semi-simple Lie algebra are inner.

Lemma 2.4. *Let L be an algebraic Lie group containing $s = \exp (-1)^{1/2}W$, where the matrix of W relative to some base for the underlying (real) linear space is skew-symmetric. Then* $\exp (-1)^{1/2}tW \in L$ *for all real t.*

Proof. Choose a base $\mathcal{D}$ for the underlying linear space relative to which the matrix of $W = \text{diag } \{(-1)^{1/2}\lambda_1, \cdots, (-1)^{1/2}\lambda_n\}$, λ_i real. Relative to $\mathcal{D}$ the matrix of s^k is diag $\{e^{-k\lambda_1}, \cdots, e^{-k\lambda_n}\}$.

If f is a polynomial and $f(e^{-k\lambda_1}, \cdots, e^{-k\lambda_n}) = 0$ for all positive integers k, then from the fact that $\lambda_1, \cdots, \lambda_n$ are real it can be inferred that $f(e^{t\lambda_1}, \cdots, e^{t\lambda_n}) = 0$ for all real t.

Inasmuch as L is algebraic, it is seen that $s^k \in L$ for all positive integers k implies that $\exp (-1)^{1/2}tW \in L$ for all real t.

As seen in Lemma 2.1, $\mathcal{G} = \mathcal{G} \cap \mathcal{G}_K + \mathcal{G} \cap (-1)^{1/2}\mathcal{G}_K$. Let $\mathcal{J} = \mathcal{G} \cap \mathcal{G}_K$, let $\mathcal{J}^* = \{\text{ad } X \mid X \in \mathcal{J}\}$, let $\mathcal{S} = \mathcal{G} \cap (-1)^{1/2}\mathcal{G}_K$, and let $\mathcal{S}^* = \{\text{ad } X \mid X \in \mathcal{S}\}$. Let F^* denote the Lie subgroup of G_C^* corresponding to the real Lie subalgebra $\mathcal{J}^* \subset \mathcal{G}_C^*$ and let $S^* = \{\exp X \mid X \in \mathcal{S}^*\}$. Let $\mathcal{C}$ be a base for the complex linear space $\mathcal{G}_C$ relative to which the matrix of ad X is real skew-symmetric for all $X \in \mathcal{G}_K$. Then G_C^* is a subgroup of the orthogonal group relative to $\mathcal{C}$. Let $\bar{F}^*$ denote the subgroup of G^* whose matrices relative to $\mathcal{C}$ are real orthogonal.

Suppose now $g \in G^*$. Then $g = f \cdot s$ where $s = \exp (-1)^{1/2}W$, and relative to $\mathcal{C}$ the matrices of f and W are respectively real orthogonal and real skew-symmetric (cf. C. Chevalley, *Lie groups*, p. 15).

Lemma 2.5. *In the above decomposition for g, $f \in \bar{F}^*$ and $s \in S^*$.*

Proof. Let η denote the conjugation of $\mathcal{G}$ with respect to $\mathcal{G}_K$, let T_M denote the matrix of T relative to $\mathcal{C}$ if T is a complex-linear transformation of $\mathcal{G}_C$, and let $\eta^*(T)$ denote the complex-linear transformation of $\mathcal{G}_C$ with matrix $\bar{T}_M$. Since $\mathcal{G} = \mathcal{G} \cap \mathcal{G}_K + \mathcal{G} \cap (-1)^{1/2}\mathcal{G}_K$, $\eta(\mathcal{G}) = \mathcal{G}$. Moreover $\eta([X, Y]) = [\eta(X), \eta(Y)]$ implies ad $\eta(X)(\eta(Y)) = \eta(\text{ad } X(Y))$ for $X, Y \in \mathcal{G}_C$. On the other hand $\eta^*(\text{ad } X)(\eta(Y)) = \eta(\text{ad } X(Y))$. It follows that $\eta^*(\text{ad } X) = \text{ad } \eta(X)$ for $X \in \mathcal{G}_C$ and that $\eta^*(\mathcal{G}^*) = \mathcal{G}^*$ where $\mathcal{G}^* = \{\text{ad } X \mid X \in \mathcal{G}\}$. Since G^* consists of finite products of the type $\exp X_1 \cdot \exp X_2 \cdots \exp X_s$ where $X_1, \cdots, X_s \in \mathcal{G}^*$, and since $\eta^*(\exp X_1 \cdot \exp X_2 \cdots \exp X_s) = \exp \eta^*(X_1) \cdots \exp \eta^*(X_s)$ it is seen that $\eta^*(G^*) = G^*$.

Consider now the relation $g=f\cdot s$. Then, $\eta^*(g)=\eta^*(f)\cdot\eta^*(s)=f\cdot s^{-1}$. Hence $s^2=(\eta^*(g))^{-1}g$ is in G^*; that is, $\exp 2(-1)^{1/2}W\in G^*$.

Suppose $\mathcal{C}=\{Z_1,\cdots,Z_r\}$. Let $\mathcal{C}'=\{Z_1,\cdots,Z_r,(-1)^{1/2}Z_1,\cdots,(-1)^{1/2}Z_r\}$. Regard now $\mathcal{G}_C$ as a linear space over the reals and denote by T_μ the matrix relative to $\mathcal{C}'$ of a real-linear transformation T on $\mathcal{G}_C$. Since the coefficients of W_M are real, W_μ is the $2r\times 2r$ matrix

$$\begin{pmatrix} W_M & 0 \\ 0 & W_M \end{pmatrix}.$$

$${}^tW_\mu=\begin{pmatrix} {}^tW_M & 0 \\ 0 & {}^tW_M \end{pmatrix}=\begin{pmatrix} -W_M & 0 \\ 0 & -W_M \end{pmatrix}=-W_\mu.$$

Consequently W_μ is a real skew-symmetric matrix. Applying Lemma 2.4, it is seen that $\exp(-1)^{1/2}t\cdot 2W$ belongs to aut $\mathcal{G}$ for all real t and thus, by Lemma 2.3, to G^*. It follows that $(-1)^{1/2}W\in\mathcal{G}^*$. Since W_M is real, $W\in\mathcal{G}^*_K$ and hence $(-1)^{1/2}W\in\mathfrak{S}^*=\mathcal{G}^*\cap(-1)^{1/2}\mathcal{G}^*_K$. Consequently $s\in S^*$ and f, being in G^*, must be in F^*. Proof of Lemma 2.5 is now complete.

Let S^* be endowed with its inner topology relative to G^* (and F^* with its topology qua Lie group).

LEMMA 2.6. *$G^*=F^*\times S^*$ topologically. F^* is a compact Lie subgroup and S^* is topologically a Euclidean space.*

PROOF. Consider the one-to-one mapping $g\rightarrow(f,s)$ defined by: $g=f\cdot s$ where $f\in F^*$, $s\in S^*$ of G^* onto the topological direct product $F^*\times S^*$.

It follows directly from Lemmas 2.2 and 2.3 and the fact that an algebraic Lie group is closed that G^* and G^*_K are closed in the group of all regular real-linear transformations on $\mathcal{G}_C$ and a fortiori in the group of all regular complex-linear transformations on $\mathcal{G}_C$. F^*, the set of all real orthogonal (relative to the base $\mathcal{C}$) complex-linear transformations in G^*, is clearly closed.

The mapping $g_M\rightarrow(f_M,s_M)$ is the restriction of the well known[5] one-to-one bicontinuous mapping of the group of all regular complex-linear transformations of $\mathcal{G}_C$ onto the topological direct product of the group of unitary matrices and the set of positive definite hermitian matrices (relative to $\mathcal{C}$). Furthermore, the exponential mapping of $\mathfrak{S}^*$ onto S^* is the restriction to $\mathfrak{S}^*$ of the well known homeomorphism[5] of the hermitian matrices onto the positive definite hermitian matrices. Since G^* and F^* are closed, their topologies as Lie groups and the given topology of S^* are equivalent to their inner topologies relative to the group of all regular complex-linear transformations of $\mathcal{G}_C$. It follows that S^* is topologically a Euclidean space and that the mapping $g \rightarrow f \cdot s$ is a homeomorphism of G^* onto $\tilde{F}^* \times S^*$.

---

[5] Cf. C. Chevalley, *Lie groups*, Princeton University Press, 1946, p. 16.

Since G^* is connected, $\tilde{F}^*$ is connected. But the Lie algebra of $\tilde{F}^*$ is clearly $\mathcal{G}^* \cap \mathcal{G}_K^*$, the Lie algebra of the Lie subgroup F^*. Consequently $\tilde{F}^* = F^*$, and F^* is closed in the full complex-linear group on $\mathcal{G}_C$. Inasmuch as $\{T_M \mid T \in F^*\}$ is a subgroup of the real orthogonal group (relative to base $\mathcal{C}$), F^* is compact. Proof of the lemma is now complete.

Let D denote the center of the semi-simple group G. D is discrete and $G^* = G/D$ topologically as well as algebraically, under the natural mapping ζ:

$g \in G \rightarrow$ automorphism of $\mathcal{G}_C$ induced by Γ_g, where Γ_g is the inner automorphism $x \rightarrow g^{-1}xg$ of G. Let $S = \{\exp X \mid X \in \mathsf{S},\ \mathsf{S} = \mathcal{G} \cap (-1)^{1/2}\mathcal{G}_K\}$, and let F denote the connected Lie subgroup of G determined by $\mathcal{J} = \mathcal{G} \cap \mathcal{G}_K$.

$\zeta(\exp X) = \exp \operatorname{ad} X$ if $X \in \mathcal{G}$. Inasmuch as the mappings $X \rightarrow \operatorname{ad} X$ and $\operatorname{ad} X \rightarrow \exp \operatorname{ad} X$ are one-to-one if $X \in \mathsf{S}$, the mapping $X \rightarrow \exp X$ is one-to-one if $X \in \mathsf{S}$. Give S the topology which makes the mapping $X \rightarrow \exp X$ of S (endowed with its natural Euclidean topology) onto S a homeomorphism and F its topology qua Lie group. Clearly ζ maps S homeomorphically onto S^*.

LEMMA 2.7. *$G = F \times S$ topologically.*

PROOF. Let $\tilde{F} = \zeta^{-1}(F^*)$. The homogeneous space $G/\tilde{F}$ is homeomorphic to $G^*/F^* = S^*$, a simply connected space. It follows that $\tilde{F}$ is connected, $\tilde{F} = F$, and $D \subset F$.

Let $g \in G$. Since $\zeta(g) = f^* \cdot s^*$ where $f \in F^*$ and $s \in S^*$, $g = d \cdot f' \cdot s$ where $f' \in F$, $s \in S$, $d \in D$. But $D \subset F$; hence $g = f \cdot s$ where $f \in F$, $s \in S$.

Suppose that $f_1 \cdot s_1 = f_2 \cdot s_2$ where f_1, $f_2 \in F$ and s_1, $s_2 \in S$. Then $\zeta(f_1)\zeta(s_1) = \zeta(f_2)\zeta(s_2)$ implies that $\zeta(s_1) = \zeta(s_2)$. Since ζ is one-to-one on S, $s_1 = s_2$ and hence $f_1 = f_2$, that is, the mapping

$$\phi: \quad (f, s) \rightarrow g$$

defined by $g = f \cdot s$ is a one-to-one mapping of $F \times S$ onto G.

We now prove that ϕ is a homeomorphism. Inasmuch as the exponential mapping of a Lie algebra (endowed with its natural Euclidean topology) into a Lie group is a continuous mapping, and inasmuch as group multiplication is continuous, the mapping ϕ is continuous.

Let U, V be connected neighborhoods in F, S respectively. Since ζ

is a homeomorphism on S, $\zeta(V)$ is a connected open set of S^*. Since F and F^* are closed subgroups of G and G^* respectively, it follows that their topologies qua Lie groups are equivalent to their inner topologies relative to G and G^* respectively. Consequently $\zeta(U)$ is a connected open set of F^* and $\zeta(\phi(U\times V))$ is a connected open set of G^*.

$$\zeta(\phi(U \times V)) = \zeta(U\cdot V) = \zeta(U)\cdot\zeta(V).$$

Hence by selecting U and V sufficiently small, $\zeta(\phi(U\times V))$ can be made arbitrarily small. Since G is a covering space of G^* under ζ, $\zeta^{-1}(N)$ is a union of disjoint connected open sets if N is a sufficiently small connected open set. Inasmuch as $\phi(U\times V)$ is connected, it is a (connected) open set of G if U and V are sufficiently small. The mapping ϕ is therefore open.

LEMMA 2.8. *$\mathcal{J}^*=\mathcal{L}^*\oplus\mathcal{A}^*$ (directly) where $\mathcal{L}^*$ and $\mathcal{A}^*$ are respectively semi-simple and abelian ideals of $\mathcal{J}^*$. Let L^*, A^* be the Lie subgroups of F^* determined by $\mathcal{L}^*$, $\mathcal{A}^*$ respectively. Then L^* and A^* are compact.*

PROOF. Consider the adjoint group of F acting on $\mathcal{J}$. Since F^* is compact, the adjoint group of F (a homomorphic image of F^*) is compact and is therefore completely reducible, that is, if the linear transformations of the adjoint group keep a subspace invariant, they keep invariant a complementary subspace. It follows that $\mathcal{J}$ is the direct union of minimal ideals. Consequently $\mathcal{J}=\mathcal{L}\oplus\mathcal{A}$ (directly) where $\mathcal{L}$ and $\mathcal{A}$ are respectively semi-simple and abelian ideals of $\mathcal{J}$. Let $\mathcal{L}^*$, $\mathcal{A}^*$, be the images of $\mathcal{L}$, $\mathcal{A}$ respectively under the adjoint representation of $\mathcal{G}$. Then $\mathcal{J}^*=\mathcal{L}^*\oplus\mathcal{A}^*$ (directly).

Since $\mathcal{L}^*$ contains no abelian ideals, $\mathcal{A}^*$ can clearly be characterized as the maximum abelian ideal of $\mathcal{J}^*$. It follows that A^* is the maximum connected normal abelian Lie subgroup of F^*. Since $\bar{A}^*$, the topological closure of A^*, is a connected normal abelian Lie subgroup, $\bar{A}^*=A^*$, that is, A^* is a closed subgroup of F^* and is thus compact.

Consider now F^*/A^*. This is a compact Lie group whose Lie algebra is isomorphic to $\mathcal{J}^*/\mathcal{A}^*=\mathcal{L}^*$. Since there is a compact Lie group with Lie algebra isomorphic to $\mathcal{L}^*$, by a theorem of Weyl[6] any Lie group whose Lie algebra is isomorphic to $\mathcal{L}^*$ is compact. Consequently L^* is compact.

THEOREM 2. *$G=K\times E$ topologically, where K is a maximal compact subgroup and E is topologically a Euclidean space.*

---

[6] H. Weyl, loc. cit. p. 170.

PROOF. As was seen in the proof of Lemma 8, $\mathcal{J}=\mathcal{L}\oplus\mathcal{A}$ where $\mathcal{L}$ is semi-simple and $\mathcal{A}$ abelian. Let L, A be the Lie subgroups of F corresponding to $\mathcal{L}$, $\mathcal{A}$ respectively, $\zeta(L)=L^*$ and $\zeta(A)=A^*$. A, being the component of the identity of the closed subgroup $\zeta^{-1}(A)$, is closed.

Since $\mathcal{L}\cong\mathcal{L}^*$ and L^* is compact, by Weyl's theorem L is compact.

Let $T\times V$ be a decomposition of the connected abelian Lie group A into the direct product of a toroidal group T and a vector group V. V is closed in A and hence in F.

Set $K=LT$, that is, $K=\{lt \mid l\in L,\ t\in T\}$. K is a compact normal subgroup of F and $F=KV$. K and V being closed subgroups of F, $K\cap V$ is both compact and a subgroup of V; it follows that K and V intersect in only the identity of F. Inasmuch as K and V are closed normal subgroups of F, $F=K\times V$ topologically as well as algebraically. Finally, let $E=V\cdot S$. Clearly E is topologically the direct product of two Euclidean spaces and is hence itself Euclidean. It may now be asserted that $G=K\times E$ topologically.

HARVARD UNIVERSITY AND
INSTITUTE FOR ADVANCED STUDY

L-S-HOMOTOPY CLASSES ON THE TOPOLOGICAL IMAGE OF A PROJECTIVE PLANE

MARSTON MORSE

1. **Introduction.** Models for the L-S-(locally simple) homotopy classes of closed p-curves (p = parameterized) on any 2-manifold S have been announced in Morse [1].[1] Proofs have been given only for the case in which S is orientable. The present paper will treat the case in which S is the top. (topological) image of a projective plane. The proofs in the case of a general non-orientable surface can be given by an appropriate modification of methods of Morse [1] and of the present paper.

Recall that one writes $f \approx 0$ when f is a closed p-curve homotop. to zero. Deferring technical definitions until later sections, we can state the principal theorem as follows.

THEOREM 1.1. *Let h be a simple closed p-curve on the top. image S of a projective plane with h not ≈ 0 on S. Let $h^{(n)}$ $(n > 0)$ be a closed p-curve on S which traces h n times. Any L-S-closed p-curve f on S is in the L-S-homotopy class of $h^{(1)}$ or $h^{(3)}$ if h not ≈ 0, and of $h^{(2)}$ or $h^{(4)}$ if $h \approx 0$. No two of the p-curves $h^{(1)}$, $h^{(2)}$, $h^{(3)}$, $h^{(4)}$ are in the same L-S-homotopy class.*

For theorems on regular closed curves in the plane see Whitney, and H. Hopf. For L-S-closed curves in the plane see Morse [2] and Morse and Heins [1]. For a use of L-S-curves in studying deformation classes of meromorphic functions see Morse and Heins [2].

2. **L-S-curves and deformations.** Let C represent the unit circle on which $|z| = 1$ in the plane of the complex variable $z = u + iv$. With $z = e^{i\theta}$ on C, we assign C the *sense* of increasing θ. Let S be an arbitrary 2-manifold. A *closed p-curve* on S is a continuous mapping f of C into S such that the image of z in C is a point $f(z)$ in S. Two p-curves f_1 and f_2 are regarded as the *same* if and only if

$$f_1(z) = f_2(z)$$

for every z in C. The union of the points $f(z)$ in S as z ranges over C is called the *carrier* of f. The simplest case arises when the points $f(z)$ are in 1–1 correspondence with their antecedents z in C, and in this case f is termed *simple*.

Let f be a continuous mapping of C into S. Let λ be any sense

---

Received by the editors August 13, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

preserving top. (that is, 1–1 and continuous) mapping of C onto C with $\lambda(z)$ the image in S of z in C. Then f and $f\lambda$ are termed *equivalent* mappings of C into S or equivalent p-curves on S. Here $f\lambda$ symbolizes the function whose value at z is $f[\lambda(z)]$. In earlier papers the class of mappings equivalent to a given mapping has been termed a *curve*, as distinguished from a p-curve. We shall here find it simpler to rely on p-curves and make use of Lemma 2.1, according to which any two equivalent closed p-curves are in the same L-S-homotopy class.

A closed p-curve f on S will be termed L-S if there is a positive constant e so small that the mapping under f of any arc of C with length less than e is top. Hence there exists a constant $e_1 > 0$ so small that any subarc of f whose carrier has a diameter less than e_1 is simple. Such a constant e_1 is called a *norm of local simplicity* of f.

The L-S-homotopy class $[f]$. Let J be the interval $0 \leqq t \leqq 1$, t the "time." A *deformation* of f on S is a continuous mapping D of $C \times J$ into S such that the image of a point (z, t) in $C \times J$ is a point $D(z, t)$ in S, with

$$D(z, 0) = f(z) \qquad (z \in C)$$

initially (that is, when $t = 0$). For t fixed in J, D defines a mapping[2] $D(\cdot, t)$ of C into S, and thus a p-curve f^t termed the *deform* of f at the time t. We say that f is deformed through the family f^t into f^1. If the deforms f^t, $0 \leqq t \leqq 1$, possess a common norm of local simplicity, D will be said to be L-S. The class of p-curves f^1 into which f can be L-S-deformed on S is termed the L-S-homotopy class $[f]$ of f on S. By virtue of a proof similar to that of Lemma 28.1 of Morse [2] we can affirm the following.

LEMMA 2.1. *Any two equivalent closed p-curves on S are in the same L-S-homotopy class on S.*

A first objective of this paper is the proof of Theorem 1.1. Models for the L-S-homotopy class $[f]$ on the top. image S of a projective plane will be determined as indicated. In case the given p-curve $f \approx 0$, the M-order of f, as defined in Morse [1, §4], is 1, or 2 mod 2, according as $[f] = [h^{(2)}]$ or $[f] = [h^{(4)}]$. Here M is the top. sphere covering S.

In case the given p-curve f not ≈ 0, a new *S-difference order* $d_S(f)$ is defined in §7 and $[f] = [h^{(1)}]$ or $[h^{(3)}]$ according as

$$d_S(f) = 1 \text{ or } 3 \qquad (\text{mod } 4).$$

The value of $d_S(f)$ will be shown to be independent of f in its L-S-homotopy class, and to be a top. invariant in the following sense. If S' is a top. image of S with f' on S' the top. image of f on S, then

---

[2] For fixed t, $D(\cdot, t)$ symbolizes the function whose value at z in C is $D(z, t)$.

$$d_S(f) = d_{S'}(f') \qquad \pmod 4.$$

An important special result in the case f not ≈ 0 is that $d_S(f) = 1 \bmod 4$ if and only if $[f]$ contains a simple closed p-curve.

3. **The 2-sphere M covering Π.** We shall begin with a special model Π of a projective plane obtained by identifying diametrically opposite points

$$x = (x_1, x_2, x_3), \quad -x = (-x_1, -x_2, -x_3)$$

of a 2-sphere

$$(3.1) \qquad M: \quad x_1^2 + x_2^2 + x_3^2 = 1.$$

A point in Π can accordingly be given by a pair $(x, -x)$ of diametrically opposite points x and $-x$ in M. We understand that the point $(x, -x)$ in Π equals the point $(-x, x)$ of Π. We say that x in M *covers* the point $(x, -x) = (-x, x)$ in Π, and denote this point in Π by $A(x)$. We say also that x on M *projects* into $A(x)$ on Π.

The mapping A of M onto Π has two fundamental properties: (1) *for x and y in M*

$$(3.2) \qquad A(x) = A(y)$$

if and only if $x = \pm y$; (2) *the mapping A is locally top.*

Let ϕ be any mapping of an abstract set E into M. Then[3] $A\phi$ is a mapping of E into Π termed the *A-projection* of ϕ into Π.

A closed p-curve F mapping C into M will be termed R-invariant (reflection invariant) if for every z in C

$$(3.3) \qquad F(-z) = -F(z).$$

The mapping μ of C onto the semi-circle C_1. We shall make frequent use of a mapping μ of C onto the semi-circle

$$(3.4) \qquad C_1: \quad z = e^{\theta i}, \qquad 0 \leqq \theta < \pi,$$

of C. Explicitly, with $z = e^{i\theta}$, set

$$(3.5) \qquad \mu(z) = e^{i(\theta/2)} \qquad (0 \leqq \theta < 2\pi).$$

The mapping μ of C into C_1 is continuous, except at $z = 1$. Let C_2 be the residual semi-circle $C - C_1$ of C. When F is R-invariant the mapping $F\mu$ of C into M determines the mapping F of C into M in accordance with the equations

---

[3] Here and elsewhere a functional operation such as A on ϕ is indicated by $A\phi$. The value of $A\phi$ at a point x of E will be denoted by $A\phi(x)$ and not by $A[\phi(x)]$.

$$F\mu(z^2) = F(z) \qquad [z \in C_1], \tag{3.6}$$
$$F\mu(z^2) = -F(z) \qquad [z \in C_2].$$

By definition μ is single-valued; it is one branch of $z^{1/2}$ on C, and to avoid confusion is not to be continued into the other branch. The mapping $F\mu$ of C into M is discontinuous at the point $z=1$ of C. In fact,

$$F\mu(1) = -F\mu(1^-) \tag{3.7}$$

where $F\mu(1^-)$ indicates the limit of $F\mu(z)$ as z tends to $z=1$ on C from the arc of C on which $\pi < \text{arc } z < 2\pi$.

When F is an R-invariant closed p-curve on M, the mapping

$$f^F = AF\mu \tag{3.8}$$

of C into Π is a *closed* p-curve on Π by virtue of (3.7). The p-curve f^F is the A-projection of $F\mu$ on Π.

A *deformation* Δ of a closed p-curve F on M is a continuous mapping of $C \times J$ into M in which the image of a point (z, t) of $C \times J$ is a point $\Delta(z, t)$ in M, with

$$\Delta(z, 0) = F(z) \qquad [z \in C] \tag{3.9}$$

initially, that is, for $t=0$. Such a deformation is termed *R-invariant* if

$$\Delta(-z, t) = -\Delta(z, t)$$

for each point (z, t) in $C \times J$. Suppose then that Δ is R-invariant. Then F is necessarily R-invariant. The deform F^t of F under Δ is the R-invariant mapping $\Delta(\cdot, t)$ of C into M. The A-projection $AF\mu$ of $F\mu$ is a closed p-curve f^F on Π, and $A\Delta\mu$ is a deformation of f^F on Π. We draw the following conclusion:

LEMMA 3.1. *If an R-invariant L-S-closed p-curve F on M admits an R-invariant L-S-deformation on M through p-curves F^t, then the A projection on* Π *of $F\mu$ admits a L-S-deformation on* Π *through the closed p-curves $AF^t\mu$ on* Π.

4. **The cases $f \approx 0$ and f not ≈ 0.** Let f be a closed p-curve on Π. Let x_0 and $-x_0$ be the points in M which cover $f(1)$. Let a be either one of the points x_0, $-x_0$. Given f and a it follows from the local top. character of the projection A of M onto Π that there exists a unique continuous mapping of ϕ_a^f of the θ-axis into M such that (with $z=e^{i\theta}$)

$$(4.1) \qquad \phi_a^f(0) = a \qquad A\phi_a^f(\theta) = f(z) \qquad (-\infty < \theta < +\infty).$$

It follows from (4.1) that

$$(4.2) \qquad A\phi_a^f(\theta + 2\pi) = A\phi_a^f(\theta).$$

Recall that for points x and y in M, $A(x) = A(y)$ if and only if $x = \pm y$. From (4.2) then

$$(4.3) \qquad \phi_a^f(\theta + 2\pi) = \pm\, \phi_a^f(\theta).$$

The sign in (4.3) is independent of θ and is + if and only if $f \approx 0$ in accordance with the following lemma.

LEMMA 4.1. *If f is a closed p-curve on* Π *and a a point in M such that $A(a) = f(1)$, and if ϕ_a^f is the unique continuous mapping of the θ-axis into M such that* (4.1) *holds given a, then a necessary and sufficient condition that*

$$(4.4) \qquad \phi_a^f(\theta + 2\pi) = \phi_a^f(\theta) \qquad (-\infty < \theta < +\infty)$$

is that $f \approx 0$ on Π.

If (4.4) holds the equation

$$(4.5) \qquad F_a^f(z) = \phi_a^f(\theta) \qquad (z = e^{i\theta})$$

defines a single-valued mapping F_a^f of C into M. Thus F_a^f is a closed p-curve on M, and as such is deformable on M through a continuous family $F_a^{f^t}$, $0 \leq t \leq 1$, of closed p-curves on M into a p-curve, whose carrier is a point of M. The A-projections $AF_a^{f^t}$ into Π of these p-curves deform f on Π into a p-curve on Π whose carrier is a point. Hence $f \approx 0$ on Π if (4.4) holds.

Conversely suppose that $f \approx 0$ on Π, or more specifically that f on Π is deformed into a p-curve whose carrier is a point, through a family f^t $(0 \leq t \leq 1)$ of closed p-curves on Π. With ϕ_a^f given in terms of f by (4.1) a continuous mapping $\phi_a^{f^t}$ of the θ-axis into M can be determined by continuation with respect to increasing t, with

$$\phi_a^{f^0}(\theta) \equiv \phi_a^f(\theta)$$

initially, and

$$(4.6) \qquad A\phi_a^{f^t}(\theta) \equiv f^t(z) \qquad [\text{for } z = e^{i\theta}]$$

where $\phi_a^{f^t}(\theta)$ is continuous in (θ, t) for $0 \leq t \leq 1$ and arbitrary θ.

As a consequence of (4.6)

$$\phi_a^{ft}(2\pi) = \pm\, \phi_a^{ft}(0) \qquad (0 \leqq t \leqq 1) \tag{4.7}$$

where the sign in (4.7) is independent of t on $[0, 1]$. For t sufficiently near 1 on $[0, 1]$ the + sign must hold in (4.7) since the carrier of f^1 in (4.6) is a point and A is locally top. Hence the + sign must hold in (4.7) when $t=0$ as well. Thus (4.4) holds when $f \approx 0$ on Π.

Antecedents and μ-antecedents on M of p-curves on Π. If f is a closed p-curve on Π, any closed p-curve F on M such that $AF=f$ will be called an antecedent on M to f on Π. If f not ≈ 0 no closed p-curve F on M can be antecedent to f, because (4.4) cannot hold in this case. However we shall verify the following. When f not ≈ 0 there always exists (Lemma 4.2) an R-invariant closed p-curve F on M such that $AF\mu=f$. Such a closed p-curve F will be called a *μ-antecedent* of f.

LEMMA 4.2. *Let f be a closed p-curve on* Π. *If $f \approx 0$ on* Π *there exist just two closed p-curves F^f antecedent on M to f on* Π. *If f not ≈ 0 there exist just two closed p-curves F^f, μ-antecedent on M to f on* Π.

The two antecedents (μ-antecedents) F and F^* of f satisfy the relation $F(z) = -F^*(z)$.

Case I. $f \approx 0$. We start with ϕ_a^f as defined in (4.1), with $a=x_0$ or $-x_0$ where $A(x_0)=f(1)$. Let F_a^f then be defined as in (4.5). In case I, (4.4) holds, so that F_a^f is closed on M. The A-projection of F_a^f is f in accordance with (4.1). There are accordingly at least two closed p-curves F_a^f ($a=x_0$ or $-x_0$) antecedent to f on Π.

Any other closed p-curve F on M such that $AF=f$ must satisfy the condition

$$F(1) = a \qquad (a = x_0 \text{ or } -x_0).$$

By virtue of the continuity of the mapping F^f and the top. character of A, F is thereby uniquely determined by f and a, and so must equal F_a^f. There are accordingly just two closed p-curves F antecedent on M to f on Π when $f \approx 0$.

Case II. f not ≈ 0. We start again with ϕ_a^f as defined in (4.1). In Case II we define F_a^f as a p-curve on M such that for $w=e^{i\alpha}$

$$F_a^f(w) = \phi_a^f(2\alpha) \qquad [-\infty < \alpha < \infty]. \tag{4.8}$$

That F_a^f is single-valued for each w in C, and R-invariant, follows from the relation

$$\phi_a^f(\alpha + 2\pi) = -\, \phi_a^f(\alpha) \tag{4.9}$$

which holds by virtue of Lemma 4.1 and (4.3). In fact

$$F^f_a(-w) = \phi^f_a(2[\alpha + \pi]) = -\phi^f_a(2\alpha) = -F^f_a(w)$$

for every w in C. Relation (4.8) implies that

$$F^f_a\mu(z) = \phi^f_a(\theta) \qquad [z = e^{i\theta}, 0 \leqq \theta < 2\pi] \tag{4.10}$$

and it follows from (4.1) that $AF^f_a\mu = f$. While $F^f_a\mu$ is discontinuous at $z=1$ on account of (4.9), $AF^f_a\mu$ is continuous at $z=1$. In Case II there accordingly exists two closed p-curves F^f_a $(a = \pm x_0)$, μ-antecedent on M to f on Π.

Any other closed p-curve F on M such that $AF\mu = f$ must satisfy the condition $F(1) = a$, $[a = \pm x_0]$ and by a process of continuation be uniquely determined by the relation $AF\mu = f$ as the R-invariant closed p-curve F^f_a. The p-curves $F^f_a[a = \pm x_0]$ are accordingly the only closed p-curves μ-antecedent on M to f on Π when f not ≈ 0.

The preceding lemma can be extended to deformations D on M as follows.

LEMMA 4.3. *Corresponding to any continuous deformation D on Π of a closed p-curve f such that $f \approx 0$ [f not ≈ 0] on Π, and to either of the two antecedents [μ-antecedents] F^f of f, there exists a unique deformation Δ of F^f on M such that $A\Delta = D$ when $f \approx 0$, while Δ is R-invariant and $A\Delta\mu = D$ when f not ≈ 0.*

We term the deformation Δ of the lemma *antecedent* on M to D on Π when $A\Delta = D$, and μ-antecedent when $A\Delta\mu = D$.

The proof of Lemma 4.3 is so similar to that of Lemma 4.2 that it need only be indicated.

Let f^t, $0 \leqq t \leqq 1$, be the deform of f under D at the time t. Let $X_0(t)$ and $-X_0(t)$ be the two points in M which cover $f^t(1)$ on Π, X_0 being chosen so that it is continuous for t in $[0, 1]$. Let Θ represent the θ-axis. Let a be either of the functions $\pm X_0$. There exists a unique continuous mapping Φ_a of $\Theta \times J$ into M such that (with $z = e^{i\theta}$)

$$\Phi_a(0, t) = a(t), \qquad A\Phi_a(\theta, t) = D(z, t) \tag{4.11}$$

for each point (θ, t) on $\Theta \times J$. Equation (4.11) replaces (4.1) while the analogue of (4.3) is

$$\Phi_a(\theta + 2\pi, t) = \pm\, \Phi_a(\theta, t) \tag{4.12}$$

where the sign is + if and only if $f \approx 0$ on Π.

Case I. $f \approx 0$. In this case the required deformation Δ is defined by the equation

$$\Delta_a(z, t) = \Phi_a(\theta, t) \qquad [z = e^{i\theta}].$$

Then $A\Delta_a = D$ in accordance with (4.11).

Case II. f not ≈ 0. One here sets

$$\dot{\Delta}_a(w, t) = \Phi_a(2\alpha, t) \qquad [w = e^{i\alpha}] \tag{4.13}$$

and observes that Δ_a is single-valued and R-invariant by virtue of (4.12), the sign $-$ prevailing in (4.12). Finally (4.13) implies that

$$\Delta_a[\mu(z), t] = \Phi_a(\theta, t) \qquad (z = e^{i\theta}, 0 \leqq \theta < 2\pi)$$

for each t on $[0, 1]$, so that $A\Delta_a\mu = D$.

In either case the uniqueness of a deformation Δ satisfying the lemma when an antecedent (μ-antecedent) F^f of f is given, follows as in the proof of Lemma 4.3.

If F is an R-invariant closed p-curve on M the class of all R-invariant closed p-curves which admit R-invariant L-S-deformations into F on M will be called an R-invariant *L-S-homotopy class* on M.

The following theorem reduces the problem of determining the L-S-homotopy classes on Π to a problem on M. It is a consequence of the preceding lemmas including Lemma 3.1.

THEOREM 4.1. *A necessary and sufficient condition that L-S-closed p-curves f_1 and $f_2 \approx 0$ [not ≈ 0] on Π be in the same L-S-homotopy class on Π is that an antecedent [μ-antecedent] F^{f_1} and F^{f_2} of f_1 and f_2 respectively be in the same L-S-homotopy class [R-invariant L-S-homotopy class] on M.*

In case $f \approx 0$ on Π models for the L-S-homotopy classes of f can accordingly be inferred from those on the 2-sphere M. Such models on M are given in Theorem 4.2 of Morse [1].

The model p-curve k on Π, *and* Γ *on* M. We shall introduce a simple closed p-curve k on Π, with carrier on Π covered by a great semicircle on M. More definitely we suppose that k has *a* μ-antecedent Γ on M given by the mapping

$$x_1 + ix_2 = z, \qquad x_3 = 0 \qquad [z \in C], \tag{4.14}$$

of the circle C into M. For each integer $n > 0$ let closed p-curves $k^{(n)}$ on Π and $\Gamma^{(n)}$ on M be defined by the equations

$$k^{(n)}(z) = k(z^n), \qquad \Gamma^{(n)}(z) = \Gamma(z^n) \qquad [z \in C].$$

The p-curves $k^{(1)}$ and $k^{(3)}$ have Γ and $\Gamma^{(3)}$ as μ-antecedents on M, while $k^{(2)}$ and $k^{(4)}$ have Γ and $\Gamma^{(2)}$ as antecedents on M. Theorem 4.2 of Morse [1] gives the following.

THEOREM 4.2. *Any L-S-closed p-curve* $f \approx 0$ *on* Π *is in the L-S-homotopy class of* $k^{(2)}$ *or* $k^{(4)}$ *on* Π, *while* $[k^{(2)}] \neq [k^{(4)}]$ *on* Π.

That $[k^{(2)}] \neq [k^{(4)}]$ on Π follows from the fact that $[\Gamma] \neq [\Gamma^{(2)}]$ on M. For the equality $[k^{(2)}] = [k^{(4)}]$ on Π would imply that $[\Gamma] = [\Gamma^{(2)}]$ on M by virtue of Theorem 4.1.

Given $f \approx 0$ on Π the problem of determining to which of the two homotopy classes, $[k^{(2)}]$ or $[k^{(4)}]$, f belongs is equivalent to the problem of determining to which of the two homotopy classes, $[\Gamma]$ or $[\Gamma^{(2)}]$ on M, an antecedent F of f belongs on M. This problem is resolved by the determination of the M-order $p(F)$, of F, as shown in §4 of Morse [1]. In fact

$$[F] = [\Gamma] \quad \text{or} \quad [\Gamma^{(2)}]$$

according as $p(F) = 1$ or $2 \bmod 2$. As shown in Morse [1], $p(F)$ is a topological invariant of M and F, and in particular is invariant under any "R-invariant" homeomorphism T of M, that is, one for which

$$T(-x) = -T(x) \qquad (x \in M),$$

and is accordingly a top. invariant of Π. Finally if F and F' are L-S-closed p-curves on M

$$p(F) = p(F')$$

if and only if $[F] = [F']$ on M, and accordingly if and only if $[AF] = [AF']$ on Π.

We turn accordingly to the case f not ≈ 0 on Π.

5. **L-S-homotopy classes when f not ≈ 0 on Π.** Let F be an R-invariant L-S-closed p-curve on M. In accordance with Theorem 4.1 we seek a model for the R-invariant L-S-homotopy classes of F on M. To that end we refer to the semi-circle C_1 defined by $z = e^{\theta i}$ for $0 \leqq \theta < \pi$, and to the residual semi-circle C_2 defined by $z = e^{\theta i}$ when $\pi \leqq \theta < 2\pi$. Let F_1 and F_2 be submappings of F defined by the equations (a superimposed bar indicates closure)

$$F_1(z) = F(z) \qquad [z \in \overline{C}_1],$$
$$F_2(z) = F(z) \qquad [z \in \overline{C}_2]$$

and term F_1 the ***kernel*** of F_1 and F_2 the ***kernel residue***. We shall be concerned with various continuous mappings of $\overline{C}_1$ into M and will term such mappings p-arcs on M.

Various elementary p-arcs and p-curves on M will be defined and analyzed for later use. In defining kernels F_1 the path which $F(z)$ traces as z traces $\overline{C}_1$ will be given. These paths will be ordered finite sequences of simple, sensed arcs successively joined to form a continuous curve. The paths used will be rectifiable. From a path α a kernel $F_1 = \{\alpha\}$ will be formed by making z in $\overline{C}_1$ correspond to that point $F_1(z)$ in α which divides α in the same ratio with respect to arc length as that in which z divides $\overline{C}_1$ with respect to arc length. The kernel residue F_2 will be defined by setting

$$F_2(-z) = -F_1(z) \qquad [z \in \overline{C}_1].$$

In order that F so defined be L-S it is sufficient that F_1 be L-S and that the images on M under F of sufficiently small neighborhoods of $z=1$ in C be simple.

A symbolism is needed for a path α which is a *product*

$$\alpha = a_1 a_2 \cdots a_n$$

of simple, regular, sensed, closed curves $a_1, \cdots, a_n$, with a_k positively tangent to a_{k+1} $(k=1, \cdots, n-1)$ at a prescribed point P_k. If a is a simple, sensed arc or closed curve, and P and Q are two points in a, $a(P, Q)$ shall denote the subarc (if any exists) of a leading from P to Q on a. With this understood α shall denote the path defined by the sequence of simple arcs (with $a_1(P_1, P_1)$ the arc a_1 cut at P_1),

$$\begin{gathered} a_1(P_1, P_1)a_2(P_1, P_2) \cdots a_{n-1}(P_{n-2}, P_{n-1}), \\ a_n(P_{n-1}, P_{n-1})a_{n-1}(P_{n-1}, P_{n-2}) \cdots a_2(P_2, P_1). \end{gathered} \tag{5.1}$$

We admit the possibility that a_1 is not a closed curve, but rather the closure of a simple arc, while $a_2 \cdots a_n$ remain simple closed curves. In such a case P_1 is to be an inner point of a_1. If P and P' are the initial and final points of a_1, the preceding sequence (5.1) is to be altered by replacing $a_1(P_1, P_1)$ by $a_1(P, P_1)$ and $a_1(P_1, P')$ is to be added to the sequence.

The elementary p-arcs on M to be used in defining model kernels F_1 on M can now be defined. Let γ be the simple arc

$$x_1 = \cos\theta, \qquad x_2 = \sin\theta, \qquad x_3 = 0 \qquad (0 \leqq \theta < \pi) \tag{5.2}$$

taken in the sense of increasing θ. Let λ be a small sensed circle of diameter <1, with $x_3 \geqq 0$ thereon, positively tangent to γ at the mid point $(0, 1, 0)$ of γ. Let λ^{-1} be the reflection of λ in the plane $[x_3=0]$. For n a positive integer λ^n shall formally symbolize $\lambda \cdots \lambda$ with n factors λ, while λ^{-n} shall formally symbolize $\lambda^{-1} \cdots \lambda^{-1}$ with n factors λ^{-1}. Let q be any nonvanishing integer. We introduce a product path $\gamma\lambda^q$ in which $(0, 1, 0)$ is the point of contact of successive factors. Then $\{\gamma\lambda^q\}$ is a well-defined p-arc on M which, taken as a kernel F_1, leads to a L-S-closed p-curve F on M. We shall prove the following lemma.

LEMMA 5.1. *The p-arc $\{\gamma\lambda^q\}$ admits a L-S-deformation on M in which sufficiently short initial and final subarcs of γ remain simple with invariant carriers, and in which $\{\gamma\lambda^q\}$ is deformed into $\{\gamma\}$ when q is even, and into $\{\gamma\lambda\}$ when q is odd.*

For simplicity we begin with $\gamma\lambda^2$. The circle λ can be deformed on M through circles tangent to γ at (0, 1, 0) into λ^{-1}, so that $\{\gamma\lambda^2\}$ is L-S-deformable on M into $\{\gamma\lambda\lambda^{-1}\}$. The point of contact of λ^{-1} with λ can be continuously regressed on λ to the point of maximum x_3 on λ, varying λ^{-1} through circles λ_t^{-1}, $0 \leq t \leq 1$, of fixed radius. Then λ_1^{-1} is the terminal circle in this deformation of λ^{-1}. Observe that $\lambda\lambda_1^{-1}$ is a figure eight with $x_3 > 0$ thereon, except at the point of contact of λ with γ at (0, 1, 0). It is clear that $\{\gamma\lambda\lambda_1^{-1}\}$ is L-S-deformable into $\{\gamma\}$, and that the whole deformation of $\{\gamma\lambda^2\}$ into $\{\gamma\}$ can be so made that sufficiently short initial and final arcs of γ remain simple with invariant carriers.

In the same way, it is clear that for $q>2$ $[\gamma\lambda^q\}$ is L-S-deformable successively into

$$\{\gamma\lambda^{q-2}\lambda\lambda_1^{-1}\} \qquad \{\gamma\lambda^{q-2}\},$$

so that an induction with respect to q shows that the lemma is true if $q>2$. A reflection in the plane at $x_3=0$ makes it appear that for $q<0$, $\{\gamma\lambda^q\}$ is L-S-deformable in the required manner into $\{\gamma\}$ when q is even, and into $\{\gamma\lambda^{-1}\}$, when q is odd. But the above deformation of λ into λ^{-1} shows that $\{\gamma\lambda^{-1}\}$ is L-S-deformable into $\{\gamma\lambda\}$, and the proof of the lemma is complete.

The succeeding proofs will be simplified if one can suppose that the mappings F of C into M are regular, that is, that the representation of the point $F(z)$ in terms of the parameter θ defining $z=e^{\theta i}$ has a form

$$F(z) = [a_1(\theta), a_2(\theta), a_3(\theta)] \tag{5.3}$$

in which a_i ($i=1, 2, 3$) has a continuous derivature $\dot{a}_i$ and

$$\dot{a}_1^2(\theta) + \dot{a}_2^2(\theta) + \dot{a}_3^2(\theta) \neq 0. \tag{5.4}$$

This and more is needed, and is supplied by the following lemma.

LEMMA 5.2. *Let e be a positive constant. Any R-invariant L-S-closed p-curve F on M admits an R-invariant L-S-deformation on M into a p-curve F' on M with no point $F(z)$ thereby displaced a distance more* *than e, and with F' regular.*

The method of proof of this lemma is entirely similar to the methods used in proving Theorems 28.2 and 28.3 of Morse [2], except for the conditions of R-invariance of the p-curves used. Disregarding this condition for the moment recall that the component deformations used in Morse [2] are local in character, involving among other procedures the use of conformal transformations. All this is essentially the same on the sphere M. Short straight arcs used in the plane are here replaced by short geodesics on M. If the successive local deformations D are applied to sufficiently restricted arcs h on M, it will be possible to accompany each D by a simultaneous deformation of the reflection h' of h in the origin through a reflection of the deforms h^t of h under D. In this way the resultant deformations will be made R-invariant as required.

The order $Q(F_1, E)$. We shall refer to the given system of coordinates (x_1, x_2, x_3) on M as the system E. The points

$$Z_1 = (0, 0, 1), \qquad Z_{-1} = (0, 0, -1)$$

will be called the *poles* of E. A p-curve or arc on M whose carrier does not intersect the poles of E will be termed *E-pole free.* A p-curve on Π will be termed E-pole free if no point of its carrier is covered by a pole of E on M. Let F be an R-invariant closed p-curve of M which is E-pole free. With $F(z)$ of the form

$$F(z) = [x_1(z), x_2(z), x_3(z)] \qquad (z \in C)$$

we set

$$Q(F_1, E) = \underset{C_1}{\text{variation}} \left[\frac{\operatorname{arc} x_1(z) + i x_2(z)}{\pi} \right] \tag{5.5}$$

as z traverses C_1 from $z=1$ to $z=-1$. Observe that $x_1(z)$ and $x_2(z)$ do not vanish simultaneously since F is E-pole free. Thus $Q(F_1, E)$ is well defined. Moreover $Q(F_1, E)$ is an odd integer since

$$x_1(-1) = -x_1(1), \qquad x_2(-1) = -x_2(1).$$

We define $Q(F, E)$ similarly with C_1 replaced by C in (5.5), and obobserve that $Q(F, E) = 2Q(F_1, E)$.

The following lemma is of the nature of a procedural simplification.

LEMMA 5.3. *Let F be an R-invariant closed p-curve on M. The order $Q(F_1, E)$ is an invariant of any R-invariant deformation of F on M in which the deforms F^t of F remain E-pole free. The R-invariant L-S-homotopy class of F contains p-curves F^* such that*

$$Q(F^*, E) = 2. \tag{5.6}$$

The first affirmation of the lemma is immediately clear. In establishing the concluding statement of the lemma no generality will be lost if F is assumed regular.

We shall deform an arc of F_1 over one of the poles of E. More definitely we start with an open simple arc g of F_1 and deform the middle third g_1 of g, leaving the carrier of the residue of g invariant in order that the deforms g^t of g may cause no failure of F_1^t to remain L-S, apart from a failure of g^t itself to remain L-S. We deform g_1 through tongue shaped curves g_1^t with two end points fixed on g, and with semi-circular tips τ^t. We suppose τ^t moves across $(0, 0, -1)$ so that at the moment of crossing $(0, 0, -1)$ is at the mid point of τ^t. By virtue of such a crossing $Q(F_1^t, E)$ will change by 2 or -2 according as the sense of τ^t just after the moment t_0 of crossing is or is not the sense in which arc (x_1+ix_2) increases on τ^t. By an appropriate deformation in which the tongue remains L-S, either case can be made to happen. It should be observed that the tongue can be made self-intersecting provided it remains L-S. Since any finite number of such tongues can be used, it is clear that (5.6) can be made to hold provided the deformation of F_1 through the above p-arcs F_1^t be converted into an R-invariant L-S deformation of F by deforming the kernel residue F_2 of F through p-arcs F_2^t for which

$$F_2^t(-z) = -F_1^t(z) \qquad [z \in C_1]. \tag{5.7}$$

Canonical p-curves on M. These curves are special p-curves introduced to simplify the proof of Theorem 5.1. Such p-curves are to be regular R-invariant closed p-curves with the following properties:

(I) $F(\pm 1) = (\pm 1, 0, 0)$.

(II) *The positive tangent to the path of F at the points corresponding to $z = \pm 1$ on C shall have the direction cosines* $(0, \pm 1, 0)$ *respectively.*

(III) *The p-curve F shall be E-pole free.*

(IV) *The order* $Q(F, E) = 2$.

It follows from Lemmas 5.2 and 5.3 that there exists a canonical p-curve F in the R-invariant L-S-homotopy class of any given R-invariant L-S-p-curve. Cf. proof of Lemma 7.1. We term a kernel F_1 of a canonical p-curve F, a *canonical kernel* F_1. Canonical kernels lie on the open sub-manifold of M

$$M_1 = M - Z_1 - Z_{-1} \qquad [Z_{\pm 1} = (0, 0, \pm 1)].$$

We shall make several uses of the following mapping.

A mapping W of M_1 into a complex w-plane. Under this mapping x in M_1 has an image $w = W(x)$ in the w-plane where

$$W(x) = \exp\,[x_3 + 2i \operatorname{arc}\,(x_1 + ix_2)] \qquad (x \in M_1). \tag{5.8}$$

This mapping can be equivalently given in the form

$$|\,w\,| = \exp\,[x_3], \qquad \operatorname{arc} w = 2 \operatorname{arc}\,(x_1 + ix_2). \tag{5.9}$$

The mapping W is single-valued and continuous, and carries M_1 into a ring in the w-plane on which

$$e^{-1} < |\,u\,| < e. \tag{5.10}$$

Each point w in this ring has just two distinct points on M_1 of the form

$$(a_1, a_2, a_3) \qquad (-\,a_1, -\,a_2, a_3) \quad [(a_1, a_2) \neq (0, 0)]$$

as antecedents. The mapping W is locally top. The inverse W^{-1} is single-valued on a two-sheeted Riemann surface covering the ring (5.10) twice without branch points.

Canonical p-curves in the w-plane. If F_1 is a canonical kernel on M, there exists a unique regular, closed p-curve Ω mapping the circle C into the ring (5.10) on the w-plane, and such that

$$\Omega(z) = WF_1\mu(z) \qquad [z \in C]. \tag{5.11}$$

Such a p-curve has the following properties, paralleling the properties I–IV of canonical p-curves on M.

(I′) $\Omega(1) = 1$.

(II′) *The positive tangent to the path of Ω at the point corresponding to $z = 1$ is parallel to the positive v-axis $(u + iv = w)$.*

(III′) *The carrier of Ω is on the ring* (5.10).

(IV′) *The ordinary plane order of Ω with respect to $w = 0$ is* 1.
Conversely any closed regular p-curve Ω in the w-plane which is canonical in the above sense determines a unique canonical kernel F_1 on M such that (5.11) holds. We then term F the μ-antecedent on M of Ω in the w-plane. Any L-S-deformation of a canonical kernel on M_1 or p-curve on the ring (5.10) through such curves will be called *canonical.*

Let Ω then be a canonical p-curve Ω on the ring (5.10) and F its canonical μ-antecedent on M. Any canonical L-S-deformation Ω^t, $0 \leqq t \leqq 1$, of Ω on the ring (5.10) implies a canonical L-S-deformation F_1^t, $0 \leqq t \leqq 1$, of F_1 on M such that

$$\Omega^t = WF_1^t\mu.$$

Such deformations of Ω on the ring (5.10) are 0-deformations in the sense of Theorem 33.1 of Morse [2]. Since the ordinary plane order q of Ω is 1 the proof[4] of Lemma 33.1 and Theorem 33.1 shows that Ω admits a canonical L-S-deformation on the ring (5.10) into a p-curve Ω^1 whose μ-antecedent on M has the kernel $\{\gamma\}$ or $\{\gamma\lambda^{p-1}\}$ according as the angular order (cf. Morse [2]) p of Ω is 1 or not 1. Lemma 5.1 thus permits the following conclusion.

LEMMA 5.4. *A canonical kernel* F_1 *on* M *admits a canonical L-S-deformation on* M *into* $\{\gamma\}$ *or* $\{\gamma\lambda\}$.

Observe that $\{\gamma\}$ is the kernel Γ_1 of the R-invariant p-curve Γ on M defined at the end of §4. Recall that $A\Gamma\mu = k^{(1)}$. Observe further that the circle λ can be deformed through circles which remain tangent to γ at (0, 1, 0) into a great circle C' on which $x_3 = 0$, so that $\{\gamma\lambda\}$ is L-S-deformable among canonical kernels into $\{\gamma C'\}$. This is the kernel of $\Gamma^{(3)}$. Recall that $A\Gamma^{(3)}\mu = k^{(3)}$. We are thus led to the basic theorem.

THEOREM 5.1. *Any L-S-closed p-curve* f *not* ≈ 0 *on* Π *is in the L-S-homotopy class of* $k^{(1)}$ *or* $k^{(3)}$.

We have merely to review the various steps which lead to this result. In the first place the given f has an R-invariant closed p-curve F as a μ-antecedent on M. Cf. Lemma 4.2. Such an F admits an R-invariant L-S-deformation into a canonical p-curve F^*. Cf. Lemmas 5.2 and 5.3, and the proof of Lemma 7.1.

The kernel F_1^* admits a L-S-deformation through canonical kernels F_1^{*t} into $\{\gamma\}$ or $\{\gamma\lambda\}$ in accordance with Lemma 5.4, and hence into the canonical kernel of Γ or $\Gamma^{(3)}$. On extending these canonical kernels on M by reflection as in (5.7) we infer that F^* admits a L-S-deformation on M through R-invariant p-curves F^{*t} into Γ or $\Gamma^{(3)}$. The p-curves $AF_1^{*t}\mu$ on Π are closed and L-S, and deform $AF_1^*\mu$ into $k^{(1)}$ or $k^{(3)}$. In résumé, $f = AF_1\mu$ is first L-S-deformed on Π into $AF_1^*\mu$ and then into $k^{(1)}$ or $k^{(3)}$.

This completes the proof of the theorem.

It remains to show that $k^{(1)}$, $k^{(2)}$, $k^{(3)}$, $k^{(4)}$ are in distinct L-S-homotopy classes on Π. Part of this result is already clear. For the property of a p-curve f being null homotop. on Π is invariant of arbitrary continuous deformations of f on Π and in particular invariant of L-S-deformations. Thus the null homotop. p-curves $k^{(2)}$ and $k^{(4)}$

---

[4] In the proof of Lemma 33.1 suppose that a line element E of g at Q is tangent to a circle C with center at $w=0$. One can hold E fast in the deformation. A preliminary L-S-deformation should be used to make g convex towards the origin near Q. One then proceeds as before identifying Q with the point $s=a$ of the proof.

are not in the L-S-homotopy classes of $k^{(1)}$ and $k^{(3)}$. Moreover $[k^{(2)}] \neq [k^{(4)}]$ as affirmed in Theorem 4.2. We must finally show that

$$[k^{(3)}] \neq [k^{(1)}]$$

and going somewhat deeper characterize the classes $[k^{(3)}]$ and $[k^{(1)}]$ topologically.

In §6 a numeral invariant $d(f, E)$ of a L-S-homotopy class $[f]$ is defined in case f not ≈ 0 and f is E-pole free. In §7, $d(f, E)$ is replaced by a topological invariant $d_S(f)$ defined for an arbitrary top. image S of the projective plane, thereby freeing $d(f, E)$ from its dependence on the special coordinate system E and the special representation Π of a projective plane.

6. The difference order $d(f, E)$ when f not ≈ 0. Let F be an R-invariant closed p-curve on M. In the case in which F is E-pole free an angular order $P(F_1, E)$ of the kernel F_1 will be defined. For this purpose it is necessary that M receive an orientation from E.

The E-orientation of M. Corresponding to the coordinate system E of M, M will be oriented as follows. Let $C(x)$ be an arbitrarily small circle on M with center at x in M. As previously, let $Z_{\pm 1} = (0, 0, \pm 1)$. The positive sense of $C(Z_{-1})$ shall be such that a continuous branch of the multiple-valued function

$$\text{arc } (y_1 + iy_2) \qquad [y = (y_1, y_2, y_3)]$$

increases as y traces $C(Z_{-1})$ in its positive sense. The sense of $C(x)$ at other points x in M will be obtained by a continuous variation of $C(x)$ from $C(Z_{-1})$. In particular it should be noted that as $C(Z_1)$ is traced in its positive sense by a point y any continuous branch of arc $(y_1 + iy_2)$ decreases.

Reference directions for the measurement of angles at a point x of $M_1 = M - Z_1 - Z_{-1}$ must be defined. For each x in M_1 let $C^1(x)$ be the circle through x parallel to the plane on which $x_3 = 0$. Let the positive sense of $C^1(x)$ be that of increasing arc $(y_1 + iy_2)$ for y in $C^1(x)$. The reference direction at x shall be the positive tangent to $C^1(x)$ at x. The sign of an angle at x measured from the reference direction will be determined by the orientation of M at x as defined by $C(x)$.

The angular order $P(F_1, E)$. Let F be R-invariant L-S, closed and E-pole free. Set $f = AF_1\mu$. Let ϵ_1 be a positive constant so small that the submappings of F on which $z = e^{i\theta}$ with $\alpha \leqq \theta \leqq \alpha + \epsilon_1$ are top. mappings for each constant α. Given $z = e^{i\theta}$ let z_ϵ denote the point $e^{(\theta+\epsilon)i}$. We suppose that $0 < \epsilon < \epsilon_1$. Let $H_F(z, \epsilon)$ denote the angle at the point $x = F(z)$ in M, measured from the reference direction at x to the positive tangent at x to the great circle on M leading from $F(z)$ to

$F(z_e)$. For fixed ϵ let $H_F(z, \epsilon)$ be chosen so as to vary continuously with z in $\bar{C}_1$. Since $F(-z) = -F(z)$ it is clear that

$$H_F(-1, \epsilon) = -H_F(1, \epsilon) \pmod{2\pi} \tag{6.1}$$

so that

$$H_F(-1, \epsilon) + H_F(1, \epsilon) = 2r\pi \tag{6.2}$$

where r is an integer. We set

$$P(F_1, E) = \frac{H_F(-1, \epsilon) + H_F(1, \epsilon)}{\pi} \tag{6.3}$$

and note the following.

The value of $P(F_1, E)$ is independent mod 4 of the choice of ϵ in $(0, \epsilon_1)$, of the choice of F between the two μ-antecedents of f, and of the choice of H_F among the possible continuous branches of this angle function. For $H_F(z, \epsilon)$ can be chosen as to vary continuously with (z, ϵ) for ϵ in $(0, \epsilon_1)$ and z in $\bar{C}_1$, so that the left member of (6.2) is independent of ϵ in $(0, \epsilon_1)$. If F and F^* are the two μ-antecedents of f, $F(z) = -F^*(z)$, so that one can take

$$H_{F^*}(z, \epsilon) = -H_F(z, \epsilon).$$

Hence

$$P(F_1^*, E) = -P(F_1, E) = P(F_1, E) \pmod{4}.$$

Finally a change of the continuous branch of H_F will change the left member of (6.2) by an integral multiple of 4π and so leave $P(F_1, E)$ unchanged mod 4.

The difference order $d(f, E)$. Let f not ≈ 0 be a L-S-closed p-curve on Π which is E-pole free. Let F be a μ-antecedent on M of f on Π. We set

$$d(f, E) \equiv Q(F_1, E) - P(F_1, E) \pmod{4}$$

and observe that $d(f, E)$ is independent of the choice of F as μ-antecedent of f, and of any R-invarient L-S-deformation of F on M_1.

One sees that

$$\begin{aligned} d(k^{(1)}, E) &\equiv Q(\Gamma_1, E) - P(\Gamma_1, E) = 1 - 0 \pmod{4}, \\ d(k^{(3)}, E) &\equiv Q(\Gamma_1^{(3)}, E) - P(\Gamma_1^{(3)}, E) = 3 - 0 \pmod{4}. \end{aligned} \tag{6.4}$$

An immediate conclusion is that $k^{(1)}$ admits no L-S-deformation on Π into $k^{(3)}$ through p-curves which are E-pole free. To remove the latter condition, deformations must be made through the poles of E and the effect on $d(f, E)$ determined. For this purpose p-curves on Π whose μ-antecedents on M are broken geodesics are useful.

Admissible broken geodesics on M. In Morse and Heins [1] use has been made of L-S-curves composed of sequences of a finite number of straight arcs. The analogous p-curves on M are sequences of a finite number of geodesic arcs each less than π in length, with nonzero angles at the vertices (the junction points of successive arcs). A p-curve on M of this character will be called an admissible broken geodesic. Admissible broken geodesics are L-S. A deformation on M of an R-invariant closed p-curve F through admissible broken geodesics F^t will be termed *admissible* if the number of vertices is independent of t, if the vertices vary continuously with t and remain distinct on any p-curve F^t, if the point $F^t(1)$ is a vertex of F^t, and if each p-curve F^t is R-invariant. The methods of Morse and Heins [1] suffice to prove the following lemma.

LEMMA 6.1. *Let e be a positive constant. Any R-invariant L-S-p-curve F on M admits an R-invariant L-S-deformation into an admissible broken geodesic displacing each point $F(z)$ on M at most e in this process.*

Any two R-invariant broken geodesic closed p-curves F and F' which are in the same R-invariant L-S-homotopy class, can be admissibly deformed on M into each other through R-invariant broken geodesics, provided a suitable number of vertices are initially added to F and F'.

With this lemma as an aid, the following theorem can be proved:

THEOREM 6.1. *If f not ≈ 0 and f' not ≈ 0 are two closed L-S-p-curves on Π in the same homotopy class on Π and if f and f' are E-pole free, then*

$$d(f, E) = d(f', E) \pmod 4. \tag{6.5}$$

The theorem follows at once from the definition of $d(f, E)$ if f can be L-S-deformed into f' on Π through p-curves which are E-pole free. In any other case we can suppose, without loss of generality, that the μ-antecedents F and F' on M, of f and f' respectively on Π, are admissible broken geodesics which are E-pole free. In accordance with Lemma 6.1, F can be admissibly deformed into F' through a family F^t of broken geodesics. If use is made of the freedom of small displacements of the vertices of F^t, we can be assured that F^t is E-pole free except for a finite set of values $t_1, \cdots, t_n$ of t, that no vertex of F^{t_i} $(i=1, \cdots, n)$ is at a pole of E, and that F^{t_i} has just one point in common with the poles of E.

The conventions as to the measurement of angles are such that as a geodesic arc of F_1^t moves across the pole $(0, 0, 1)$

$$\Delta P(F_1^t, E) = \Delta Q(F_1^t, E) = \pm 2$$

so that $d(f^t, E)$ is unchanged by such a passage. When a geodesic arc of F_1^t moves across the pole $(0, 0, -1)$

$$\Delta P(F_1^t, E) = -\Delta Q(F_1^t, E) = \pm 2.$$

The difference $d(f^t, E)$ mod 4 is accordingly invariant as t increases from 0 to 1. This completes the proof of the theorem.

Reference to (6.4) gives the following corollary of the theorem.

COROLLARY 6.1. *The models* $k^{(1)}$ *and* $k^{(3)}$ *on* Π *are not in the same L-S-homotopy class on* Π.

By virtue of Theorems 5.1 and 6.1 any L-S-p-curve f not ≈ 0 on Π which is E-pole free is in the L-S-homotopy class of $k^{(1)}$ or $k^{(3)}$ according as $d(f, E) = 1$ or 3 mod 4. This determination of the L-S-homotopy class of f depends upon our special model Π of the projective plane and upon the coordinate system E. We shall remove this dependency.

7. **Invariant orders and models.** We begin with the following lemma:

LEMMA 7.1. *Any simple closed* p*-curve* f *not* ≈ 0 *on* Π *can be deformed on* Π *into any other such* p*-curve on* Π *through simple, closed* p*-curves.*

It will be sufficient to show that f can be deformed into $k^{(1)}$ in the manner required. If F is a μ-antecedent of f it will be sufficient to show that F can be deformed on M into Γ through R-invariant simple, closed p-curves on M. The required deformation will be defined as a sequence of five deformations.

(1) We first deform F in the required manner into an R-invariant, simple, regular closed p-curve $F^{(1)}$. With obvious precautions to maintain a simple curve the proof of Lemma 5.2 will suffice.

(2) We next rotate M in such a manner that $F^{(1)}$ is deformed into a p-curve $F^{(2)}$ for which $F^{(2)}(1) = 1$.

(3) A suitable rotation of M about the x_1 axis will then carry $F^{(2)}$ into a p-curve $F^{(3)}$ which is tangent to Γ at the point $(1, 0, 0)$.

(4) If $F_1^{(3)}$ is not E-pole free its kernel $F_1^{(3)}$, with its simple projection on Π, intersects $(0, 0, 1)$ in a point $F_1^{(3)}(z)$ for just one value of z on C_1. A suitable L-S-deformation of $F_1^{(3)}$ near this point of intersection and a corresponding R-invariant L-S-deformation of $F^{(3)}$ will yield a simple closed p-curve $F^{(4)}$ which is E-pole free. Moreover

$$Q(F^{(4)}, E) = \pm 2 \tag{7.0}$$

as one sees on projecting M_1 stereographically from $(0, 0, 1)$ onto the plane tangent to M at $(0, 0, -1)$. Finally we can suppose that the $+$ sign holds in (7.0). For if one keeps $F_1^{(4)}$ simple and regular and deforms a tongue once over $(0, 0, 1)$, the order (7.0) of $F^{(4)}$ will be changed from -2 to 2, if initially -2.

(5) The resultant p-curve $F^{(4)}$ is canonical in the sense of §5. Use can be made of the mapping W of M_1 into the ring (5.10) of the w-plane. On this ring there exists a canonical p-curve Ω such that

$$\Omega = WF_1^{(4)}\mu.$$

In particular Ω has the plane order 1 relative to $w=0$ in the w-plane. It follows that there exists a deformation of Ω on the ring (5.10) through simple, canonical p-curves Ω^t, $0 \leqq t \leqq 1$, on the ring into the p-curve $w=z=e^{\theta i}$. On M, F can accordingly be L-S-deformed through simple canonical p-curves into Γ.

Hence f can be deformed on Π in the manner required into $k^{(1)}$.

Various methods (including conformal mapping) are available to prove the following.

(i) Let g^t, $0 \leqq t \leqq 1$, be a 1-parameter family of simple closed p-curves in the (u, v)-plane of which g^1 is the circle $C: z = e^{i\theta}$. Let G^t be the closure of the interior of g^t. There exists a continuous 1-parameter family of top. mappings T^t of G^t into the closed disc bounded by C, such that T^t maps $g^t(z)$ into z and G^1 is the identity.

Recall that an *isotopic* deformation of a manifold S is defined by a continuous 1-parameter family of top. mappings of S onto S. With this understood we state the following lemma. In proving this lemma it will be convenient to denote the carrier of a p-curve F by $|F|$.

LEMMA 7.2. *Any homeomorphism H of Π can be isotopically deformed into the identity on Π.*

Let K be an R-invariant homeomorphism of M such that $AK=H$. Set $F=K^{-1}\Gamma$. By virtue of Lemma 7.1 there exists a continuous 1-parameter family of R-invariant, simple, closed p-curves F^t on M which deform F into Γ. Let Σ^t, $0 \leqq t \leqq 1$, be a continuous 1-parameter family of closed domains on M bounded by the respective Jordan curves $|F^t|$. Observe that Σ^1 is a hemisphere of M bounded by $|\Gamma|$. It follows from (i) that there exists a continuous 1-parameter family of top. mappings T^t of Σ^t onto the hemisphere Σ^1, which, in particular, map the Jordan curve $|F^t|$ onto the circle $|\Gamma|$ in such a manner that $\Gamma(z)$ is the image of $F^t(z)$ and T^1 is the identity. The mappings T^t can be extended over M by reflection, that is, so that

$$T^t(-x) = -T^t(x) \qquad (X, t) \in (M \times J).$$

So extended T^t, $0 \leqq t \leqq 1$, defines an isotopic deformation of T^0 into the identity T^1.

It remains to deform K isotopically into T^0. By definition $KF=\Gamma$ so that $K(x)=T^0(x)$ when x is in the Jordan curve $|F|$. By a theorem of Tietsze there is an isotopic deformation of the mapping K restricted to Σ^0, into T^0, likewise restricted to Σ^0, leaving $|F|$ pointwise fixed. This deformation can be extended to all of K by reflection in the origin, so as to yield an R-invariant isotopic deformation of K into T^0. Hence K is isotopically deformable into the identity through R-invariant top. mappings of M onto M.

The lemma follows.

Proof of Theorem 1.1 *of the introduction.* The p-curve k whose multiple tracings $k^{(1)}$, $k^{(2)}$, $k^{(3)}$, $k^{(4)}$ appear in Theorems 4.2 and 5.1 is a simple, closed p-curve on Π with k not≈ 0 on Π. It follows from Lemma 7.1 that in these theorems k can be replaced by any other simple, closed p-curve h such that h not≈ 0. This completes the proof of the fundamental Theorem 1.1.

Further invariance of $d(f, E)$. We now admit any coordinate system E' obtained from E by a rotation of E about the origin, or by a reflection of E' in the origin. We have seen that $d(f, E)$ is independent of the L-S-deformation class of f provided only that $d(f, E)$ is well defined, that is, provided that f is E-pole free. The following theorem shows the essential top. invariance of $d(f, E)$.

THEOREM 7.1. *Let f be a L-S-closed p-curve on Π, H a homeomorphism of Π and $f'=Hf$ the transform of f under H. If E and E' are admissible coordinate systems such that f and f' are respectively E and E'-pole free, then*

$$d(f, E) = d(f', E'). \tag{7.1}$$

We shall first show that

$$d(f, E) = d(Hf, E) \tag{7.2}$$

provided f and Hf are E-pole free. Relation (7.2) follows from Lemma 7.2 according to which H can be isotopically deformed into the identity, thus implying a L-S-deformation of Hf into f. From the invariance of $d(f, E)$ under such deformations of f, (7.2) must hold.

We shall next show that

$$d(f, E) = d(f, E') \tag{7.3}$$

provided f is E and E'-pole free. To that end let T be the orthogonal transformation by virtue of which $E'=TE$. It is trivial that

$$d(f, E) = d(Tf, TE). \tag{7.4}$$

But f and f' are E'-pole free so that

$$d(Tf, TE) = d(f, TE) = d(f, E')$$

according to (7.2). Hence (7.3) holds.

To establish (7.1) let E'' be chosen (as is possible) so that f and f' are E''-pole free. By hypothesis f is E-pole free, and f' is E'-pole free. Hence

$$d(f, E) = d(f, E'') = d(f', E'') = d(f', E')$$

in accordance with (7.2) and (7.3). This completes the proof of the theorem.

Definition of an invariant S-order of f when f not≈ 0. Let S be an arbitrary top. model of the projective plane, and f an L-S-closed p-curve on S with f not≈ 0 on S. Then $d(Zf, E)$ is independent mod 4 of the choice of Z among top. mappings of S onto Π and of the choice of E among admissible rectangular coordinate systems for M provided Zf is E-pole free.

For each L-S-closed p-curve of not≈ 0 *on S we set*

$$d(Zf, E) = d_S(f) \qquad (\text{mod } 4)$$

provided Zf is E-pole free, and term $d_S(f)$ *the S-difference order of f.*

The fundamental nature of the top. invariance of $d_S(f)$ is specified in the following theorem.

THEOREM 7.2. *The S-difference order* $d_S(f)$ *of a L-S-closed p-curve f not*≈ 0 *on S is independent of the choice of f in its L-S-homotopy class. If S is mapped top. onto S' under a mapping K and if* $f' = Kf$, *then*

$$d_S(f) = d_{S'}(f') \qquad (\text{mod } 4). \tag{7.5}$$

The difference order $d_S(f)$ *has but two possible values* 1 *and* 3, mod 4. *A necessary and sufficient condition that* $d_S(f) = 1$ mod 4 *is that the L-S-homotopy class of f contain a simple, closed p-curve* f_1 *not*≈ 0 *on S.*

Let Z and Z' be arbitrary top. mappings of S and S' respectively onto Π. Then by definition

$$d_S(f) = d(Zf, E), \qquad d_{S'}(f') = d(Z'f', E')$$

provided E and E' are admissible coordinate systems for M such that Zf and $Z'f'$ are respectively E and E'-pole free. Observe that

$$Z'f' = (Z'KZ^{-1})(Zf)$$

and that the transformation

$$Z'KZ^{-1} = H,$$

is a top. mapping of Π onto Π. Hence

$$d(Zf, E) = d(Z'f', E')$$

in accordance with (7.1). Thus (7.5) holds.

The first statement in the theorem is a consequence of Theorem 6.1.

To establish the last statement in the theorem suppose first that $d_S(f)=1$. Recall that $[f]=[k^{(1)}]$ or $[k^{(3)}]$ by Theorem 5.1. But we have seen in (6.4) that

$$d(k^{(1)}, E) = 1, \qquad d(k^{(3)}, E) = 3 \qquad \pmod 4$$

so that $[f]=[k^{(1)}]$. Thus $k^{(1)}$ is a simple closed p-curve in $[f]$ as affirmed.

Conversely, suppose that $[f]$ contains a simple, closed p-curve f_1. Recall that $[f_1]=[k^{(1)}]$ as a consequence of Lemma 7.1. Hence

$$d_S(f) = d(k^{(1)}, E) = 1 \qquad \pmod 4.$$

This completes the proof of the theorem.

The part of Theorem 1.1 which concerns the case f not ≈ 0 can be completed as follows.

THEOREM 7.3. *If f not ≈ 0 is a L-S-closed p-curve on the top. image S of a projective plane $[f]=[k^{(1)}]$ or $[k^{(3)}]$ according as $d_S(f)=1$ or 3 mod 4.*

REFERENCES

HEINZ HOPF

Ueber die Drehung der Tangenten und Lehnen ebener Kurven, Compositio Math. vol. 2 (1935) pp. 50–62.

H. WHITNEY

On regular closed curves in the plane, Compositio Math. vol. 4 (1937) pp. 276–284.

M. MORSE

1. *L-S-homotopy classes of locally simple curves*, Annales de la Société Polonaise de Mathématique vol. 21 (1948) pp. 236–256.
2. *Topological methods in the theory of functions of a complex variable*, Princeton University Press, 1947.

M. MORSE AND M. HEINS

1. *Topological methods in the theory of functions of a single complex variable*, I. *Deformation types of locally simple curves*, Ann. of Math. vol. 46 (1945) pp. 600–624.
2. *Deformation classes of meromorphic functions and their extensions to interior transformations*, Acta Math. vol. 79 (1946) pp. 51–103.

INSTITUTE FOR ADVANCED STUDY

TABLE OF THE ZEROS AND WEIGHT FACTORS OF THE FIRST FIFTEEN LAGUERRE POLYNOMIALS

HERBERT E. SALZER AND RUTH ZUCKER

The chief use of the present table of zeros and weight factors of the Laguerre polynomials is in the performance of quadratures over the interval $[0, \infty]$, when the integrand behaves like the product of e^{-x} by a polynomial. From the well known theory of orthogonal polynomials, the quadrature formula is exact for any polynomial $P(x)$ up to the $(2n-1)$th degree, provided one uses the n-point quadrature formula. Thus if $\alpha_i^{(n)}$ denotes the "weight factors" or "Christoffel numbers," corresponding to $L_n(x)$, the nth Laguerre polynomial, and $x_i^{(n)}$ denotes the zeros of $L_n(x)$, then

$$\int_0^\infty e^{-x}P(x)dx = \sum_{i=1}^{n} \alpha_i^{(n)} P(x_i^{(n)}). \tag{1}$$

Besides problems involving direct quadratures, there are those arising in the numerical solution of linear integral equations, range $[0, \infty]$, where the unknown function occurs both inside and outside the integral sign. By considering the product of e^x by the integrand as a polynomial, and making use of (1), the approximation problem reduces to the solution of a set of only n linear equations. Hence only n points are needed to give accuracy obtainable by approximating the product of e^x by the integrand as a polynomial of the $(2n-1)$th degree. For a full description, including examples, see A. Reiz [3],[1] especially pp. 1–12.

For many purposes, the report of the Admiralty Computing Service [2], which furnishes zeros to 8 decimals and weight factors to 8 significant figures as far as $L_{10}(x)$, will suffice. This present table is intended to cope with problems requiring higher degree and accuracy. Thus there are given here the zeros and weight factors of the first fifteen Laguerre polynomials, the zeros to 12 decimals and the weight factors to 12 significant figures. (The zeros and weight factors are available in manuscript form to two extra places for $n \leqq 10$, and to one extra place for $10 < n \leqq 15$.) Also the example of A. Reiz is followed in that the quantities $\alpha_i^{(n)}e^{x_i^{(n)}}$ are also tabulated to 12 significant figures, to facilitate the quadratures when the integrand does not contain e^{-x} explicitly. Thus in

---

Received by the editors July 19, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

$$\int_0^\infty F(x)dx = \int_0^\infty e^{-x}[e^{x}F(x)]dx \sim \alpha_i^{(n)} e^{x_i^{(n)}} F(x_i^{(n)}),$$

having $\alpha_i^{(n)}e^{x_i^{(n)}}$ saves the labor of computing $e^{x_i^{(n)}}$ and then multiplying $e^{x_i^{(n)}}$ by either $\alpha_i^{(n)}$ or $F(x_i^{(n)})$. In addition, the exact values of the coefficients of the first twenty Laguerre polynomials have been computed, because of their fundamental importance.[2] They were calculated from the recursion formula (6) below, which was used to obtain $C_m^{(n+1)}$, the coefficient of x^m in $L_{n+1}(x)$, by the relation

$$C_m^{(n+1)} = (1+2n)C_m^{(n)} - C_{m-1}^{(n)} - n^2 C_m^{(n-1)},$$

and as a check, it was verified that

$$C_m^{(n)} = (-1)^m(n-m)!\,{}_nC_{n-m}^2.$$

The Laguerre polynomial $L_n(x)$ is defined as

$$L_n(x) = e^x\left[\frac{d^n}{dx^n}(e^{-x}x^n)\right], \tag{2}$$

from which

$$L_n(x) = (-1)^n\left(x^n - \frac{n^2}{1!}x^{n-1} + \frac{n^2(n-1)^2}{2!}x^{n-2} - \cdots\right). \tag{3}$$

The polynomial $L_n(x)$ satisfied the orthogonality relations

$$\int_0^\infty e^{-x}x^iL_n(x)dx = 0, \qquad i = 0, 1, \cdots, n-1. \tag{4}$$

Also, $L_n(x)$ satisfies the differential equation

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0, \tag{5}$$

and the recursion formula

$$L_{n+1}(x) - (1 + 2n - x)L_n(x) + n^2L_{n-1}(x) = 0. \tag{6}$$

The weight factors $\alpha_i^{(n)}$ are given by

$$\alpha_i^{(n)} = \frac{1}{L_n'(x_i^{(n)})}\int_0^\infty \frac{e^{-x}L_n(x)}{x - x_i^{(n)}}dx. \tag{7}$$

The calculation of $\alpha_i^{(n)}$ is facilitated by the following relation (given in the Admiralty Report [2] and A. Reiz [3], the latter crediting it to J. Deruyts [6]):

---

[2] These coefficients are available in manuscript form at the Computation Laboratory.

$$\int_0^\infty \frac{e^{-x}L_n(x)}{x - x_i^{(n)}}\,dx = \frac{(n!)^2}{x_i^{(n)}L_n'(x_i^{(n)})}, \tag{8}$$

from which there follows the very convenient formula for $\alpha_i^{(n)}$, namely,

$$\alpha_i^{(n)} = \frac{1}{x_i^{(n)}}\left[\frac{n!}{L_n'(x_i^{(n)})}\right]^2. \tag{9}$$

The zeros of $L_n(x)$ were calculated by Newton's method, from a first approximation. In most cases, for a given n, a first approximation to $x_i^{(n)}$ was obtained by either extrapolation as a function of n for the same value of i, or when i was considerably greater than 1, by extrapolation for fixed n, based upon the previous values of i (or a combination of both these processes). Then the first approximation was refined still further by 1), noting by how much previously obtained roots differed from their values obtained by this method of extrapolation, and 2), using another operation of extrapolation upon those deviations to find the deviation of the first approximation. All roots were checked by substitution into the polynomials $L_n(x)$, and noting how closely they satisfied $L_n(x_i^{(n)}) = 0$. In addition the roots were checked by the relations

$$\sum_{i=1}^{n} x_i^{(n)} = n^2 \tag{10}$$

and

$$\prod_{i=1}^{n} x_i^{(n)} = n!. \tag{11}$$

The first step in finding the weight factors (or Christoffel numbers) was to obtain $L_n'(x_i^{(n)})$. Because the calculations of $L_n'(x_i^{(n)})$ were lengthy, as a check upon $L_n'(x_i^{(n)})$, the second derivatives $L_n''(x_i^{(n)})$ were also computed, and were employed in the checking equation

$$x_i^{(n)}L_n''(x_i^{(n)}) = (x_i^{(n)} - 1)L_n'(x_i^{(n)}). \tag{12}$$

Then the weight factors $\alpha_i^{(n)}$ were calculated from (9) and checked by the relation $(n!)^2 = \alpha_i^{(n)}x_i^{(n)}\{L_n'(x_i^{(n)})\}^2$, and also by substitution into (1), with $P(x) = x^m$, where $m!$ must be obtained. For each $L_n(x)$, m was taken successively as every number in the set 0, 1, 2, 4, 8, 16, and 24 which did not exceed $2n-1$. This wide range of m was necessary in order to be sure of including all the desired significant figures of each $\alpha_i^{(n)}$ in the checking.

The quantities $e^{x_i^{(n)}}$ were obtained by interpolation in the Mathematical Tables Project's *Tables of the exponential function e^x* and checked by interpolation in the Project's *Table of natural logarithms*, vols. III and IV. Finally, to guarantee the quantities $\alpha_i^{(n)}e^{x_i^{(n)}}$, they were checked both by division by $e^{x_i^{(n)}}$, and by duplicate calculation.

Zeros of Laguerre Polynomials	Weight Factors[3]	Weight Factors ×Exponential of Zeros
$n=1$	$n=1$	$n=1$
1.00000 00000 00	1.00000 00000 00	2.71828 18284 6
$n=2$	$n=2$	$n=2$
.58578 64376 27	.85355 33905 93	1.53332 60331 2
3.41421 35623 73	.14644 66094 07	4.45095 73350 5
$n=3$	$n=3$	$n=3$
.41577 45567 83	.71109 30099 29	1.07769 28592 7
2.29428 03602 79	.27851 77335 69	2.76214 29619 0
6.28994 50829 37	.(1) 10389 25650 16	5.60109 46254 3
$n=4$	$n=4$	$n=4$
.32254 76896 19	.60315 41043 42	.83273 91238 38
1.74576 11011 58	.35741 86924 38	2.04810 24384 5
4.53662 02969 21	.(1) 38887 90851 50	3.63114 63058 2
9.39507 09123 01	.(3) 53929 47055 61	6.48714 50844 1
$n=5$	$n=5$	$n=5$
.26356 03197 18	.52175 56105 83	.67909 40422 08
1.41340 30591 07	.39866 68110 83	1.63848 78736 0
3.59642 57710 41	.(1) 75942 44968 17	2.76944 32423 7
7.08581 00058 59	.(2) 36117 58679 92	4.31565 69009 2
12.64080 08442 76	.(4) 23369 97238 58	7.21918 63543 5
$n=6$	$n=6$	$n=6$
.22284 66041 79	.45896 46739 50	.57353 55074 23
1.18893 21016 73	.41700 08307 72	1.36925 25907 1
2.99273 63260 59	.11337 33820 74	2.26068 45933 8
5.77514 35691 05	.(1) 10399 19745 31	3.35052 45823 6

---

[3] The number in the parentheses stands for the number of zeros between the decimal point and the first significant figure.

Zeros of Laguerre Polynomials	Weight Factors[3]	Weight Factors ×Exponential of Zeros
$n=6$	$n=6$	$n=6$
9.83746 74183 83	.(3) 26101 72028 15	4.88682 68002 1
15.98287 39806 02	.(6) 89854 79064 30	7.84901 59456 0
$n=7$	$n=7$	$n=7$
.19304 36765 60	.40931 89517 01	.49647 75975 40
1.02666 48953 39	.42183 12778 62	1.17764 30608 6
2.56787 67449 51	.14712 63486 58	1.91824 97816 6
4.90035 30845 26	.(1) 20633 51446 87	2.77184 86362 3
8.18215 34445 63	.(2) 10740 10143 28	3.84124 91224 9
12.73418 02917 98	.(4) 15865 46434 86	5.38067 82079 2
19.39572 78622 63	.(7) 31703 15479 00	8.40543 24868 3
$n=8$	$n=8$	$n=8$
.17027 96323 05	.36918 85893 42	.43772 34104 93
.90370 17767 99	.41878 67808 14	1.03386 93476 7
2.25108 66298 66	.17579 49866 37	1.66970 97656 6
4.26670 01702 88	.(1) 33343 49226 12	2.37692 47017 6
7.04590 54023 93	.(2) 27945 36235 23	3.20854 09133 5
10.75851 60101 81	.(4) 90765 08773 36	4.26857 55108 3
15.74067 86412 78	.(6) 84857 46716 27	5.81808 33686 7
22.86313 17368 89	.(8) 10480 01174 87	8.90622 62152 9
$n=9$	$n=9$	$n=9$
.15232 22277 32	.33612 64217 98	.39143 11243 16
.80722 00227 42	.41121 39804 24	.92180 50285 29
2.00513 51556 19	.19928 75253 71	1.48012 79099 4
3.78347 39733 31	.(1) 47460 56276 57	2.08677 08075 5
6.20495 67778 77	.(2) 55996 26610 79	2.77292 13897 1
9.37298 52516 88	.(3) 30524 97670 93	3.59162 60680 9
13.46623 69110 92	.(5) 65921 23026 08	4.64876 60021 4
18.83359 77889 92	.(7) 41107 69330 35	6.21227 54197 5
26.37407 18909 27	.(10) 32908 74030 35	9.36321 82377 1
$n=10$	$n=10$	$n=10$
.13779 34705 40	.30844 11157 65	.35400 97386 07
.72945 45495 03	.40111 99291 55	.83190 23010 44

Zeros of Laguerre Polynomials	Weight Factors[3]	Weight Factors ×Exponential of Zeros
$n=10$	$n=10$	$n=10$
1.80834 29017 40	.21806 82876 12	1.33028 85617 5
3.40143 36978 55	.(1) 62087 45609 87	1.86306 39031 1
5.55249 61400 64	.(2) 95015 16975 18	2.45025 55580 8
8.33015 27467 64	.(3) 75300 83885 88	3.12276 41551 4
11.84378 58379 00	.(4) 28259 23349 60	3.93415 26955 6
16.27925 78313 78	.(6) 42493 13984 96	4.99241 48721 9
21.99658 58119 81	.(8) 18395 64823 98	6.57220 24851 3
29.92069 70122 74	.(12) 99118 27219 61	9.78469 58403 7
$n=11$	$n=11$	$n=11$
.12579 64421 88	.28493 32128 94	.32312 88804 35
.66541 82558 39	.38972 08895 28	.75812 55998 10
1.64715 05458 72	.23278 18318 49	1.20864 14229 0
3.09113 81430 35	.(1) 76564 45354 62	1.68457 91671 4
5.02928 44015 80	.(1) 14393 28276 74	2.19963 34786 2
7.50988 78638 07	.(2) 15188 80846 48	2.77348 97457 5
10.60595 09995 47	.(4) 85131 22435 47	3.43712 14165 2
14.43161 37580 64	.(5) 22924 03879 57	4.24484 02908 0
19.17885 74032 15	.(7) 24863 53702 77	5.30682 60194 8
25.21770 93396 78	.(10) 77126 26933 69	6.90421 54788 3
33.49719 28471 76	.(13) 28837 75868 32	10.17671 27469
$n=12$	$n=12$	$n=12$
.11572 21173 58	.26473 13710 55	.29720 96360 44
.61175 74845 15	.37775 92758 73	.69646 29804 31
1.51261 02697 76	.24408 20113 20	1.10778 13946 2
2.83375 13377 44	.(1) 90449 22221 17	1.53846 42390 4
4.59922 76394 18	.(1) 20102 38115 46	1.99832 76062 7
6.84452 54531 15	.(2) 26639 73541 87	2.50074 57691 0
9.62131 68424 57	.(3) 20323 15926 63	3.06532 15182 8
13.00605 49933 06	.(5) 83650 55856 82	3.72328 91107 8
17.11685 51874 62	.(6) 16684 93876 54	4.52981 40299 8
22.15109 03793 97	.(8) 13423 91030 52	5.59725 84618 4
28.48796 72509 84	.(11) 30616 01635 04	7.21299 54609 3
37.09912 10444 67	.(15) 81480 77467 43	10.54383 74619

Zeros of Laguerre Polynomials	Weight Factors[3]	Weight Factors ×Exponential of Zeros
$n=13$	$n=13$	$n=13$
.10714 23884 72	.24718 87084 30	.27514 39554 71
.56613 18990 40	.36568 88229 01	.64413 90765 43
1.39856 43364 51	.25256 24200 58	1.02272 17785 8
2.61659 71084 06	.10347 07580 24	1.41641 76056 6
4.23884 59290 17	.(1) 26432 75441 56	1.83252 46178 6
6.29225 62711 40	.(2) 42203 96040 27	2.28058 03822 9
8.81500 19411 87	.(3) 41188 17704 73	2.77382 63658 6
11.86140 35888 11	.(4) 23515 47398 15	3.33192 99564 3
15.51076 20377 04	.(6) 73173 11620 25	3.98648 29749 2
19.88463 56638 80	.(7) 11088 41625 70	4.79354 78219 3
25.18526 38646 78	.(10) 67708 26692 21	5.86763 13371 1
31.80038 63019 47	.(12) 11599 79959 91	7.50209 88049 2
40.72300 86692 66	.(16) 22450 93203 89	10.88961 00139
$n=14$	$n=14$	$n=14$
.09974 75070 33	.23181 55771 45	.25613 11547 37
.52685 76488 52	.35378 46915 98	.59917 04774 95
1.30062 91212 51	.25873 46102 45	.94997 15024 75
2.43080 10787 31	.11548 28935 57	1.31280 78114 7
3.93210 28222 93	.(1) 33192 09215 93	1.69326 59907 5
5.82553 62183 02	.(2) 61928 69437 01	2.09840 90842 7
8.14024 01415 65	.(3) 73989 03778 67	2.53763 26549 4
10.91649 95073 66	.(4) 54907 19466 84	3.02415 99177 0
14.21080 50111 61	.(5) 24095 85764 09	3.57780 43557 2
18.10489 22202 18	.(7) 58015 43981 68	4.23056 50349 1
22.72338 16282 69	.(9) 68193 14692 49	5.03941 33684 6
28.27298 17232 48	.(11) 32212 07751 89	6.12094 79283 3
35.14944 36605 92	.(14) 42213 52440 52	7.77429 53670 1
44.36608 17111 17	.(18) 60523 75022 29	11.21683 42167
$n=15$	$n=15$	$n=15$
.09330 78120 17	.21823 48859 40	.23957 81703 11
.49269 17403 02	.34221 01779 23	.56010 08427 93
1.21559 54120 71	.26302 75779 42	.88700 82629 19
2.26994 95262 04	.12642 58181 06	1.22366 44021 5
3.66762 27217 51	.(1) 40206 86492 10	1.57444 87216 3
5.42533 66274 14	.(2) 85638 77803 61	1.94475 19765 3

Zeros of Laguerre Polynomials	Weight Factors[8]	Weight Factors ×Exponential of Zeros
$n=15$	$n=15$	$n=15$
7.56591 62266 13	.(2) 12124 36147 21	2.34150 20566 4
10.12022 85680 19	.(3) 11167 43923 44	2.77404 19268 3
13.13028 24821 76	.(5) 64599 26762 02	3.25564 33464 0
16.65440 77083 30	.(6) 22263 16907 10	3.80631 17142 3
20.77647 88994 49	.(8) 42274 30384 98	4.45847 77538 4
25.62389 42267 29	.(10) 39218 97267 04	5.27001 77844 3
31.40751 91697 54	.(12) 14565 15264 07	6.35956 34697 3
38.53068 33064 86	.(15) 14830 27051 11	8.03178 76321 2
48.02608 55726 86	.(19) 16005 94906 21	11.52777 21009

BIBLIOGRAPHY

NOTE: A detailed bibliography on Laguerre polynomials is really unnecessary, since a very exhaustive one up to the year 1940 is given in J. Shohat [1]. However, the few references given below afford a representative sample and cover many essential aspects of Laguerre polynomials.

1. J. A. Shohat, *A bibliography on orthogonal polynomials*, Bulletin of the National Research Council, no. 103, 1940. Lists over 2000 articles, books, and theses, by 643 authors, on the subject of orthogonal polynomials. The topical index has a special section on Laguerre polynomials, where more than 250 references are given.

2. Department of Scientific Research and Experiment, Admiralty Computing Service, *Zeros of Laguerre polynomials and the corresponding Christoffel numbers*, June, 1945.

3. A. Reiz, *On the numerical solution of certain types of integral equations*, Arkiv för Matematik, Astronomi och Fysik, vol. 29A, no. 29.

4. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939. Includes a large bibliography.

5. D. Jackson, *Fourier series and orthogonal polynomials*, Carus Mathematical Monographs, no. 6, Mathematical Association of America, 1941. Chapters VII (general) and X (on Laguerre polynomials).

6. J. Deruyts, *Sur le calcul approché de certaines intégrales définies*, Bulletins de l'Académie Royale de Belgique (3) vol. 11 (1886) pp. 307–311.

7. W. Hahn, *Bericht über die Nullstellen der Laguerreschen und der Hermiteschen Polynome*, Jber. Deutschen Math. Verein. vol. 44 (1934) pp. 215–236. An elaborate bibliography is included in this paper.

8. E. Moecklin, *Asymptotische Entwicklungen der Laguerreschen Polynome*, Comment. Math. Helv. vol. 7 (1934) pp. 24–46.

9. F. Tricomi, *Generalizzazione di una formula asintotica sui polinomi di Laguerre e sue applicazioni*, Atti Accad. Sci. Torino vol. 76 (1941) pp. 288–316.

10. E. C. Kemble, *The fundamental principles of quantum mechanics*, McGraw-Hill, 1937, pp. 160, 585–587.

11. J. Cossar and A. Erdélyi, *Dictionary of Laplace transforms*, Part 2B (issued by Dept. of Scientific Research and Experiment, Admiralty Computing Service), pp. 84–85.

12. H. Margenau and G. M. Murphy, *The mathematics of physics and chemistry*, New York, Van Nostrand, 1943, pp. 77–78, 122–128, 350.

13. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. I, Berlin, Springer, 1931, pp. 79–80, 284–285, 294–296, 324–326, 440–441. (Reprinted by Interscience Publishers, 1943.)

14. N. Košliakov, *The calculation by a formula of mechanical quadrature of definite integrals with infinite limits* (Russian), Bull. Acad. Sci. URSS. (7) (1933) pp. 801–808.

Of the above references, [2], [3], and [14] include zeros and Christoffel numbers. Those in [2] agree exactly with the table above as far as comparison can be made. The zeros and Christoffel numbers in [3], both given to *7D*, up to $n=5$, also agree with this table as far as comparison can be made, save for a few values that differ by no more than 2 units in the last place. The zeros and Christoffel numbers in [14], given only for $n=5$, to *7D*, are all incorrect (except for A_4) up to as much as 63 units in the 7th decimal place. For details on [14], see MTAC III, no 21, Jan. 1948, item 121 on p. 41. In a footnote to [9], there are given 5-decimal values of the zeros of the first few Laguerre polynomials, correct to within several units of the last place. But in the text on p. 308, $x_{10}^{(10)}$ is erroneously given as 29.9*315* instead of 29.9*207*. In [3], there occurs $\alpha_i^{(n)}e^{x_i^{(n)}}$, up to $n=5$, to *7D*.

Computation Laboratory, National Bureau of Standards

MEMOIRS OF THE AMERICAN MATHEMATICAL SOCIETY

The American Mathematical Society will issue in the near future the first number of a new medium of publication to be known as the Memoirs of the American Mathematical Society. This new series is designed particularly for long papers and for groups of cognate papers. The Memoirs are to be under the supervision of the Editorial Committee of the Transactions of the American Mathematical Society and to be subject to the same standards for acceptance of manuscripts as those now used for the Transactions.

The Memoirs are to be printed by the photo-offset process from copy fully prepared by the authors, following instructions formulated in a manual. A committee consisting of B. P. Gill (Chairman), A. W. Tucker, G. T. Whyburn, and J. W. T. Youngs is now considering the various problems concerning the manual of instructions. Until this manual appears, authors desiring to submit manuscripts for the new series are advised to consult this committee.

The Memoirs are to be nonperiodical, but serial, publications; in this respect, they are like the volumes in Mathematical Surveys or the Colloquium Publications. They will be issued and sold separately. As they are ready for distribution, members will receive suitable announcements and order cards. Individuals and institutions will be given the opportunity to place standing orders for all numbers in the series. As the manuscripts are to be fully prepared by the authors for photo-offsetting, there will be no expense to the Society for typesetting. Therefore it is hoped that it will be possible to sell the Memoirs at a low price.

One of the factors which has caused the heavy backlog of papers in the Transactions is the number of very long manuscripts worthy of publication which have been and are being accepted by the Editorial Committee of the Transactions. With the existence of this new series, the editors of the Transactions are setting increasingly higher standards for longer papers and the publication of these papers may be deferred at the discretion of the editors. Papers of more than a specified size are not to be printed in the Transactions, *except in unusual cases*, but are to be published in the Memoirs. In this manner authors whose papers are accepted for the Memoirs can expect prompt publication. This is a great advantage, both to the author and to mathematics in general, for at the present time there is often a period of eighteen months between acceptance and publication. By relieving the pressure on the Transactions, the Council hopes that the Bulletin will receive some corresponding relief.

The new series will be inaugurated by the publication of two Memoirs. The first will be entitled *Open mappings on locally compact spaces* by G. T. Whyburn and the second will contain papers by Jean Dieudonné and L. K. Hua, entitled *On automorphisms of the classical groups* and *Supplement to the preceding paper by Dieudonné*, respectively.

J. R. KLINE,
Secretary

MEASURABLE TRANSFORMATIONS

PAUL R. HALMOS

1. **Introduction.** The purpose of this paper is to review the progress made in the study of measurable and measure preserving transformations during the last 17 years. The interest of mathematicians in this subject was aroused at the end of 1931 by von Neumann's and Birkhoff's proofs of their respective versions of the ergodic theorem [**8, 9, 101**].[1] It was very quickly recognized that the proper general framework for von Neumann's mean ergodic theorem lay in the direction of Hilbert spaces and Banach spaces, whereas the extent of generality suitable to Birkhoff's theorem was to be found in the concept of a measure space. A measure space is a set possessing no intrinsic algebraic, analytic, or topological structure—all that is necessary is that a concept of measurability and a numerical measure be defined in it. Perhaps the best known nontrivial example of a measure space is one which, to be sure, has many essential non measure theoretic properties, but which may, nevertheless, be considered typical of measure spaces in general—namely the closed unit interval $X = [0, 1]$. For the sake of definiteness I shall begin the discussion by considering a one-to-one transformation T of this space X onto itself, such that, for every measurable subset E of X, both TE and $T^{-1}E$ are measurable and $\mu(E) = \mu(TE) = \mu(T^{-1}E)$ (where μ denotes Lebesgue measure in X). In much of what follows the space X and the transformation T can be replaced by more general spaces and transformations respectively. I shall indicate some of these generalizations in what might be called the geometric direction (that is, generalizations that retain something like an underlying measure space and a transformation acting on it), but I shall not enter at all into the analytic generalizations which constitute the current theory of the mean ergodic theorem.

2. **Asymptotic properties.** The problems that were first treated, and that are still of interest and importance, are connected with the behavior of the sequence $\{T^n\}$ of powers of T. One of the first results in this direction is the Poincaré recurrence theorem, which asserts that, for every measurable set E and for almost every point x in E,

---

An address delivered before the Chicago meeting of the Society on November 26, 1948, by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings; received by the editors November 26, 1948.

[1] Bold face numerals refer to the bibliography at the end.

there are an infinite number of positive values of n such that $T^n x \in E$ [18, 79]. An equivalent way of phrasing this assertion, in terms of the characteristic function χ_E of E, is to say that for every measurable set E the series $\sum_{n=0}^{\infty}\chi_E(T^n x)$ diverges for almost every point x in E. A related result, proved by Hopf [67], is that if f is any positive measurable function, then the series $\sum_{n=0}^{\infty} f(T^n x)$ diverges for almost every point x in X. From the Poincaré recurrence theorem it follows easily that if E is any measurable set of positive measure, then there are an infinite number of positive values of n such that $\mu(E \cap T^n E) > 0$, and, conversely, this formulation of Poincaré's theorem implies the original one. In this direction the result has been strengthened by Khintchine who showed that if E is any measurable set of positive measure and if ϵ is any positive number, then the set of values of n for which

$$\mu(E \cap T^n E) > (\mu(E))^2 - \epsilon$$

is relatively dense in the sense of Bohr [90, 126].

Birkhoff's ergodic theorem may also be viewed as a statement concerning the recurrence properties of $\{T^n\}$. In its most primitive form the ergodic theorem says that if E is any measurable set and if, for each positive integer n and for every point x in X, $s_n(x)$ is the number of values of i for which $T^i x \in E$, $0 \leqq i \leqq n-1$, then $\lim (1/n)s_n(x) = s^*(x)$ exists almost everywhere and, moreover, $\int s^*(x)d\mu(x) = \mu(E)$. If the discrete parameter n is interpreted as time, then $s^*(x)$ is the relative amount of time (time of sojourn) that x spends in E, and the assertion is that the mean sojourn time is almost everywhere defined and that its value depends in an obvious way on the size of E. In terms of characteristic functions the statement of the ergodic theorem concerns the existence almost everywhere of $\lim (1/n) \sum_{i=0}^{n-1}\chi_E(T^i x)$ and the value of the integral of the limit function. A more general statement asserts that, for any integrable function f, $\lim (1/n) \sum_{i=0}^{n-1} f(T^i x) = f^*(x)$ exists almost everywhere and that, moreover, $\int f^*(x)d\mu(x) = \int f(x)d\mu(x)$.

3. **Generalizations.** The first attempts to generalize the ergodic theorem consisted of replacing the underlying space X by abstract measure spaces [88] and dropping the hypothesis $\mu(X) < \infty$ [57, 122]. A slightly more interesting generalization was given by Khintchine [89] who proved that if f is any integrable function on X and g is any periodic function of n, then $\lim (1/n) \sum_{i=0}^{n-1} f(T^i x)g(i)$ exists almost everywhere. The most powerful result in this direction is due to Wiener and Wintner [129] who proved that, if f is, as before, an integrable function, then not only does $\lim (1/n) \sum_{j=0}^{n-1} e^{2\pi i\lambda j} f(T^j x)$

exist almost everywhere for each real number λ (here $i=(-1)^{1/2}$), but in fact the exceptional set of measure zero can be chosen to be independent of λ.

The ergodic theorem is a statement about a space, a function, and a transformation. In the preceding paragraph I mentioned the possibility of generalizing the space and the function; there remains the possibility of generalizing the transformation. What, for instance, can be said if the transformation is not necessarily one-to-one, but is still, in one sense or another, measure preserving? Suppose, for the sake of definiteness, that T is a transformation of X onto itself such that the inverse image of every measurable set is a measurable set of the same measure. (A nontrivial example of such a transformation T is defined in the unit interval by $Tx=2x$ [mod 1].) If T is such a transformation, if f is any measurable function, and if $g_n(x)=f(T^nx)$, $n=0, 1, 2, \cdots$, then the sequence $\{g_n\}$ has the homogeneity property that the joint k-dimensional distribution of any k of its terms depends only on the relative distances between the indices and not on their values. More precisely, it is true that if A is any Borel set in k-dimensional space, and if $(n_1, \cdots, n_k)$ is any k-tuple of distinct non-negative integers, then

$$\mu(\{x: (g_{n_1+m}(x), \cdots, g_{n_k+m}(x)) \in A\})$$

is independent of m. It is a nontrivial extension of the ergodic theorem to say that for any sequence $\{g_n\}$ of integrable functions with this homogeneity property $\lim (1/n)\sum_{i=0}^{n-1}g_i(x)$ exists almost everywhere—this extension is due to Doob [24].

The final generalization that will be mentioned here is one in which the transformation is not required to be measure preserving. The first significant result here was obtained by Hurewicz [73]. I shall state a special case of Hurewicz's theorem—a special case which, however, is typical of the general case and from which, in fact, the general case can easily be derived. Suppose that the transformation T is a one-to-one transformation of a measure space X onto itself such that, for every measurable subset E of X, both TE and $T^{-1}E$ are measurable and such that if $\mu(E)=0$, then $\mu(TE)=\mu(T^{-1}E)=0$. Suppose moreover that T is incompressible in the sense that, for every measurable set E, $E\subset TE$ implies $\mu(E-TE)=0$. It follows from the Radon-Nikodym theorem that, for each integer n, there exists a positive measurable function ω_n such that $\mu(T^nE)=\int_E\omega_n d\mu$ for every measurable set E. In the notation so established Hurewicz's ergodic theorem says that, for any integrable function f, the weighted averages

$$\left(\sum_{i=1}^{n-1} f(T^i x)\omega_i(x)\right) \Big/ \left(\sum_{i=0}^{n-1} \omega_i(x)\right)$$

converge to a finite limit almost everywhere [42].

4. **Indecomposability.** If the interval X is the union of two disjoint measurable sets E and F of positive measure, each of which is invariant under the measure preserving transformation T, then the study of any property of T on X reduces to the separate studies of the corresponding property of T on E and T on F. In such a situation the transformation T may be called decomposable. The most significant transformations are the indecomposable ones—they are usually called metrically transitive [7] or ergodic. In the early days of the theory many special examples of indecomposable transformations were presented in the literature—they occur in fields as apparently diverse as geometry [51, 52, 53, 54, 55, 56, 66, 69, 70, 71, 100, 120, 121, 123, 124, 125], probability [22, 74], and topological groups [39].

For indecomposable transformations the statement of the ergodic theorem can be strengthened by adjoining to it a description of the limit function. Precisely speaking, if f is an integrable function, if $f^*(x) = \lim (1/n) \sum_{i=0}^{n-1} f(T^i x)$, and if the measure preserving transformation T is indecomposable, then $f^*(x)$ is equal almost everywhere to a constant, and the value of that constant is $\int f d\mu$. This assertion is at the basis of the celebrated and sometimes misunderstood interchangeability of time means and phase means. It is of interest to observe that the principle of interchangeability is in fact equivalent to indecomposability for measure-preserving transformations on a finite measure space; if, in other words, for each integrable function f, f^* is equal almost everywhere to a finite constant, then T is indecomposable [103].

A question might be raised as to the extent to which Birkhoff's theorem could be extended to functions which are not necessarily integrable. It is clear that if a transformation is sufficiently decomposable (meaning that there exists a disjoint infinite sequence of measurable invariant sets of positive measure), then the conclusion of Birkhoff's theorem is true for many non integrable functions. The identity transformation is from this point of view the extreme case—if T is the identity, then the conclusion of Birkhoff's theorem is true for every function. For indecomposable transformations T, on the other hand, it can be proved that if f is a non-negative measurable function, or, more generally, a measurable function with the property that either its positive part or its negative part is integrable, and if $\lim (1/n) \sum_{i=0}^{n-1} f(T^i x)$ exists and is finite almost everywhere, then f

must be integrable. M. Gerstenhaber has recently shown me an example which proves that for arbitrary measurable functions this result is not necessarily true.

5. **Decompositions.** A natural problem in connection with the concept of indecomposability is whether or not every measure preserving transformation may be decomposed into indecomposable components. In order to clarify the question and motivate the answer it is helpful to consider an example. Suppose that the measure space X is the unit square in the Cartesian plane, and that the measure preserving transformation T on X has the property that it leaves unaltered the first coordinate of each point. This implies that every subset E of X which depends on the first coordinate alone, that is, every set E which is the union of a class of vertical segments, is invariant under T. Suppose moreover that the invariant sets so obtained exhaust essentially all possibilities—that is, that every measurable invariant set is, modulo sets of measure zero, a union of vertical segments. Since each vertical segment may be considered as a measure space on its own right (with linear Lebesgue measure) and since, in the presence of the conditions described above, the transformation T is indecomposable on each such segment, this situation is an example of a decomposable transformation which is in a certain intuitively obvious sense made up of many little indecomposable components.

The general situation exemplified by the preceding paragraph may be described as follows. Suppose that to each point x of a measure space X (with measure μ) there corresponds a measure space Y_x (with measure μ_x) so that the spaces corresponding to distinct points are disjoint. Suppose that a concept of measurability and a numerical measure λ are introduced into the set Z of all those pairs (x, y) for which $x \in X$ and $y \in Y_x$, in such a way that whenever A is a measurable subset of X, then the set $A^* = \{(x, y) : x \in A\}$ is a measurable subset of Z. If, for every measurable subset E of Z,

$$\lambda(E) = \int_X \mu_x(E \cap Y_x) d\mu(x),$$

then the measure space Z (with measure λ) is called a direct sum of the measure spaces Y_x with respect to the measure space X. The best possible theorem on the decomposability of a measure preserving transformation T on a measure space Z would presumably assert that Z may be represented as a direct sum in such a way that the class of measurable invariant sets coincides, except possibly for sets of measure zero, with the class of all sets of the form A^*; this would imply that the transformation T on Y_x is indecomposable for almost every x in X. The first theorem of this type (in case Z is a complete metric space) was proved by von Neumann [103]; extensions to more general spaces were later given by Dieudonné [21] and myself [36, 44].

6. **Density and category theorems.** How likely is a measure preserving transformation to be indecomposable? Birkhoff conjectured that in some sense the indecomposable case is the general case. One possible way of establishing this conjecture was to introduce a suitable metric or topology into the set $\boldsymbol{T}$ of all measure preserving transformations (after identifying two transformations which differ only on a set of measure zero) and then to show that the subset $\boldsymbol{D}$ of decomposable transformations is of the first category. This was first done by Oxtoby and Ulam for measure preserving homeomorphisms of certain subsets of Euclidean spaces and later by me for arbitrary measure preserving transformations [40, 107]. These topological investigations had some interesting byproducts of which at least one is worth mentioning. A class of particularly simple measure preserving transformations of the interval is obtained by dividing the interval into a finite number of subintervals of equal length and sending each such interval into another such interval by translation. The assertion concerning the class $\boldsymbol{P}$ of transformations so obtained is that it is everywhere dense in $\boldsymbol{T}$—in other words every measure preserving transformation is the limit of a sequence of permutations of intervals.

7. **Strong mixing.** It is possible to define the concept of indecomposability of a transformation T in terms of the asymptotic behavior of the sequence of powers of T. It is in fact an easy consequence of the ergodic theorem that if T is indecomposable, then

$$(*) \qquad \lim \frac{1}{n} \sum_{i=0}^{n-1} \mu(E \cap T^i F) = \mu(E)\mu(F)$$

for every pair of measurable sets E and F and that, conversely, the validity of this relation for every pair of measurable sets implies that T is indecomposable. Since the condition $\mu(E \cap F) = \mu(E)\mu(F)$ is the usual requirement in the definition of independence in the sense of the theory of probability, the equation (*) may be viewed as asserting that asymptotically, in the sense of Cesaro convergence, any two measurable sets, of which one is held fixed and the other is allowed to move under the influence of the transformation T, tend to become independent of each other.

The equation (*) has another natural physical interpretation. Suppose that the transformation T is visualized as a particular way of stirring a container (of total volume 1) full of an incompressible fluid which may be thought of as 99.44 per cent water and .56 per cent red ink. If F is the region occupied by the red ink, then, for any part E of the container, the relative amount of red ink in E, after n repetitions of the act of stirring, is given by $\mu(E \cap T^n F)/\mu(E)$. The indecomposability of the transformation T implies therefore that on the average this relative amount of redness is exactly equal to .56. In general, in physical situations like this one, one expects to be justified in making a much stronger statement, namely that, after the liquid has been stirred sufficiently often, every part E of the container will contain approximately .56 per cent red ink. In mathematical language this pious hope amounts to replacing Cesaro convergence by ordinary convergence, that is, replacing (*) by

$$(**) \qquad \lim \mu(E \cap T^n F) = \mu(E)\mu(F).$$

Transformations T satisfying (**) for every pair E and F of measurable sets are called strongly mixing.

It may be worth while to give an example of a strongly mixing transformation on the interval X. The definition of such a transformation is surprisingly simple. For each x in X, let $x = \sum_{i=1}^{\infty} a_i/2^i$ be the binary expansion of x, and let p be a cyclic permutation of the set of all positive integers; for instance p may be defined by

$$p(1) = 2, \ p(2n) = 2n + 2, \text{ and } p(2n + 1) = 2n - 1, \qquad n = 1, 2, \cdots .$$

If S is defined by $Sx = \sum_{i=1}^{\infty} a_{p(i)}/2^i$, then, except for some easily rectifiable trouble caused by the non uniqueness of the binary expansion, it follows that S is one-to-one from X onto X and measure preserving. Strongly mixing transformations similar to this one occur frequently in probability theory.

8. **Weak mixing.** Between indecomposable transformations and strongly mixing transformations there is room for another concept—the concept of a weakly mixing transformation. This apparently artificial concept is of great technical significance. A measure preserving transformation T is, by definition, weakly mixing if

$$(***) \qquad \lim \frac{1}{n} \sum_{i=0}^{n-1} \left| \mu(E \cap T^i F) - \mu(E)\mu(F) \right| = 0$$

for every pair E and F of measurable sets. In mathematical language, the definition of weak mixing substitutes strong Cesaro convergence for the Cesaro convergence occurring in the definition of indecomposability and for the ordinary convergence occurring in the definition of strong mixing. It is an analytic exercise to show that (***) is satisfied if and only if there exists a set N of positive integers such that N has density zero and such that

$$\lim_{n \notin N} \mu(E \cap T^n F) = \mu(E)\mu(F).$$

If indecomposability is expressed by saying that on the average E is .56 per cent red, and if strong mixing is expressed by saying that after a while E will be .56 per cent red, then weak mixing can be expressed by saying that after a while E will be .56 per cent red, with the exception of a few rare instants during which it may be either too scarlet or else too pale a pink.

For any two measure preserving transformations S and T, the direct product $S \times T$ is defined as that transformation on the Cartesian product of the space X with itself which sends each point (x, y) into (Sx, Ty). The first indication that weak mixing is more than an analytic artificiality is in the assertion that T is weakly mixing if and only if its direct product with itself is indecomposable [65, **83**, **84**].

The physical intuition which motivated the conjecture that in general a measure preserving transformation is indecomposable seems also to indicate that in general a measure preserving transformation is weakly and even strongly mixing. With the usual (category) interpretation of the phrase "in general," I proved that this conjecture is right for weak mixing [41]. In a recent paper Rokhlin [115] showed, by a very simple and elegant argument, that for strong mixing on the other hand the conjecture is just as wrong as it can be, and that, in fact the set $\mathbf{S}$ of strongly mixing transformations is a set of the first category in the set $\mathbf{T}$ of all measure preserving transformations.

9. Automorphisms and unitary operators.

A measure preserving transformation T on a measure space X induces in an obvious way an automorphism of the measure algebra of measurable sets modulo sets of measure zero, or, in other words, a set transformation which assigns to each class of sets, any two members of which differ only on a set of measure zero, another such class. The question of whether or not the converse is true has received some attention [38, **102**]. The answer is yes except in pathological measure spaces. Since this pathology is not of very much interest from the point of view of this paper, I shall from now on require that the space X be non pathological in this sense. It has been proved that all the well known measure spaces satisfy this requirement, that is, that for them every automorphism of the measure algebra of measurable sets modulo sets of measure zero is indeed induced by a one-to-one measure preserving transformation of the space onto itself.

Another, very similar, problem is suggested by the consideration of the complex Hilbert space L_2. If, for each f in L_2, an element Uf in L_2 is defined by $(Uf)(x)=f(Tx)$, then the fact that T is measure preserving implies by an easy and familiar argument (proceeding through finite linear combinations of characteristic functions of measurable sets) that

$$\|Uf\|^2 = \int |f(Tx)|^2 d\mu(x) = \int |f(x)|^2 d\mu(x) = \|f\|^2.$$

Since the transformation U is a one-to-one linear transformation of L_2 onto itself, the last written relation means that U is unitary [**93**, 133]. In other words, to each measure preserving transformation T on X there corresponds a unitary operator U on L_2; the first problem that has to be settled is the characterization of the unitary operators that can arise in this way. A more or less satisfactory answer is known: a unitary operator is induced by a measure preserving transformation T in the way just now outlined if and only if it sends each bounded measurable function into a bounded measurable function and is such that, for any two bounded measurable functions f and g, $U(fg) = Uf \cdot Ug$, where the indicated multiplications denote the pointwise product of the factors [**103**].

It is natural to hope that at least some of the measure theoretic properties of T can be described in the language of Hilbert space and that, conversely, the ideas suggested by the Hilbert space point of view may have measure theoretic significance. There is a fact quite near the surface which seems at least partially to fulfill this hope: a necessary and sufficient condition that T be indecomposable is that the complex number 1 be a simple proper value of U [**103**]. (A proper function of proper value 1 is of course simply an invariant function. Each function in the one-dimensional family of constant functions is invariant; the theorem just stated characterizes indecomposability by the absence of any other invariant functions.) A considerably deeper fact, known as the mixing theorem, is that T is weakly mixing if and only if 1 is a simple proper value of U and moreover U has no other proper values [**62**, **63**, **65**, **94**].

10. **Isomorphism and pure point spectrum.** A fundamental prob-

lem of the theory of measurable transformations is the problem of isomorphism. Two measure preserving transformations T_1 and T_2 are called isomorphic if there exists a measure preserving transformation T which carries T_1 into T_2, that is, for which $TT_1T^{-1}=T_2$. An obvious solution of the isomorphism problem is suggested by the first result mentioned in the preceding section—two measure preserving transformations are isomorphic if and only if the automorphisms they induce are conjugate elements in the group $\boldsymbol{T}$ of all automorphisms.

One might make the more promising and less trivial conjecture that all the measure theoretic properties of a transformation T are reflected by the operatorial properties of its unitary operator U, that is, that T_1 and T_2 are isomorphic if and only if the corresponding unitary operators U_1 and U_2 are spectrally equivalent, or, in other words, if and only if there exists a unitary operator U for which $UU_1U^*=U_2$.

There is an interesting class of measure preserving transformations for which the conjecture expressed in the preceding paragraph is true. In order to motivate the introduction of this class, let the measure space X be a compact abelian group (with Haar measure) and let α be any fixed element of X. If T is defined, for every x in X, by $Tx=x+\alpha$, then T is a measure preserving transformation of X onto itself. If ϕ is any character of the group X, then the equation $\phi(x+\alpha)=\phi(\alpha)\phi(x)$ shows that ϕ is a proper function of the unitary operator U corresponding to T, with proper value $\phi(\alpha)$. Since the set of all characters is a complete orthonormal set in L_2, the transformation T is said to have pure point spectrum—the general definition of a measure preserving transformation with pure point spectrum is that the set of proper functions of its induced unitary operator is large enough to contain a complete orthonormal set. The conjecture concerning the relation of isomorphism and spectral equivalence is true in this sense: two indecomposable measure preserving transformations with pure point spectrum are isomorphic if and only if their induced unitary operators are spectrally equivalent—this was proved by von Neumann [103]. In this connection it should also be mentioned that every indecomposable measure preserving transformation with pure point spectrum is known to be of the type described in the example above—that is, each such transformation is necessarily a translation by a suitable element in a suitable compact abelian group [38].

11. Isomorphism and mixed spectrum. The result of the preceding section is not true for arbitrary measure preserving transformations—it is possible, in other words, to construct an example of two measure preserving transformations T_1 and T_2 such that T_1 and T_2 are not isomorphic but such that the corresponding unitary operators are spectrally equivalent. This construction has not been published so far—it is the result of joint work by von Neumann and myself. I proceed to sketch the details of the theory behind the construction.

With each measure preserving transformation T it is possible to associate a sequence $\{G_n\}$ of classes of bounded measurable functions as follows. The initial class G_0 contains only the function which is identically equal to 1; for $n \geqq 1$, the class G_n consists of those functions f which satisfy almost everywhere an equation of the form $f(Tx) = g(x)f(x)$, with $g \in G_{n-1}$. The class G_1 is then the class of invariant functions; if T is indecomposable, so that G_1 is the class of constant functions, then G_2 is the class of proper functions; for positive values of n the functions of G_n may be viewed as generalized proper functions belonging to proper values which instead of being necessarily constants are elements of G_{n-1}. It may happen that, for a suitable positive integer n, $G_n = G_{n+1}$ (and hence $G_n = G_{n+p}$ for every positive integer p); the least positive integer for which this happens is then denoted by $n(T)$. If for instance T is weakly mixing, then (since T is a fortiori indecomposable) G_1 is the class of constant functions and therefore, by the mixing theorem, $n(T) = 1$.

An example which shows that isomorphism is not the same as spectral equivalence is now easy to construct. Let the measure space be the torus represented as all pairs (x, y) of real numbers modulo 1. If $T_1(x, y) = (x+\alpha, y+x)$ [mod 1], and if $T_2(x, y) = (x+\alpha, Sy)$ [mod 1], where α is an irrational number and S is the transformation described in §8 as an example of a strongly mixing transformation, then a reasonably straightforward calculation shows that the induced unitary operators are spectrally equivalent. The proof that, nevertheless, T_1 and T_2 are not isomorphic leans on the concepts introduced in the preceding paragraph; it can in fact be shown that $n(T_1) = 3$ and $n(T_2) = 2$.

12. **Invariant measures.** With the exception of the ergodic theorem, most of the considerations in the preceding sections deal with measure preserving transformations. How restrictive is the requirement that a transformation preserve some measure? One of the best precise and nontrivial formulations of this question is the following one. Suppose that T is a one-to-one transformation of a measure space X onto itself such that for every measurable subset E of X both TE and $T^{-1}E$ are measurable and such that if $\mu(E) = 0$, then $\mu(TE) = \mu(T^{-1}E) = 0$; does there then exist a measure λ on the class of all measurable subsets of X such that $\lambda(E)$ vanishes if and only if $\mu(E)$

vanishes, such that X is the union of countably many measurable sets on each of which λ is finite, and such that $\lambda(TE)=\lambda(E)$ for every measurable set E?

An interesting necessary and sufficient condition for the existence of a finite λ was given by Hopf [64]. In order to describe this condition it is necessary to introduce some new concepts. Two measurable sets E and F are primitively equivalent with respect to a transformation T if there exists an integer n such that $T^nE=F$. Two measurable sets E and F are equivalent with respect to a transformation T if (after possibly omitting a set of measure zero from both of them) both E and F may be written as unions of disjoint sequences of measurable sets so that each term in the sequence representing E is primitively equivalent to the corresponding term in the sequence representing F. In imitation of Dedekind's definition of finiteness, it is customary to say that a measurable set E is bounded (or finite) with respect to T if whenever E is equivalent to a subset F of itself, then $\mu(E-F)=0$. Hopf's condition for the existence of a finite invariant measure is that X be bounded.

The concept of boundedness, while it gives quite a bit of insight into the structure of measurable transformations, is not very easy to apply. Until recently, for example, it was not known whether or not the condition was vacuously satisfied, that is, whether or not there existed any transformations at all which did not satisfy it. In 1947 I succeeded in showing that such a transformation does exist and hence that the search for invariant measures must be conducted among measures which are allowed to take infinite values [43]. The proof is based on the fact that it is sufficient to exhibit an indecomposable measure preserving transformation on a non atomic measure space of infinite measure; the first transformation with these properties was constructed by Oxtoby. The condition of Hopf has a natural extension to the infinite case, but is then even harder to apply [43, 85]. While it seems plausible to conjecture that an invariant measure need not always exist, the question is still open. Either an example or, if the conjecture is wrong, a general existence theorem would be of considerable interest. In the meantime the theory of measurable but not necessarily measure preserving transformations deserves a little investigation. The first steps of such an investigation were carried out by Rademacher in 1916 [110] and a few fragmentary results have been obtained since then [43, 46].

13. Flows. Virtually all the problems discussed above for a single transformation T and its iterates T^n, $n=0, \pm 1, \pm 2, \cdots$, make sense for a one-parameter family of transformation $\{T_t\}$, $-\infty<t$

$<+\infty$, which have the group property $T_{t+s}=T_tT_s$ for all t and s. Such a family of measure preserving transformations is called a flow; a flow is called measurable if T_tx is a measurable function of (x, t). If sums are replaced by integrals and the problems of measurability receive a modicum of attention, then the ergodic theorem, the mixing theorem, the decomposition into indecomposable parts, and many other essential results can be extended from transformations to flows without any new conceptual difficulties.

There is, however, at least one notion which is suggested by the study of flows and not by the automatic process of generalization. This is the concept of a flow built under a function. Suppose that f is any positive measurable function on a measure space X and that T is any one-to-one measure preserving transformation of X onto itself. Let $\overline{X}$ be the ordinate set of f, that is, $\overline{X}$ is a subset of the product space of X with the real line and is defined by

$$\overline{X} = \{(x, t): 0 \leqq t < f(x)\}.$$

For convenience of language conceive X as a horizontal interval, and the parameter t as time; it will then make sense to speak of points of $\overline{X}$ going "up" with a certain "velocity." Let each point (x, t) of $\overline{X}$ move up, in this sense, with a uniform unit velocity, until it hits the "ceiling" $f(x)$; let it then be put back at the place $(Tx, 0)$ and continue on up from there. This procedure defines a flow in $\overline{X}$; the flow so defined is called the flow built on T under f. The study of such a flow may obviously be reduced to the study of the transformation T and the function f; it is consequently a very useful thing to know that every measurable indecomposable flow is isomorphic to a flow built under a function. This result was proved by Ambrose [2] and later extended to all non pathological flows by Ambrose and Kakutani [3].

It is interesting to observe that the concept of a flow built under a function has a discrete analogue which promises to be of interest even in the study of a single transformation. Let f be a measurable function from a measure space X to the set of non-negative integers, and write $E_n=\{x: f(x)\geqq n\}$, $n=0, 1, 2, \cdots$. The set $\overline{X}$ of all pairs (x, n), where $n=0, 1, 2, \cdots$ and $x\in E_n$, is an analogue of what in the continuous case was called the ordinate set of f. If T is a transformation on X, then a transformation $\overline{T}$ may be defined on $\overline{X}$ by setting $\overline{T}(x, n)=(x, n+1)$ whenever $x\in E_{n+1}$ and $\overline{T}(x, n)=(Tx, 0)$ otherwise. The example (mentioned in the preceding section) of a transformation which does not preserve any suitable finite measure was constructed with the aid of such methods; a systematic study of the relation between T and $\bar{T}$ was recently begun by Kakutani [82].

14. Unsolved problems. In connection with the discussion above I have already had occasion to mention two or three unsolved problems. The most important one among these is the problem of the existence of invariant measures. In this section I shall mention a few more directions in which further progress would be desirable.

Quite a few of the results discussed above, but by no means all, can be extended to measure spaces of not necessarily finite measure; a systematic investigation of measure preserving transformations on infinite measure spaces is still lacking. Another obvious problem is to extend the ergodic theorem (known for the additive group of the integers and the additive group of the real numbers) to more general groups of transformations; some results in this direction are known for finite dimensional vector groups [109, 127, 128].

In order to state the next problem, it is convenient to introduce one more concept. If, for each of a countable set of indices i, X_i is a measure space with measure μ_i such that $\mu_i(X_i)=1$, then it is possible to make the Cartesian product X of all the X_i into a measure space in a way which is a natural extension of the concept of Cartesian product for a finite number of spaces. Such product spaces are well known in probability theory. Suppose in particular that the spaces X_i are all equal to a fixed space X_0 and that the domain of the index i is the set of all integers. A point x of X is in this case a sequence, $x=(\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots)$, $x_i \in X_0$. If a transformation T_0 is defined on X by $T_0x=(\cdots, y_{-2}, y_{-1}, y_0, y_1, y_2, \cdots)$, $y_n=x_{n+1}$, then this coordinate shift T_0 is a one-to-one measure preserving transformation of X onto itself. If X_0 is the real line (with a suitable measure μ_0 satisfying $\mu_0(X_0)=1$) and if the function f_0 on X is defined by $f_0(x)=x_0$, then results much more precise than the ergodic theorem are known about the asymptotic behavior of the sequence $\{f_0(T_0^n x)\}$. It would be of interest to obtain analogues of these results (for example, the law of the iterated logarithm and the central limit theorem) for a wider class of transformations and functions than the ones here described.

An attempt in this direction has been made by Izumi [75]. Motivated by the fact that (under a mild restriction on the measure μ_0) not only do the averages $(1/n)\sum_{i=0}^{n-1} f_0(T_0^i x)$ converge to zero almost everywhere but even the series $\sum_{n=1}^{\infty} (1/n)f_0(T_0^n x)$ is convergent almost everywhere, Izumi formulated a set of conditions on the measure preserving transformation T (and on the function f) sufficient to ensure the convergence almost everywhere of $\sum_{n=1}^{\infty} (1/n)f(T^n x)$. Unfortunately Izumi's requirements turned out to be so stringent that the only transformation which can satisfy them is the identity transformation on a measure space consisting of exactly one point [45].

In conclusion I emphasize again that the isomorphism problem is almost completely untouched in most nontrivial cases. It is, of course, hard to formulate precisely what is meant by the injunction: "find necessary and sufficient conditions for the isomorphism of two measure preserving transformations." There are, however, many concrete examples of measure preserving transformations concerning which it is not known whether or not any two are isomorphic—the vague task of finding a complete set of invariants can at least at the beginning be replaced by the specific task of finding sufficiently many invariants to sort out into isomorphism classes these known examples [39]. If, for instance, the coordinate shift on the Cartesian product of countably many k-point spaces (in which each point has measure $(1/k)$) is denoted by T_k, then is T_2 isomorphic to T_3 or is it not? Nobody knows.

BIBLIOGRAPHY

The following is a list, as nearly complete as I could make it, of the papers and books directly relevant to the theory of measurable and measure preserving transformations. It does *not* contain works concerning applications to physics or stochastic processes, generalizations to Markoff chains or mean ergodic theorems, and analytical or topological results about dynamical systems.

1. W. Ambrose, *Change of velocities in a continuous ergodic flow*, Duke Math. J. vol. 8 (1941) pp. 425–440.

2. ———, *Representation of ergodic flows*, Ann. of Math. vol. 42 (1941) pp. 723–739.

3. W. Ambrose and S. Kakutani, *Structure and continuity of measurable flows*, Duke Math. J. vol. 9 (1942) pp. 25–42.

4. W. Ambrose, P. R. Halmos, and S. Kakutani, *The decomposition of measures*, II, Duke Math. J. vol. 9 (1942) pp. 43–47.

5. M. Bebutoff and W. Stepanoff, *Sur le changement du temps dans les systèmes dynamiques possédant une mesure invariante*, C. R. (Doklady) Acad. Sci. URSS vol. 24 (1939) pp. 217–219.

6. ———, *Sur la mesure invariante dans les systèmes dynamiques qui ne diffèrent que par le temps*, Rec. Math. (Mat. Sbornik) N.S. vol. 7 (1940) pp. 143–166.

7. G. D. Birkhoff and P. A. Smith, *Structure analysis of surface transformations*, J. Math. Pures Appl. vol. 7 (1928) pp. 345–379.

8. G. D. Birkhoff, *Proof of a recurrence theorem for strongly transitive systems*, Proc. Nat. Acad. Sci. U.S.A. vol. 17 (1931) pp. 650–655.

9. ———, *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci. U.S.A. vol. 17 (1931) pp. 656–660.

10. G. D. Birkhoff and B. O. Koopman, *Recent contributions to ergodic theory*, Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) pp. 279–282.

11. G. D. Birkhoff, *Probability and physical systems*, Bull. Amer. Math. Soc. vol. 38 (1932) pp. 361–379.

12. ———, *Some unsolved problems of theoretical dynamics*, Science vol. 94 (1941) pp. 598–600.

13. ———, *What is the ergodic theorem?*, Amer. Math. Monthly vol. 49 (1942) pp. 222–226.

14. ———, *The ergodic theorems and their importance in statistical mechanics*, Revista de Ciencias vol. 44 (1942) p. 251.

15. N. Bogoliouboff and N. Kryloff, *Les mesures invariantes et la transitivité*, C. R. Acad. Sci. Paris vol. 201 (1935) pp. 1454–1456.

16. ———, *Les mesures invariantes et transitives dans la mécanique non linéaire*, Rec. Math. (Mat. Sbornik) N.S. vol. 1 (1936) pp. 707–711.

17. ———, *La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non linéaire*, Ann. of Math. vol. 38 (1937) pp. 65–113.

18. C. Carathéodory, *Über den Wiederkehrsatz von Poincaré*, Sitzungsberichte der Preussischen Akademie der Wissenschaften vol. 32 (1919) pp. 580–584.

19. ———, *Bemerkungen zum Riesz-Fischerschen Satz und zur Ergodentheorie*, Abh. Math. Sem. Hansischen Univ. vol. 14 (1941) pp. 351–389.

20. ———, *Bemerkungen zum Ergodensatz von G. Birkhoff*, Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Klasse der Bayerischen Akademie der Wissenschaften zu München (1944) pp. 189–208.

21. J. Dieudonné, *Sur le théorème de Lebesgue-Nikodym* (III), Annales de l'Université de Grenoble vol. 23 (1948) pp. 25–53.

22. J. L. Doob, *Probability and statistics*, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 759–775.

23. ———, *One-parameter families of transformations*, Duke Math. J. vol. 4 (1938) pp. 752–774.

24. ———, *The law of large numbers for continuous stochastic processes*, Duke Math. J. vol. 6 (1940) pp. 290–306.

25. J. L. Doob and R. A. Leibler, *On the spectral analysis of a certain transformation*, Amer. J. Math. vol. 65 (1943) pp. 263–272.

26. Y. N. Dowker, *Invariant measure and the ergodic theorems*, Duke Math. J. vol. 14 (1947) pp. 1051–1061.

27. N. Dunford, *Spectral theory*. I. *Convergence to projections*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 185–217.

28. ———, *Spectral theory*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 637–651.

29. N. Dunford and D. S. Miller, *On the ergodic theorem*, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 538–549.

30. K. Fan, *Les fonctions asymptotiquement presque-périodiques d'une variable entière et leur application à l'étude de l'itération des transformations continues*, Math. Zeit. vol. 48 (1943) pp. 685–711.

31. S. Fomin, *Finite invariant measures in the flows*, Rec. Math. (Mat. Sbornik) N.S. vol. 12 (1943) pp. 99–108.

32. M. Fréchet, *Sur le théorème ergodique de Birkhoff*, C. R. Acad. Sci. Paris vol. 213 (1941) pp. 607–609.

33. ———, *Sur le problème ergodique*, La Revue Scientifique (Revue Rose Illustrée) vol. 81 (1943) pp. 155–157.

34. ———, *Une application des fonctions asymptotiquement presque-périodiques à l'étude des familles de transformations ponctuelles et au problème ergodique*, La Revue Scientifique (Revue Rose Illustrée) vol. 79 (1941) pp. 407–417.

35. M. Fukamiya, *On dominated ergodic theorems in* $L_p(p \geqq 1)$, Tôhoku Math. J. vol. 46 (1940) pp. 150–153.

36. P. R. Halmos, *The decomposition of measures*, Duke Math. J. vol. 8 (1941) pp. 386–392.

37. ———, *Square roots of measure preserving transformations*, Amer. J. Math. vol. 64 (1942) pp. 153–166.

38. P. R. Halmos and J. von Neumann, *Operator methods in classical mechanics*, II, Ann. of Math. vol. 43 (1942) pp. 332–350.

39. P. R. Halmos, *On automorphisms of compact groups*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 619–624.

40. ———, *Approximation theories for measure preserving transformations*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 1–18.

41. ———, *In general a measure preserving transformation is mixing*, Ann. of Math. vol. 45 (1944) pp. 786–792.

42. ———, *An ergodic theorem*, Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 156–161.

43. ———, *Invariant measures*, Ann. of Math. vol. 48 (1947) pp. 735–754.

44. ———, *On a theorem of Dieudonné*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 38–42.

45. ———, *A non homogeneous ergodic theorem*, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 284–288.

46. ———, *Measure theory*, New York, 1949.

47. P. Hartman and A. Wintner, *Asymptotic distributions and the ergodic theorem*, Amer. J. Math. vol. 61 (1939) pp. 977–984.

48. ———, *Statistical independence and statistical equilibrium*, Amer. J. Math. vol. 62 (1940) pp. 646–654.

49. ———, *Integrability in the large and dynamical stability*, Amer. J. Math. vol. 65 (1943) pp. 273–278.

50. P. Hartman, *On the ergodic theorems*, Amer. J. Math. vol. 69 (1947) pp. 193–199.

51. G. A. Hedlund, *On the metrical transitivity of the geodesics on a surface of constant negative curvature*, Proc. Nat. Acad. Sci. U.S.A. vol. 20 (1934) pp. 136–140.

52. ———, *On the metrical transitivity of the geodesics on closed surfaces of constant negative curvature*, Ann. of Math. vol. 35 (1934) pp. 787–808.

53. ———, *A metrically transitive group defined by the modular group*, Amer. J. Math. vol. 57 (1935) pp. 668–678.

54. ———, *The dynamics of geodesic flows*, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 241–260.

55. ———, *Fuchsian groups and mixtures*, Ann. of Math. vol. 40 (1939) pp. 370–383.

56. ———, *A new proof for a metrically transitive system*, Amer. J. Math. vol. 62 (1940) pp. 233–242.

57. H. Hilmy, *Sur le théorème ergodique*, C. R. (Doklady) Acad. Sci. URSS vol. 24 (1939) pp. 213–216.

58. ———, *Sur le récurrence ergodique dans les systèmes dynamiques*, Rec. Math. (Mat. Sbornik) N.S. vol. 7 (1940) pp. 101–109.

59. E. Hopf, *Zwei Sätze über den wahrscheinlichen Verlauf der Bewegungen dynamischer Systeme*, Math. Ann. vol. 103 (1930) pp. 710–719.

60. ———, *On the time average theorem in dynamics*, Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) pp. 93–100.

61. ———, *Über lineare Gruppen unitärer Operatoren im Zusammenhange mit den*

Bewegungen dynamischer Systeme, Sitzungsberichte der Preussischen Akademie der Wissenschaften vol. 14 (1932) pp. 182–190.

62. ———, *Complete transitivity and the ergodic principle*, Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) pp. 204–209.

63. ———, *Proof of Gibbs' hypothesis on the tendency toward statistical equilibrium*, Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) pp. 333–340.

64. ———, *Theory of measure and invariant integrals*, Trans. Amer. Math. Soc. vol. 34 (1932) pp. 373–393.

65. ———, *On causality, statistics, and probability*, Journal of Mathematics and Physics vol. 13 (1934) pp. 51–102.

66. ———, *Fuchsian groups and ergodic theory*, Trans. Amer. Math. Soc. vol. 39 (1936) pp. 299–314.

67. ———, *Ergodentheorie*, Berlin, 1937.

68. ———, *Statistische Probleme und Ergebnisse in der klassischen Mechanik*, Actualités Scientifiques et Industrielles, no. 737, 1938, pp. 5–16.

69. ———, *Beweis des Mischungscharakters der geodätischen Stromung auf Flächen der Krümmung minus Eins und endlicher Oberfläche*, Sitzungsberichte der Preussischen Akademie der Wissenschaften (1938) pp. 333–344.

70. ———, *Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung*, Berichte uber die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig vol. 91 (1939) pp. 261–304.

71. ———, *Statistik der Lösungen geodatischen Probleme vom unstabilen Typus*, II, Math. Ann. vol. 117 (1940) pp. 590–608.

72. ———, *Über eine Ungleichung der Ergodentheorie*, Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Klasse der Bayerischen Akademie der Wissenschaften zu München (1944) pp. 171–176.

73. W. Hurewicz, *Ergodic theorem without invariant measure*, Ann. of Math. vol. 45 (1944) pp. 192–206.

74. K. Ito, *On the ergodicity of a certain stationary process*, Proc. Imp. Acad. Tokyo vol. 20 (1944) pp. 54–55.

75. S. Izumi, *A non-homogeneous ergodic theorem*, Proc. Imp. Acad. Tokyo vol. 15 (1939) pp. 189–192.

76. ———, *A remark on ergodic theorems*, Proc. Imp. Acad. Tokyo vol. 19 (1943) pp. 102–104.

77. B. Jessen, *Abstract theory of measure and integration*. IX, Matematisk Tidsskrift. B (1947) pp. 21–26.

78. ———, *Abstrakt Maal-og Integralteori*, Copenhagen, 1947.

79. M. Kac, *On the notion of recurrence in discrete stochastic processes*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1002–1010.

80. S. Kakutani and K. Yosida, *Birkhoff's ergodic theorem and the maximal ergodic theorem*, Proc. Imp. Acad. Tokyo vol. 15 (1939) pp. 165–168.

81. S. Kakutani, *Representation of measurable flows in Euclidean 3-space*, Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) pp. 16–21.

82. ———, *Induced measure preserving transformations*, Proc. Imp. Acad. Tokyo vol. 19 (1943) pp. 635–641.

83. Y. Kawada, *Über die masstreuen Abbildungen vom Mischungstypus im weiteren Sinne*, Proc. Imp. Acad. Tokyo vol. 19 (1943) pp. 520–524.

84. ———, *Über die masstreuen Abbildungen in Produkträumen*, Proc. Imp. Acad. Tokyo vol. 19 (1943) pp. 525–527.

85. ———, *Über die Existenz der invarianten Integrale*, Jap. J. Math. vol. 19 (1944) pp. 81–95.

86. A. Khintchine, *Zur mathematischen Begründung der statistischen Mechanik*, Zeitschrift für Angewandte Mathematik und Mechanik, vol. 13 (1933) pp. 101–103.

87. ———, *The method of spectral reduction in classical dynamics*, Proc. Nat. Acad. Sci. U.S.A. vol. 19 (1933) pp. 567–573.

88. ———, *Zu Birkhoffs Lösung des Ergodenproblems*, Math. Ann. vol. 107 (1933) pp. 485–488.

89. ———, *Fourierkoeffizienten längs einer Bahn im Phasenraum*, Rec. Math. (Mat. Sbornik) vol. 41 (1934) pp. 14–15.

90. ———, *Eine Verschärfung des Poincaréschen "Wiederkehrsatzes,"* Compositio Math. vol. 1 (1934) pp. 177–179.

91. ———, *Korrelationstheorie der stationären stochastischen Prozesse*, Math. Ann. vol. 109 (1934) pp. 604–615.

92. A. Kolmogoroff, *Ein vereinfachter Beweis des Birkhoff-Khintchineschen Ergodensatzes*, Rec. Math. (Mat. Sbornik) N.S. vol. 2 (1937) pp. 366–368.

93. B. O. Koopman, *Hamiltonian systems and transformations in Hilbert space*, Proc. Nat. Acad. Sci. U.S.A. vol. 17 (1931) pp. 315–318.

94. B. O. Koopman and J. von Neumann, *Dynamical systems of continuous spectra*, Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) pp. 255–263.

95. T. Levi-Civita, *A general survey of the theory of adiabatic invariants*, Journal of Mathematics and Physics vol. 13 (1934) pp. 18–40.

96. F. Maeda, *Application of the theory of set functions to the mixing of fluids*, Journal of Science of the Hirosima University. Ser. A vol. 5 (1935) pp. 1–6.

97. ———, *Transitivities of conservative mechanism*, Journal of Science of the Hirosima University. Ser. A vol. 6 (1936) pp. 1–18.

98. D. Maharam, *On homogeneous measure algebras*, Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) pp. 108–111.

99. P. T. Maker, *The ergodic theorem for a sequence of functions*, Duke Math. J. vol. 6 (1940) pp. 27–30.

100. M. H. Martin, *Metrically transitive point transformations*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 606–612.

101. J. von Neumann, *Proof of the quasi-ergodic hypothesis*, Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) pp. 70–82.

102. ———, *Einige Sätze über messbare Abbildungen*, Ann. of Math. vol. 33 (1932) pp. 574–586.

103. ———, *Zur Operatorenmethode in der klassischen Mechanik*, Ann. of Math. vol. 33 (1932) pp. 587–642.

104. ———, *Zusätze zur Arbeit "Zur Operatorenmethode . . . ,"* Ann. of Math. vol. 33 (1932) pp. 789–791.

105. J. C. Oxtoby, *Note on transitive transformations*, Proc. Nat. Acad. Sci. U.S.A. vol. 23 (1937) pp. 443–446.

106. J. C. Oxtoby and S. M. Ulam, *On the existence of a measure invariant under a transformation*, Ann. of Math. vol. 40 (1939) pp. 560–566.

107. ———, *Measure-preserving homeomorphisms and metrical transitivity*, Ann. of Math. vol. 42 (1941) pp. 874–920.

108. J. C. Oxtoby, *On the ergodic theorem of Hurewicz*, Ann. of Math. vol. 49 (1948) pp. 872–884.

109. H. R. Pitt, *Some generalizations of the ergodic theorem*, Proc. Cambridge Philos. Soc. vol. 38 (1942) pp. 325–343.

110. H. Rademacher, *Eineindeutige Abbildungen und Messbarkeit*, Monatshefte fur Mathematik und Physik vol. 27 (1916) pp. 183–290.

111. F. Riesz, *Sur quelques problèmes de la théorie ergodique*, Matematikai és Fizikai Lapok vol. 49 (1942) pp. 34–62.

112. ———, *Sur la théorie ergodique*, Comment. Math. Helv. vol. 17 (1945) pp. 221–239.

113. V. Rokhlin, *On the classification of measurable decompositions*, Doklady Akademii Nauk SSSR vol. 58 (1947) pp. 29–32.

114. ———, *On the problem of the classification of automorphisms of Lebesgue spaces*, Doklady Akademii Nauk SSSR vol. 48 (1947) pp. 189–191.

115. ———, *A "general" measure preserving transformation is not mixing*, Doklady Akademii Nauk SSSR vol. 60 (1948) pp. 349–351.

116. ———, *Unitary rings*, Doklady Akademii Nauk SSSR vol. 59 (1948) pp. 643–646.

117. G. Scorza Dragoni, *Sul teorema ergodico*, Rend. Circ. Mat. Palermo vol. 58 (1934) pp. 311–325.

118. ———, *Transitività metrica e teoremi di media*, Rend. Circ. Mat. Palermo vol. 59 (1935) pp. 235–255.

119. ———, *Sul fondamento matematico della teoria degli invarianti adiabatici*, Annali di Mathematica Pura ed Applicata vol. 13 (1935) pp. 335–362.

120. W. Seidel, *Note on a metrically transitive system*, Proc. Nat. Acad. Sci. U.S.A. vol. 19 (1933) pp. 453–456.

121. ———, *On a metric property of Fuchsian groups*, Proc. Nat. Acad. Sci. U.S.A. vol. 21 (1935) pp. 475–478.

122. W. Stepanoff, *Sur une extension du théorème ergodique*, Compositio Math. vol. 3 (1936) pp. 239–253.

123. M. Tsuji, *On Hopf's ergodic theorem*, Proc. Imp. Acad. Tokyo vol. 20 (1944) pp. 640–647.

124. ———, *Some metrical theorems on Fuchsian groups*, Proc. Imp. Acad. Tokyo vol. 21 (1945) pp. 104–109.

125. ———, *On Hopf's ergodic theorem*, Jap. J. Math. vol. 19 (1945) pp. 259–284.

126. C. Visser, *On Poincaré's recurrence theorem*, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 397–400.

127. N. Wiener, *The homogeneous chaos*, Amer. J. Math. vol. 60 (1938) pp. 897–936.

128. ———, *The ergodic theorem*, Duke Math. J. vol. 5 (1939) pp. 1–18.

129. N. Wiener and A. Wintner, *Harmonic analysis and ergodic theory*, Amer. J. Math. vol. 63 (1941) pp. 415–426.

130. ———, *On the ergodic dynamics of almost periodic systems*, Amer. J. Math. vol. 63 (1941) pp. 794–824.

131. ———, *The discrete chaos*, Amer. J. Math. vol. 65 (1943) pp. 279–298.

132. A. Wintner, *Remarks on the ergodic theorem of Birkhoff*, Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) pp. 248–251.

133. ———, *Dynamische Systeme und unitäre Matrizen*, Math. Zeit. vol. 36 (1933) pp. 630–637.

134. ———, *On the ergodic analysis of the remainder term of mean motions*, Proc. Nat. Acad. Sci. U.S.A. vol. 26 (1940) pp. 126–129.

135. K. Yosida, *Ergodic theorems of Birkhoff-Khintchine's type*, Jap. J. Math. vol. 17 (1940) pp. 31–36.

136. ———, *An abstract treatment of the individual ergodic theorem*, Proc. Imp. Acad. Tokyo vol. 16 (1940) pp. 280–284.

UNIVERSITY OF CHICAGO

THE SUMMER MEETING IN BOULDER

The fifty-fifth Summer Meeting and the thirty-first Colloquium of the American Mathematical Society were held at the University of Colorado, Boulder, Colorado, Tuesday to Friday, August 30–September 2, 1949. The Mathematical Association of America met on Monday and Tuesday, and the Econometric Society and the Institute for Mathematical Statistics held meetings Monday through Thursday. Attendance at the meetings was approximately 800, including the following 394 members of the Society:

J. C. Abbott, V. W. Adkisson, O. W. Albert, E. B. Allen, R. D. Anderson, T. W. Anderson, R. V. Andree, B. H. Arnold, K. J. Arnold, M. N. Arnoldy, W. L. Ayres. Reinhold Baer, E. W. Barankin, Joshua Barlaz, I. A. Barnett, D. Y. Barrer, M. A, Basoco, P. T. Bateman, F. D. Bateman, J. W. Beach, J. E. Bearman, E. F. Beckenbach, M. M. Beenken, A. K. Bell, J. H. Bell, P. O. Bell, B. C. Bellamy, C. A. Bennett, Theodore Bennett, A. H. Berger, Herman Betz, S. F. Bibb, F. C. Biesele, E. E. Blanche, W. A. Blankinship, H. D. Block, M. L. Boas, R. P. Boas, H. F. Bohnenblust, A. W. Boldyreff, F. E. Bortle, T. A. Botts, J. W. Bradshaw, H. E. Bray, J. R. Britton, J. C. Brixey, J. R. Buchi, C. C. Buck, E. F. Buck, R. C. Buck, C. E. Buell, P. B. Burcham, Herbert Busemann, J. H. Bushey, Jewell H. Bushey, S. S. Cairns, W. D. Cairns, J. W. Calkin, E. A. Cameron, C. C. Camp, H. H. Campaigne, B. G. Carlson, F. M. Carpenter, W. B. Carver, C. R. Cassity, T. E. Caywood, Abraham Charnes, Harold Chatland, L. G. Chelius, Herman Chernoff, K. L. Chung, R. V. Churchill, H. M. Clark, H. E. Clarkson, Nathaniel Coburn, C. J. Coe, Nancy Cole, L. A. Colquitt, E. G. H. Comfort, R. M. Cook, J. A. Cooley, Max Coral, N. A. Court, M. J. Cox, F. G. Creese, E. L. Crow, H. J. Curtis, J. H. Curtiss, P. H. Daus, A. C. Davis, E. A. Davis, A. S. Day, M. M. Day, R. F. Deniston, W. W. Denton, A. H. Diamond, R. P. Dilworth, Bernard Dimsdale, W. J. Dixon, J. M. Dobbie, C. L. Dolph, M. D. Donsker, J. L. Doob, C. H. Dowker, Y. N. Dowker, M. J. Dresher, Roy Dubisch, W. L. Duren, Aryeh Dvoretzky, J. M. Earl, E. D. Eaves, Paul Eberhart, W. F. Eberlein, B. J. Eisenstadt, E. S. Elyash, Paul Erdös, G. C. Evans, H. P. Evans, Howard Eves, G. M. Ewing, A. B. Farnell, William Feller, H. H. Ferns, C. H. Fischer, L. R. Ford, W. C. Foreman, J. S. Frame, T. C. Fry, L. E. Fuller, R. E. Fullerton, M. G. Gaba, H. M. Gehman, F. C. Gentry, H. H. Germond, R. E. Gilman, Wallace Givens, A. M. Gleason, Michael Golomb, D. B. Goodner, R. N. Goss, W. H. Gottschalk, Cornelius Gouwens, R. E. Graves, J. W. Green, R. E. Greenwood, Edison Greer, F. L. Griffin, V. G. Grove, William Gustin, B. L. Hagen, D. T. Haimo, Franklin Haimo, Edwin Halfar, P. R. Halmos, O. H. Hamilton, K. E. Hazard, E. A. Hazlewood, G. A. Hedlund, E. D. Hellinger, Erik Hemmingsen, A. S. Henriques, G. A. Herr, J. G. Herriot, Fritz Herzog, M. R. Hestenes, Edwin Hewitt, T. H. Hildebrandt, J. J. L. Hinrichsen, G. P. Hochschild, J. L. Hodges, P. G. Hoel, D. L. Holl, Carl Holtom, Eberhard Hopf, R. E. Horton, L. A. Hostinsky, Harold Hotelling, E. M. Hove, W. A. Howard, H. K. Hughes, M. G. Humphreys, N. C. Hunsaker, C. C. Hurd, Witold Hurewicz, W. R. Hutcherson, C. A. Hutchinson, Børge Jessen, B. W. Jones, F. B. Jones, Shizuo Kakutani, G. K. Kalisch, L. H. Kanter, Irving Kaplansky, William Karush, Leo Katz, M. W. Keller, J. L. Kelley, Claribel Kendall, G. S. Ketchum, P. W. Ketchum, D. E. Kibbey, R. S. Kingsbury, J. R. Kline, H. S. Konijn,

H. L. Krall, M. S. Kramer, M. P. Lackey, O. E. Lancaster, Joseph Landin, R. E. Lane, R. E. Langer, H. D. Larsen, D. H. Leavens, W. G. Leavitt, J. R. Lee, Solomon Lefschetz, D. H. Lehmer, W. T. Lenser, W. J. LeVeque, Harry Levy, F. A. Lewis, M. T. Lewis, C. B. Lindquist, W. S. Loud, R. G. Lubben, R. B. McClenon, Dorothy McCoy, N. H. McCoy, L. H. McFarlan, E. J. McShane, G. W. Mackey, Saunders MacLane, Wilhelm Magnus, H. B. Mann, Morris Marden, Anna Marm, George Marsaglia, M. H. Martin, F. J. Massey, M. S. Matchett, J. R. Mayor, A. E. Meder, A. S. Merrill, B. C. Meyer, H. L. Meyer, R. R. Middlemiss, R. A. Miller, T. W. Moore, W. K. Moore, C. B. Morrey, F. R. Morris, D. R. Morrison, D. J. Morrow, S. B. Myers, W. M. Myers, Leopoldo Nachbin, E. D. Nering, J. D. Newburgh, August Newlander, C. V. Newsom, Jerzy Neyman, O. M. Nikodým, Ivan Niven, J. I. Northam, D. A. Norton, A. B. J. Novikoff, E. G. Olds, E. J. Olson, A. M. Ostrowski, J. C. Oxtoby, O. O. Pardee, L. E. Payne, Sam Perlis, R. P. Peterson, T. S. Peterson, M. M. Pihlblad, George Piranian, J. C. Polley, G. B. Price, R. C. Prim, A. L. Putnam, E. D. Rainville, J. F. Randolph, R. B. Rasmusen, L. T. Ratner, L. L. Rauch, M. O. Reade, O. H. Rechard, O. W. Rechard, M. S. Rees, P. K. Rees, W. T. Reid, Irving Reiner, I. M. Reiner, J. G. Renno, C. N. Reynolds, C. E. Rickart, F. A. Rickey, P. R. Rider, F. D. Rigby, L. G. Riggs, L. A. Ringenberg, E. K. Ritter, J. B. Robinson, R. M. Robinson, Arthur Rosenthal, J. B. Rosser, M. F. Rosskopf, Herman Rubin, W. A. Rutledge, Rafael Sanchez-Diaz, R. G. Sanger, L. J. Savage, A. C. Schaeffer, H. M. Schaerf, E. R. Schneckenburger, Lowell Schoenfeld, K. C. Schraut, Henry Schutzberger, I. E. Segal, George Seifert, B. R. Seth, A. S. Shapiro, H. C. Schaub, L. W. Sheridan, Jack Sherman, Seymour Sherman, Edward Silverman, Annette Sinclair, M. F. Smiley, A. H. Smith, F. C. Smith, G. W. Smith, H. L. Smith, S. S. Smith, H. K. Sohl, E. S. Sokolnikoff, I. S. Sokolnikoff, T. H. Southard, E. J. Specht, D. E. Spencer, V. E. Spencer, C. E. Springer, K. H. Stahl, B. M. Stewart, R. W. Stokes, E. B. Stouffer, D. M. Studley, Otto Szasz, A. H. Tappan, Alfred Tarski, A. H. Taub, Eugene Taylor, H. P. Thielman, J. M. Thomas, F. B. Thompson, R. M. Thrall, V. H. Tingey, Leonard Tornheim, H. M. Trent, A. W. Tucker, J. W. Tukey, A. R. Turquette, S. M. Ulam, J. L. Ullman, Gilbert Ulmer, W. R. Utz, H. S. Vandiver, A. H. Van Tuyl, V. J. Varineau, G. L. Walker, R. J. Walker, J. L. Walsh, S. E. Warschawski, M. A. Weber, Alexander Weinstein, P. A. White, G. W. Whitehead, A. L. Whiteman, W. F. Whitmore, Hassler Whitney, L. R. Wilcox, F. B. Wiley, S. S. Wilks, R. L. Wilson, G. M. Wing, R. M. Winger, Clement Winston, H. E. Wolfe, J. H. Wolfe, C. R. Wylie, J. L. Yarnell, L. C. Young, P. M. Young.

The Colloquium Lectures, on *Topological dynamics*, were presented by Professor G. A. Hedlund of Yale University on Tuesday afternoon and Wednesday, Thursday, and Friday mornings. Presiding were, in order, President J. L. Walsh, Vice President Hassler Whitney, Dean W. L. Ayers, and Professor W. H. Gottschalk.

On Wednesday morning an invited address, *Some recent investigations in almost periodic functions*, was delivered by Professor Børge Jessen of the University of Copenhagen and the Institute for Advanced Study. Professor E. F. Beckenbach presided. A second invited address, *Aposyndetic and non-aposyndetic continua*, was delivered Thursday afternoon by Professor F. B. Jones of the University of Texas. Professor S. B. Myers presided at this meeting.

President Walsh presided at the business meeting on Wednesday morning. It was announced that Dean R. G. D. Richardson had died during the summer, and those present stand a moment in respect for his memory. Those presiding at the sectional sessions were as follows: Professors S. E. Warschawski, A. H. Taub, N. H. McCoy, P. R. Rider (Joint session with I.M.S.), H. F. Bohnenblust, J. L. Kelley, J. M. Thomas, Reinhold Baer, R. M. Winger, J. W. Calkin, B. W. Jones, and A. H. Diamond.

Headquarters for the meeting were in the Third Dormitory. The extensive facilities of this and adjacent dormitories were open to those attending the meetings.

On Tuesday afternoon there was a tea at the home of President Stearns. Wednesday afternoon was devoted to an excursion to Rocky Mountain National Park. On Thursday evening a steak fry was held on Flagstaff Mountain. On this occasion a resolution was presented by Dr. C. V. Newsom on behalf of the four participating societies, thanking the University of Colorado, the members of the Department of Mathematics, and all those who had contributed to the arrangements and extended their hospitality.

The Council met at 8:00 P.M. on August 30 and at 10:00 A.M. on September 1.

The Secretary announced the election of the following fifty-one persons to ordinary membership in the Society:

Dr. Harry Igor Ansoff, Rand Corporation, 1500 4th St., Santa Monica, Calif.;
Professor Margaret Malone Baskerville, Shorter College, Rome, Ga.;
Mr. Martin Roy Bates, Cornell University, Ithaca, N. Y.;
Professor Raymond Frank Bell, Eastern Washington College of Education, Cheney, Wash.;
Mr. Harold Dean Brown, Department of Physics and Astronomy, University of Kansas, Lawrence, Kan.;
Mr. Charles Carpenter Buck, University of Michigan, Ann Arbor, Mich.;
Professor Pedro Gerardo Cabezas, University of Cuyo, Cuyo, Argentina;
Mr. Sanford Colbert, 500 West 140th Street, New York, N. Y.;
Miss Helen Frances Cullen, University of Michigan, Ann Arbor, Mich.;
Mr. Thomas Scott Dean, 5930 Palo Pinto, Dallas, Tex.;
Mr. George William Evans, II, New York University, New York, N. Y.;
Mr. William Buell Evans, University of Illinois, Urbana, Ill.;
Mr. William Forman, Brooklyn College, Brooklyn 10, N. Y.;
Mr. Christopher Paul Gadsden, Tulane University of Louisiana, New Orleans, La.;
Mr. Robert Doran Glauz, University of Michigan, Ann Arbor, Mich.;
Mr. James Francis Hannan, University of North Carolina, Chapel Hill, N. C.;
Mr. Julius Honig, 6801 Bay Parkway, Brooklyn 4, N. Y.;
Mr. David Warren Hullinghorst, 2525 Peniston St., New Orleans 15, La.;
Mr. William Robert Hydeman, U. S. Navy, 3810 39th St., N.W., Washington 16, D. C.;
Professor Samuel Jacob Jasper, Kent State University, Kent, Ohio;
Mr. Paul Constantine Karpov, 634 West 135th Street, New York, N. Y.;

Mr. Harold William Kuhn, Princeton University, Princeton, N. J.;
Mr. Edward David Lawler, 268 Bowman Ave., Merion Station, Pa.;
Mr. Ernest John MacKenzie, Whitney Institute School, Bermuda;
Dr. Wilhelm Magnus, California Institute of Technology, Pasadena 4, Calif.;
Professor Esteban Francisco R. Manfredi, University of LaPlata, La Plata, B. A., Argentina;
Mr. Raymond William Moller, Catholic University of America, Washington, D. C.;
Mr. Frederick Joseph Morton, 1421 North Broad Street, Philadelphia 22, Pa.;
Professor Maria Laura Moura Mousinho, University of Brazil, Rio de Janeiro, Brazil;
Sister Maria Socorro Piccirillo, Immaculata College, Immaculata, Pa.;
Mr. Henry Polowy, Ferris Hall, North Brothers Island, New York, N. Y.;
Mr. Lyle Eugene Pursell, Purdue University, Lafayette, Ind.;
Mr. Edgar Reich, Rand Corporation, Santa Monica, Calif.;
Dr. Murray Rosenblatt, Cornell University, Ithaca, N. Y.;
Miss Shirley Anne Rubenstein, University of Virginia, Charlottesville, Va.;
Mr. William Trumbo Sandlin, Box 1606, Huntington, W. Va.;
Mr. Joseph Arthur Schatz, Brown University, Providence 12, R. I.;
Dr. Francis James Scheid, Boston University, Boston 16, Mass.;
Mr. David Harry Shaftman, 1156½ E. 61st St., Chicago 37, Ill.;
Mr. Stanley Musgrave Shartle, Office of Surveyor of Hendricks County, Danville, Ind.;
Mr. Daniel Shanks, Naval Ordnance Laboratory, White Oak, Silver Spring, Md.;
Mr. Frank R. Simpson, Franklin Institute, Philadelphia, Pa.;
Mr. Irving Sklansky, City College, New York 31, N. Y.;
Mr. Thomas Joseph Smith, College of the Holy Cross, Worcester, Mass.;
Major Mariano del Rosario Talag, Philippine Military Academy, Baguio, Philippines;
Mr. Nathaniel Thompson, Seton Hall College, South Orange, N. J.;
Mr. Robert Wasserman, University of Michigan, Ann Arbor, Mich.;
Mr. William Swoll Winn, Millsaps College, Jackson, Miss.;
Mr. Otto Wolf, 2 Hinckley Pl., Brooklyn 18, N. Y.;
Mr. David Monaghan Young, Harvard University, Cambridge 38, Mass.;
Mr. Neal Zierler, 2318 Eutaw Place, Baltimore, Md.

It was reported that the following person had been elected to membership on nomination of the University of Toronto: Mr. Gerald Berman, teaching fellow.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with other mathematical organizations: London Mathematical Society: Mr. Abram Samoilovitch Besicovitch, Trinity College, Cambridge University; Mr. Arthur ap Gywnn, University College of Wales; Dr. Kurt Angustus Hirsch, King's College, University of Durham; Swiss Mathematical Society: Professor Felix Fiala, University of Neuchâtel.

The following appointments by President J. L. Walsh of representatives of the Society were reported: Professor Eric Reissner on the U. S. Committee on Theoretical and Applied Mechanics for a term ending December 31, 1952; Professor Hassler Whitney at Ceremonial Session of Semi-Centennial Meeting of American Physical

Society on June 16, 1949; Professor F. A. Lewis at the inauguration of Ralph Brown Draughon as President of Alabama Polytechnic Institute on May 12, 1949; Professor B. W. Jones at the Seventy-fifth Anniversay of the Colorado School of Mines on September 30–October 1, 1949; Professor C. B. Morrey, Jr., at the inauguration of Dr. J. E. Wallace Sterling as President of Stanford University on October 7, 1949.

The following additional appointments by the President were reported: Professor R. V. Churchill as a member of the Editorial Committee for Applied Mathematics Symposium Proceedings for three years beginning July 1, 1949 (Professor Churchill appointed by committee to serve as Chairman for one year beginning July 1, 1949); Professors A. W. Tucker (Chairman), G. T. Whyburn, and J. W. T. Youngs as a committee to consider the non-editorial problems in connection with the Memoirs; Professors Saunders MacLane (Chairman), J. L. Doob, Dean A. E. Meder, Jr., Professors G. B. Price and A. W. Tucker as a committee to consider institutional membership rates, terms of individual membership, and the relation of the Society to institutional members which publish mathematical journals.

It was reported that the Society had signed a contract with the Office of Air Research whereby a subsidy of $21,500 will be available for the support of Mathematical Reviews for a period of one year.

It was reported that an arrangement had been made with the McGraw-Hill Book Company for the publication of the Proceedings of the Third Symposium in Applied Mathematics.

The Council voted to recommend to the Board of Trustees its approval of an arrangement between the Society and the McGraw-Hill Book Company for providing a limited supply of reprints to authors of articles in the Proceedings of the Third Symposium in Applied Mathematics.

A reciprocity agreement has been established between the Society and the Polish Mathematical Society, the terms being similar to those of other reciprocity agreements.

The following resolution on the death of Dean R. G. D. Richardson was adopted by the Council:

> A long record of most distinguished service to the Society came to its close with the death, on July 17, 1949, of Roland George Dwight Richardson. Although for some years his health had been uncertain, the end came suddenly, on a trip to his native province of Nova Scotia.
>
> The first twelve years of Dean Richardson's career, after he received his doctor's degree at Yale in 1906, showed a very considerable ability in original investigation. His impetus was largely given by James Pierpont. He spent the year 1908–1909 at Göttingen where he was greatly influenced by Hilbert. Later he owed much to his

association with G. D. Birkhoff. His papers, concerned mostly with integral equations, calculus of variations, and boundary value problems, appeared in various American and foreign publications. Meantime, in 1907, he had begun his lifelong service as Professor, and later as Dean, at Brown University.

In 1915 he became Chairman of the Department of Mathematics at Brown, and turned his energies toward the upbuilding of that department to its present high position. If this had been his only service to American mathematics, it would have been a notable one. In his later years he had a leading part in making Brown a center of advanced instruction in applied mathematics. He also served for over twenty years as Dean of the Graduate School of Brown University. On his retirement in 1948 he was awarded the honorary degree of LL.D. by Brown University. Previously he had received honorary degrees from Acadia University and Lehigh University. For a year and a half before his death, he was Vice President of the American Academy of Arts and Sciences.

This burden of administrative work for his own university makes all the more impressive his devotion to the affairs of the Society. In 1920 reorganization of the offices of the Society became urgent with the retirement of Professor F. N. Cole who, with failing health, had insisted on personally attending to all clerical details. Dean Richardson was a Vice President at that time and was familiar with the business of the Society. When asked to become Secretary, he consented, though fearing that this would definitely end his career as a producing mathematician. He believed, however, that if his plans for the Society could be realized, the sacrifice would be more than worthwhile.

His unequalled services to the Society were fittingly described in a resolution adopted by the Council at the time of his retirement from the office of Secretary in 1940, as published in the Bulletin of March, 1941. This should be supplemented, however, by at least a mention of his membership on the Board of Trustees continuously since 1923.

His administration was never static. He always had a plan, well worked out, for new situations, yet never was over-insistent on his own views, and would listen courteously to others who were advancing perhaps less well-thought ideas. His sense of order and efficiency contributed to make the offices of the Society a model to other scientific organizations.

But with all due regard to what he did for this Society and his university, the highest tribute is expressed in this sentence of the 1940 resolution: "Richardson's real memorial lies, however, in the affection and admiration for him which are widespread among the present members of the Society, and universal among those who have known him personally and worked with him."

The following dates of meetings in 1950 were approved: February 25 in New York City; February 24–25 in East Lansing, Michigan; April 21–22 in Oak Ridge, Tennessee; April 28–29 in Washington, D.C., April 28–29 in Chicago, Illinois; April 29 in Berkeley, California; June 16–17 in Seattle, Washington.

Certain invitations to deliver hour addresses were announced: Professor R. H. Fox for the October, 1949, meeting in New York City; Professor Richard Bellman for the November, 1949, meeting in Pasadena; Professor R. H. Bruck for the November, 1949, meeting in

Chicago; Professors L. V. Ahlfors and S. Chowla for the 1949 Annual Meeting.

Professor G. A. Hedlund was elected (subject to confirmation by the Board of Trustees) as a member of the Bulletin Editorial Committee, to serve for the unexpired term of Professor Deane Montgomery, resigned (that is, until December 31, 1951).

Professors H. F. Bohnenblust and L. R. Ford were elected representatives of the Society on the Council of the American Association for the Advancement of Science for the years 1950 and 1951.

Professors G. C. Evans and Einar Hille were nominated as representatives of the Society in the Division of Mathematical and Physical Sciences of the National Research Council for a three-year period beginning July 1, 1950.

Professor C. B. Allendoerfer and Dr. R. P. Boas were elected representatives of the Society on the Editorial Board of the Duke Mathematical Journal for a period of three years beginning January 1, 1950.

The Council accepted the recommendation of the Committee to Nominate an Executive Director that Dr. H. M. MacNeille, now with the Atomic Energy Commission, be appointed to this position, subject to the approval of the Board of Trustees.

The Council accepted, with certain amendments, a report of the Committee on the Role of the Society in Mathematical Publication dealing with the problem of what constitutes publication.

Recommendations in connection with institutional membership dues were accepted by the Council as follows: that the page rate for institutional dues be raised to $5.00 per page (present rate is $2.75 per page); that dues be based on averages computed triennially over three-year periods, with certain provisions for the transition period; that the page counts for institutional dues cover the American Journal, Annals, Bulletin, Transactions, Proceedings (when established) and the Canadian Journal; that the privileges for institutional members be revised according to a table determined by the Council and approved by the Trustees; that the new scale for institutional members go into effect beginning January 1, 1951.

It was reported that the Memoirs would be inaugurated in the near future, as soon as arrangements could be made for the preparation of manuscripts already accepted.

The Council voted to invite Professor G. E. Uhlenbeck of the University of Michigan to deliver the Josiah Willard Gibbs Lecture in 1950.

The Committee on Revisions of By-Laws recommended certain changes in connection with (1) the time of the annual meeting of the Board of Trustees; (2) the method of arranging for temporary replacements on Editorial Committees; (3) the inauguration of the Proceedings and Memoirs. These changes will be acted upon at the October meeting of the Society in New York and will be effective January 1, 1950.

The Council voted to abolish the Committee on Publicity; this action was taken at the request of the committee and it was the sense of the meeting that the activities of this committee would now be carried on by the Executive Director.

The Council voted to adopt the following resolution in connection with the loyalty oath section of the National Science Foundation Bill being considered by the House of Representatives:

> The Council of the American Mathematical Society expresses its hearty approval of the National Science Foundation Bill (S247) as passed by the United States Senate, and confidently expects that an early passage of a similar bill by the House of Representatives will be a potent factor in the development of science in this country.
>
> With respect to the amendments introduced into the Bill by the House committee, the Council deplores the amendment inserting a new section, Section 10(b), which introduces non-scientific objectives.

Abstracts of papers presented at the meeting follow below. Papers whose titles are followed by "*t*" were read by title. Some of the abstracts of papers read by title were published in September. Mr. Conrad and Mr. Goss were introduced by Professor D. L. Holl, Professor Buchi and Professor Magnus by Professor Saunders MacLane, Professor Spencer by Professor R. B. Lindsay, Professor Dresden and Mr. Brown by Professor G. B. Price, Mr. Calderon by Professor Zygmund, Mr. Lieblein, Dr. Castellani, and Professor Chang by Professor J. W. Green. Paper 515 was presented by Professor Herzog, paper 536 by Mr. Riggs, paper 568 in the absence of the author by Dr. H. M. Trent, paper 470 by Professor Perlis, paper 577 by Mrs. Szmielew, paper 586 by Dr. Sherman, paper 581 by Professor Dolph, paper 517 by Professor Hewitt, paper 523 by Dr. Farnell, paper 472 by Professor Tucker, paper 464 by Professor Barnett, paper 473 by Professor Givens, paper 531 by Professor Piranian, paper 462 by Professor Abbott, and paper 563 by Dr. Van Tuyl.

ALGEBRA AND THEORY OF NUMBERS

462. J. C. Abbott and T. J. Benac: *Right congruences on groupoids.*

A right congruence θ on a right quasi-group G is any equivalence preserved by the group $\mathfrak{R}$ generated by the right multiplications R_y of G. If G contains a left unit, then every coset has the same power. If either the order or index of θ is finite, then any equivalence preserved by the R_y alone is a right congruence (cf. Albert, Trans. Amer. Math. Soc. vol. 54 (1943) Theorem 6, and Bates and Kiokemeister, Bull. Amer. Math. Soc. vol. 54 (1948) Theorem 4). The set of right congruences forms a complete lattice which is not modular since it does satisfy Dubreil's condition (Dubreil, *Algèbre*, 1946). If the kernel of the right congruence is called right normal, then a sub-right quasi-group will be right normal if and only if $H\mathfrak{J}_r = H$ where $\mathfrak{J}_r$ is the right inner mapping group of G (Bruck, Trans. Amer. Math. Soc. vol. 60 (1946)). $\mathfrak{J}_r$ contains no subgroup normal in $\overline{\mathfrak{R}}$, and the set of R_y's forms a complete set of representatives of G mod θ, so that $(\overline{\mathfrak{R}}/\mathfrak{J}_r; R_y)$ is a Baer canonical representation of G. Finally, every right congruence of G can be extended to a congruence on $\overline{\mathfrak{R}}$. (Received July 22, 1949.)

463. R. V. Andree: *The sequence development of g-adic rings.*

The usual definition of valuation is modified by replacing the equality in $\phi(xy) = \phi(x)\phi(y)$ by $\leq$. The g-adic numbers (g prime or composite) are then obtained as sequences of rational numbers which converge with respect to this generalized valuation. It is proved that for every g containing at least two prime factors, and for every integer n, there exists an equation of degree n having no solution in Ω_g. It is established that every g-adic ring is isomorphic to a direct sum of those p-adic fields for which p divides g. Through this isomorphism a finite process is obtained for determining the exact number of distinct solutions a given g-adic equation will have, and a non-tentative method for producing every solution is developed. In conclusion it is proved that every g-adic ring is perfect (complete) with respect to the generalized valuation. (Received June 21, 1949.)

464. I. A. Barnett and C. W. Mendel: *Fermat's last theorem in matrices with integral elements.*

The equations $X^n + Y^n + Z^n = 0$ are considered where X, Y, Z are square matrices of the same order with integral or zero elements. If X and Y are commutative, the most general solution in rationals of the equation is found when $n = 2$. Solutions for any n in integral matrices of order n are exhibited. For $n = 2$, the general solution for 2×2 integral matrices is given. Finally, with the Fermat conjecture assumed to be true, we let X be a two-rowed matrix with first row x_1, x_2 and second row x_3, x_4, and so on, and $D \neq 0$, where D is the three-rowed determinant with first row $x_1 - x_4$, $y_1 - y_4$, $z_1 - z_4$, second row x_2, y_2, z_2, and third row x_3, y_3, z_3. Then the only solution of the equation in 2×2 integral matrices for n odd are those for which the traces of two of the matrices are numerically equal with the corresponding determinants equal, while the trace and the determinant of the third matrix are both zero. (Received July 14, 1949.)

465. P. T. Bateman: *On the distribution of kth-power-free integers.*

Known prime number theory implies that the number of kth-power-free integers not exceeding x is $x/\zeta(k) + O(x^{1/k} \exp\{-A(\log x)^\alpha\})$, where A is a positive constant and α is a constant between 1/2 and 1, in fact, the same constant that appears in the error term for the number of prime numbers not exceeding x. It is pointed out here that the Riemann hypothesis implies only that the above error term can be improved to $O(x^{2/(2k+1)+\epsilon})$. These error terms give upper estimates for the differences between consecutive kth-power-free integers. On the other side it is shown that the difference between a kth-power-free integer q and the next kth-power-free integer is greater than $(\log q)/k(\log \log q)$ infinitely often. (Received July 18, 1949.)

466. J. H. Bell: *The solution of a unilateral direct product matrix equation.*

The author's Bull. Amer. Math. Soc. Abstract 54-11-417 gives necessary and sufficient conditions for the solution of the matrix equation $\sum_{m=0}^{s} A_m \cdot xX^m = 0$ over a field $\mathcal{J}$. In this paper the equation is generalized to $\sum_{m=0}^{s} A_m \cdot x(K_m X^m) = 0$. It is shown that the solution of this generalized equation may be reduced to the solution of r simultaneous unilateral equations $\sum_{m=0}^{s} a_{m,ij} K_m X^m = 0$, where $A_m = (a_{m,ij})$ and r is the number of linearly independent matrices in the set $A_s, A_{s-1}, \cdots, A_0$. If the dimensions of A_m, K_m, and X are such that $\sum_{m=0}^{s} A_m \cdot x(K_m X^m)$ is a square matrix, an alternative method of solution, patterned after Ingraham's algorithm for the solution of the unilateral matrix equation, is given. In the development of this method a remainder theorem and a divisor theorem, involving direct products, are obtained. (Received June 10, 1949.)

467. A. W. Boldyreff: *Reduction of homogeneous diophantine equations to systems of linear homogeneous equations.*

This paper considers the possibility of reduction of homogeneous diophantine equations to systems of linear homogeneous equations, directly or with the aid of algebraic identities, and develops methods of deducing the solutions of the former from those of the latter. The methods are illustrated by applications to a number of well known classical problems, and it is shown how they can be modified to apply to the solution of non-homogeneous equations. (Received August 13, 1949.)

468. J. R. Buchi: *Boolean orderings and pairings.*

The paper has its origin in the idea of an axiomatic treatment of the inclusion relation of set theory. By taking inclusion as primitive rather than the relation "is element of," it becomes unnecessary to assume that there are atoms (as the elements in ordinary set theory). The best way to carry out such an axiomatic approach is to take the disjunction relation as a second primitive. Four simple axioms complete the axiomatic system. It is shown that inclusion in the above context is a special type of partial ordering called Boolean ordering. The importance of this Boolean ordering in the theory of Boolean algebras is illustrated. For example, every Boolean ordered set can uniquely be embedded in a smallest complete Boolean algebra (analogy to the embedding of a general partially ordered set in a complete lattice). In the second part of the paper a concept called pairing is introduced. In terms of this concept the ordinary notion of mappings of sets is generalized to mappings of Boolean orderings. In other words, a definition of the notion "mapping" is given in the case where the structors which are to be mapped are not sets and therefore do not have atoms. (Received July 18, 1949.)

469t. Leonard Carlitz: *Some properties of Bernoulli numbers of higher order.*

Among the results of this paper, the following may be cited. 1. Let $k < p-1$, $m \not\equiv 0, 1, \cdots, k-1 \pmod{p-1}$, then $B_m^{(k)}/m^{(k)}$ is integral $(\text{mod } p)$, $m^{(k)} = m(m-1) \cdots (m-k+1)$. Also $\sum_{i=0}^{r}(-1)^i C_{r,i} T_{m+i(p-1)} \equiv 0 \pmod{p^r}$, where $T_m = B_m^{(k)}/m^{(k)}$. (The

$B_m^{(n)}$ notation is that of Nörlund.) 2. $B_p^{(p)} \equiv p^2/2 \pmod{p^3}$, $B_{p+2}^{(p+1)} \equiv 0 \pmod{p^3}$ $(p>3)$. The special case $B_{p+2}^{(p+1)} \equiv 0 \pmod{p^2}$ is due to S. Wachs, Bulletin des Sciences Mathématiques (2) vol. 71 (1947) pp. 219–232. (Received July 14, 1949.)

470. Roy Dubisch and Sam Perlis: *On total nilpotent algebras.*

A total nilpotent algebra $F^{(n)}$ of degree n over a field F is defined to be the totality of n-rowed square matrices over F with 0's on and above the main diagonal, or any isomorphic copy thereof. The importance of $F^{(n)}$ lies in the fact that every nilpotent associative algebra of order $n-1$ is isomorphic to a subalgebra of $F^{(n)}$, which itself is nilpotent. The authors determine all ideals of $F^{(n)}$, the group G of all automorphisms over F of $F^{(n)}$, some of the structural properties of G, all characteristic ideals C (ideals C such that $C^T = C$ for every T in G). Certain characteristic ideals D give rise to all others by the processes of intersection and join. The internal structural properties of these ideals D are investigated. (Received July 18, 1949.)

471. J. S. Frame: *A simple recursion formula for inverting a matrix.*

A simple recursion formula is obtained for computing the inverse of a matrix, and as a biproduct the characteristic equation, the λ-adjoint, and the characteristic vectors of the matrix. Let the $n \times n$ matrix A have the characteristic equation (1) $|\lambda I - A| = F(\lambda) = \lambda^n - \sum_1^n c_k \lambda^{n-k} = 0$, let the adjoint of $\lambda I - A$ be denoted by (2) $C(\lambda) = A_0 \lambda^{n-1} + A_1 \lambda^{n-2} + \cdots + A_{n-1}$, so that (3) $(\lambda I - A) C(\lambda) = F(\lambda) I$, and define $A_n = 0$. It is proved that we may compute successively $c_1, A_1, c_2, A_2, \cdots, c_{n-1}, A_{n-1}, c_n, A_n = 0$ by means of the two formulas (4) $c_k = (1/k)$ trace (AA_{k-1}), (5) $A_k = AA_{k-1} - c_k I$. In particular the determinant and inverse are obtained as $|A| = (-1)^{n-1} c_n$, $A^{-1} = A_{n-1}/c_n$, if $c_n \neq 0$. Formula (4) is derived from (6) $F'(\lambda) =$ trace $C(\lambda)$, and (7) trace $AC(\lambda) = \lambda F'(\lambda) - nF(\lambda)$. The computation is finally checked if A_n comes out 0, but a method is given for correcting small errors without repeating the calculation. The relation $AA_k = A_k A$ provides a further check at each stage. The nonvanishing columns of $C(\lambda_k)$ are characteristic vectors if λ_k is a simple root of $F(\lambda) = 0$. The characteristic vectors for multiple roots are found explicitly from the derivatives of $C(\lambda)$, and the results are applied to the solution of simultaneous linear differential equations with constant coefficients. (Received June 4, 1949.)

472. David Gale, Harold W. Kuhn, and A. W. Tucker: *Reductions of game matrices.*

Let a game matrix A of m rows and n columns be reduced to an $m-p+1$ by $n-q+1$ matrix A^* by combining linearly the first p rows of A into a single leading row and the first q columns into a single leading column, using as coefficients the components of a pair of optimal strategies for the game determined by the p by q submatrix common to the p rows and q columns. Two special cases of this reduction are noteworthy: (1) when the first p rows of A have the same last $n-q$ elements, and the first q columns have the same last $m-p$ elements; (2) when the coefficients are all positive, and the new leading element is a saddlepoint for A^*. In both cases A^* has the same game-value as A. In the first case a pair of optimal strategies for A^* unites with the two sets of coefficients to give a pair of optimal strategies for A. In the second case the two sets of (positive) coefficients themselves constitute a pair of optimal strategies for A. Results of Bohnenblust-Karlin-Shapley, Gale, and Sherman show that this reduction to a saddlepoint matrix is always possible, the maximal sets of p rows and q columns being unique. (Received July 18, 1949.)

473. Wallace Givens and W. A. Rutledge: *Symmtric and skew Hadamard matrices.* Preliminary report.

A square matrix H of order n is Hadamard if each of its elements is ± 1 and $H'H=nI_n$; n must be 2 or a multiple of 4 with existence for every such value being in doubt. If H is symmetric, it is shown that the signature is zero if n is not a square and is an even integer between $-n^{1/2}$ and $+n^{1/2}$ if n is a square. For various forms of n, symmetric H are constructed and for $n=2^{2k}$ all permissible signatures are obtained. When $H=I_n+S$ where S is skew symmetric, the invariant factors of S (over the ring of rational integers) are half $+1$ and half $(n-1)$ if $(n-1)$ is square free. If $(n-1)$ has a square factor they are restricted otherwise. The tensor set associated under the spinor representation of the orthogonal group with an Hadamard matrix of either of the above types has especially simple features. (Received July 18, 1949.)

474. L. Aileen Hostinsky: *Endomorphisms of lattices.*

The object of this paper is to develop a theory of homomorphisms for lattices by generalizing the concept of homomorphism as used in group theory. In addition to a derivation of the basic isomorphism theorems, a generalization of Fitting's lemma and of the theory of splitting endomorphisms is given. The main results are established by the introduction of the concept of an η-automorphic element. It is shown that a necessary condition for uniform splitting is that the sum of η-automorphic elements be η-automorphic and that under the assumption of uniform splitting the complement is uniquely determined as the sum of η-automorphic elements. Reduction theorems and certain necessary and sufficient conditions for uniform splitting are included. (Received July 5, 1949.)

475t. L. K. Hua: *A theorem on matrices over a sfield and its applications.*

Let Φ be a sfield (or a division ring). Suppose $1<n\leqq m$. Two matrices are called coherent if the rank of their difference is one. Any one-to-one mapping which carries $n\times m$ matrices into $n\times m$ matrices and leaves the coherence invariant is of the form $Z_1=PZ^{\sigma}Q+R$, where $P(=P^{(n)})$ and $Q(=Q^{(m)})$ are non-singular, and R is an $n\times m$ matrix and σ is an automorphism of the sfield Φ. When $m=n$, in addition to the previous one, we have also $Z_1=PZ'^{\sigma}Q+R$, where Z' is a transpose of Z and σ is an anti-automorphism of the sfield. There are quite a number of applications of the theorem. This paper contains three of them: (i) Grassmann geometry, (ii) Jordan ring, and (iii) Lie ring. For example, for the Jordan ring the author proves that any Jordan automorphism of a semi-simple ring with descending chain conditions for the left ideals is either an automorphism or an anti-automorphism. (Received July 12, 1949.)

476t. L. K. Hua: *On exponential sums over an algebraic number field.*

Let K be an algebraic field of degree n over the rational field, and d be the different of the field. Let $f(x)=\alpha_k x^k+\cdots+\alpha_1 x+\alpha_0$ be a polynomial of the kth degree with coefficients in the field K, and let a be the fractional ideal generated by $\alpha_k, \cdots, \alpha_1$. Let $ad=rq$, where r and q are two relatively prime integral ideals. Then the following result is proved: Let $S(f(x),q)=\sum e^{2\pi i\,\mathrm{tr}\,(f(x))}$, where x runs over a complete residue system, mod q. Then, for any given $\epsilon>0$, we have $S(f(x),q)=O(N(q)^{1-1/k+\epsilon})$, where the constant implied by the symbol O depends only on k, n, and ϵ. (Received July 12, 1949.)

477*t*. L. K. Hua: *On semi-automorphisms of a sfield.*

Let K be a sfield. A one-to-one mapping $x \to x^\sigma$ of K onto itself is called a semi-automorphism if $(a+b)^\sigma = a^\sigma + b^\sigma$ and $(aba)^\sigma = a^\sigma b^\sigma a^\sigma$. It is proved in the paper that any semi-automorphism is either an automorphism or an anti-automorphism. This settles a problem of Ancochea. As a consequence we solve the fundamental theorem of the projective geometry over a sfield. (Received July 12, 1949.)

478*t*. L. K. Hua: *On the automorphisms of classical groups.*

In this paper the author solves several problems which were left open by Dieudonné (C. R. Acad. Sci. Paris vol. 225 (1947) pp. 914–915) about the automorphisms of the classical groups. The groups under consideration are $GL_2(K)$, $PGL_2(K)$, $SL_4(K)$, and $PSL_4(K)$ for K any sfield, and $O_4^+(K)$ and $PO_4^+(K)$ for K any field. The situation concerning $GL_2(K)$ and its related groups is unexpectedly successful, since there is no obscure notion like semi-automorphism in the final result. Concerning $O_4^+(K)$ we found some new automorphisms which were not known before. (Received July 12, 1949.)

479*t*. L. K. Hua: *Some properties of a sfield.*

The results of the paper are initiated from an almost trivial identity: if $ab \neq ba$, then $a = (b^{-1} - (a-1)^{-1}b^{-1}(a-1))(a^{-1}ba - (a-1)^{-1}b^{-1}(a-1))^{-1}$. Among the results, there are two interesting ones which are the perfect form of two theorems due to H. Cartan and J. Dieudonné respectively. (i) Every sfield is generated by a non-central element and its conjugate. (ii) The projective special linear group over the sfield K is simple except when $n=2$ and K has 2 or 3 elements. It is easily proved that both the upper central series and the lower central series of the multiplicative group of a sfield do not exist. The author proved also that the multiplicative group of a sfield is either abelian or non-metabelian. (Received July 12, 1949.)

480*t*. Nathan Jacobson and C. E. Rickart: *Jordan homomorphisms of rings.*

A mapping $a \to a^J$ of an arbitrary ring A into a second ring B is called a Jordan homomorphism provided $(a+b)^J = a^J + b^J$, $(a^2)^J = (a^J)^2$, $(aba)^J = a^J b^J a^J$. If B is an integral domain, then $a \to a^J$ is either a ring homomorphism or an anti-homomorphism. The proof of this is elementary and is an extension of a proof given by L. K. Hua (unpublished) for the case of a Jordan automorphism of a division ring. If A and B are primitive rings with minimal ideals and $a \to a^J$ is one-to-one *onto* B, then $a \to a^J$ is either a ring isomorphism or an anti-isomorphism. If A contains a system of matrix units $e_{ij}(i, j=1, \cdots, n; n \geqq 2)$ and a subring C whose elements commute with the e_{ij} and such that every element of A is of the form $\sum c_{ij}e_{ij}$, $c_{ij} \in C$, then there exists a direct sum decomposition $E_J = E' + E''$ of the enveloping ring E_J of A^J such that $a^J = a^{J'} + a^{J''}$, where $a \to a^{J'}$ is a ring homormophism onto E' and $a \to a^{J''}$ is a ring anti-homomorphism onto E''. The proof here is suggested in part by methods used previously by F. D. Jacobson and N. Jacobson [Trans. Amer. Math. Soc. vol. 65 (1949) pp. 141–169]. (Received July 18, 1949.)

481. D. H. Lehmer: *On the distribution of totitives.*

This paper is concerned with the distribution of the numbers not exceeding n and prime to n, whose number is Euler's function $\phi(n)$. Following Sylvester, these numbers are called the totitives of n. The function $\phi(k, q, n)$ is defined (for integer q, k, n, such that $0 \leqq q < k$) as the number of totitives of n lying between nq/k exclusive, and $n(q+1)/k$ inclusive. To measure the departure from uniformity in the distribution of the totitives we define the function $E(k, q, n) = \phi(n) - k\phi(k, q, n)$. Properties of $E(k, q, n)$ are studied. This function can be evaluated explicitly for $k = 1, 2, 3, 4$, and 6 in terms of the Liouville function $\lambda(n)$ and the Euler function $\theta(n) = 2^t$, where t is the number of distinct primes dividing n. Special results hold for other values of k. $E(k, q, n)$ is uniformly small in q. Its mean value for $n \leqq x$ is, in absolute value, less than $k \log x$ for every $q < k$. Applications are given to certain problems in cyclotomy. (Received July 18, 1949.)

482t. Benjamin Lepson: *Some examples related to the $\alpha+\beta$ hypothesis in the theory of Schnirelmann density.*

The $\alpha+\beta$ hypothesis, first proved by Mann (Ann. of Math. vol. 43 (1942) pp. 523–527), states that the density γ of the sum of two sets, each containing zero, and of densities α and β respectively, is at least min $(1, \alpha+\beta)$. It is shown in the present paper that, for any two non-negative real numbers α and β with $\alpha+\beta \leqq 1$, the g.l.b. of the numbers γ, each of which is the density of a set which is the sum of a set of density α containing zero and a set of density β containing zero, is $\alpha+\beta$. If α and β are both rational, there is a set as above whose density is equal to $\alpha+\beta$. This proves that the $\alpha+\beta$ hypothesis is best possible in the strong sense that the function $\alpha+\beta$ cannot be replaced by another function $f(\alpha, \beta)$ with $f(\alpha, \beta) > \alpha+\beta$ for some pair (α, β). For the rational case, the construction involves congruence classes of integers. The general case depends upon the lemma that any set A of density α contains a subset B of density α' where α and α' are any real numbers satisfying the inequality $0 \leqq \alpha' < \alpha \leqq 1$. (Received July 18, 1949.)

483. Wilhelm Magnus: *On the Baker-Hausdorff formula.*

If $\exp x \exp y = \exp z$ where x, y, are noncommutative variables, then z is a sum of alternantes or Lie-elements of x, y with rational coefficients. The terms whose coefficients have denominators which are divisible by p (a fixed prime number) are shown to lie (possibly with a finite number of exceptions) in an ideal which can be derived from a single element by means of algebraic substitutions and Baker-Hausdorff differentiation. From this a method can be derived to construct groups for which the pth power of every element equals the unit element and which are of a class greater than p. (Received July 18, 1949.)

484. D. A. Norton: *Hamiltonian loops.*

A loop G is Hamiltonian if every subloop is normal. It is power-associative if $\{x\}$ is a group for every $x \in G$. If G is the direct product of Hamiltonian loops M, N, it is Hamiltonian if and only if for each pair H, K of subloops of M, N respectively which contain subloops H', K' such that $H/H' \cong K/K'$, then H/H' and K/K' are in the centers of M/H' and N/K' respectively. Given any series of integers excluding 4, there exists a loop which contains only one series of composition, with these as factors of composition, and which contains no other proper subloops. A finite Hamiltonian power-associative loop is the direct product of prime-power order loops. A finite Hamiltonian Moufang loop is a direct product of an Abelian group of odd order, an Abelian group of order 2^m, type $[2, 2, \cdots, 2]$, and a loop of order 2^n of which the center is the two-group, each two independent elements generate a quaternian group, and each three generate a Cayley loop (the multiplicative loop of the basis elements of the Cayley-Dickson algebra). (Received July 14, 1949.)

485*t*. K. G. Ramanathan: *The theory of units of quadratic and hermitian forms.*

Let K be a totally real algebraic number field of finite degree $f>1$ over the field of rationals. Let S be the matrix of a nondegenerate quadratic form with m variables and with elements integral in K. Let n_r, $m-n_r$ $(r=1, \cdots, f)$ be the signatures of the conjugates of S. Let Γ be the group of units u of S in which u and $-u$ are identified. The following results are proved. (1) Γ is finite if and only if S is total-definitive. (2) Γ is finitely generated. (3) There is a space H of $\sum_{r=1}^{f} n_r(m-n_r)$ dimensions in which Γ has a faithful and discontinuous representation as a group G of non-euclidean motions leaving H invariant. (4) In H there exists a fundamental domain F in regard to G bounded by a finite number of hyperplanes. (5) In H can be defined an invariant volume element measured with which F has a finite volume V. (6) V, apart from a constant factor, is the reciprocal of the order of Γ if it is finite. Exactly similar results are proved for the case when S is an Hermitian matrix with integral elements in $K(d^{1/2})$ where d is a totally-negative integer of K. (Received July 18, 1949.)

486*t*. W. T. Reid: *A note on the characteristic polynomials of certain matrices.*

This note establishes the following results for matrices with elements in an arbitrary field: If A is an $n\times m$ matrix and D is an $n\times n$ matrix, then $|\lambda I-AB| = |\lambda I-AB-D|$ for arbitrary $m\times n$ matrices B if and only if D is nilpotent and $DA=0$. In particular, it follows that if C is an $m\times n$ matrix then the condition $ACA=0$ is necessary and sufficient for $|\lambda I-AB| = |\lambda I-A(B+C)|$ for arbitrary $m\times n$ matrices B; the sufficiency of this condition has been obtained recently by W. V. Parker [Bull. Amer. Math. Soc. vol. 55 (1949) pp. 115–116]. (Received July 18, 1949.)

487*t*. C. E. Rickart: *Isomorphism of groups of linear transformations.* II.

Consider a system $(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*: \mathcal{G})$ in which $\mathfrak{X}$, $\mathfrak{X}^*$ are respectively right and left linear vector spaces over the division ring $\mathcal{D}$, which are dual in the sense of Jacobson [Ann. of Math. vol. 48 (1947) pp. 8–21] with dimension greater than two, and $\mathcal{G}$ is a group of linear transformations on $\mathfrak{X}$ under the operation $A\circ B=A+B-AB$. Assume further that every $A\in\mathcal{G}$ possesses an adjoint (loc. cit., p. 16) and that $\mathcal{G}$ contains every one-dimensional involution which possesses an adjoint (T is an involution if $T\circ T=0$). Let $(\mathfrak{Y}, \mathcal{E}, \mathfrak{Y}^*: \mathcal{H})$ be a second such system and assume $\mathcal{G}$ and $\mathcal{H}$ isomorphic as groups under the mapping $G\rightarrow g(G)$. It is proved here that there exists a homomorphism $G\rightarrow G^c$ of $\mathcal{G}$ into the center of $\mathcal{E}$ (under the operation $\alpha\circ\beta=\alpha+\beta-\alpha\beta$) such that $g(G)$ either has the form $g(G)=(\phi^{-1}G\phi)\circ IG^c$, where $x\rightarrow x\phi$ is an isomorphism of the space $\mathfrak{X}$ onto the space $\mathfrak{Y}$, or $g(G)^*$ has an analogous form involving the inverse of G and an anti-isomorphism $x\rightarrow x\phi^*$ of $\mathfrak{X}$ onto $\mathfrak{Y}^*$. This result essentially includes a similar result of Dieudonné for the finite-dimensional case [C. R. Acad. Sci. Paris vol. 225 (1947) pp. 914–915] and uses a result previously announced by the author [Amer. Math. Soc. Abstract 54-11-440]. (Received July 18, 1949.)

488t. C. E. Rickart: *Representation of certain Banach algebras on Hilbert space.*

Let A denote a real Banach algebra which contains a minimal ideal R and in which there is defined a mapping $x \to x^*$ satisfying the following three conditions: (1) $(x^*)^* = x$, (2) $(xy)^* = y^*x^*$, (3) $xx^* = 0$ implies $x = 0$. Then there exists an idempotent $e \in R$ such that $e^* = e$, $R = eA$, and eAe is either the reals, quaternions, or complexes. Let R have dimension greater than one, greater than four, or infinite according as eAe is the reals, quaternions, or complexes. Then a real inner product (x, y) can be introduced into R such that $(xa, y) = (x, ya^*)$ for all $x, y \in R$ and $a \in A$. In general, additional conditions are needed to ensure that the norm $|x| = (x, x)^{1/2}$ be equivalent to the given norm in R. The results here include those of Kakutani and Mackey [Ann. of Math. vol. 45 (1944) pp. 50–58; Bull. Amer. Math. Soc. vol. 52 (1946) pp. 727–733] on the characterization of Hilbert space in terms of its ring of bounded operators. (Received July 18, 1949.)

489. R. M. Robinson: *Undecidable rings.*

The solution to the decision problem is shown to be negative for the ring of algebraic integers in a quadratic field, for a polynomial ring over a field, and for various other rings. These results follow from the fact that all problems of the arithmetic of natural numbers can be stated in the arithmetical theory of each of the rings mentioned. (Received June 27, 1949.)

490. J. B. Rosser: *Real roots of Dirichlet L-series.*

Let k be a positive integer and take any real character (mod k) and let L be the corresponding Dirichlet L-series (see Landau, *Vorlesungen über Zahlentheorie*, vol. 1, pp. 83–88). In the present paper it is proved that for $2 \leqq k \leqq 67$, L has no positive real zeros. (Received July 5, 1949.)

491t. W. E. Roth: *On the equations $AX - YB = C$ and $AX - XB = C$ in matrices.*

The necessary and sufficient condition that the equation $AX - YB = C$, where A, B, and C are rectangular matrices with elements in any field F, have a solution is that the 2-rowed matrix with first row A, C and second row 0, B and the two-rowed matrix with first row A, 0 and second row 0, B be equivalent. In case $X = Y$ and the matrices A, B, and C are square, the condition is that these matrices be similar. (Received July 14, 1949.)

492t. W. E. Roth: *On the matric equation $X^2 + AX + XB + C = 0$.*

The equation $X^2 + AX + XB + C = 0$, where A, B, and C are $n \times n$ matrices with elements in F, the field of complex numbers or a subfield thereof, is solved for $n \times n$ matrices X with elements in F. The methods are entirely rational under the operations of F. Though the equation becomes unilateral if X be replaced by $Y - A$ or by $Z - B$, this fact is not utilized for it was noted by the author only after the results of the present paper had been discovered. (Received July 5, 1949.)

493. Herman Rubin: *Postulates for the existence of measurable utility and pyschological probability.*

Von Neumann and Morgenstern in *Theory of games and economic behavior* show that under certain axioms a rational man will maximize a function, called the *utility* *function*, of probability distributions of future histories of nature. This function is unique except for a monotonic linear transformation. The present paper considers, instead of distributions of future states of nature, prospects, that is, functions from the present state to them. It is assumed that prospects are partially ordered, that given any two prospects, a rational man can select a unique random combination of them which he prefers, that if he cannot influence the consequences of a random event E, his choice will be independent of E, and that if a prospect is between two others, it is equivalent to a random combination of them. These postulates yield a utility function. If it is further assumed that every value of a prospect is the value of a constant prospect, then we can map the prospects linearly into real-valued functions on the states of nature. If these functions form a lattice, and utility of a prospect is monotone, then a rational man will act as if he believes in psychological probability. (Received July 16, 1949.)

494t. Peter Scherk: *Decomposition of orthogonalities into symmetries.*

Let A, G, T denote $n\times n$ matrices over a field of characteristic $\neq 2$. The fixed matrix G is assumed to be regular and symmetric. T is called orthogonal if $T'GT=G$. If A is orthogonal and rank $(A-I)=1$, A is called a symmetry [I=unit matrix]. From a theorem by Cartan every orthogonality can be decomposed into a product of n or less than n symmetries [cf. Dieudonné, *Sur les groupes classiques*, Actualités Scientifiques et Industrelles, no. 1040, 1948, pp. 20–22]. Let $m=\text{rank }(T-I)$. The author proves: If $G(T-I)$ is not skew-symmetric, then T can be written as a product of m but not less than m symmetries. If $G(T-I)$ is skew-symmetric, then $m\leqq n/2$—and, of course, $m\equiv 0 \pmod 2$—and this minimum number is equal to $m+2$. This exceptional case is studied in detail. (Received July 11, 1949.)

495. M. F. Smiley: *Special alternative rings.*

A study, initiated in a previous paper (Ann. of Math. (2) vol. 49 (1948) pp. 702–709), of the possibility of extending Jacobson's structure theory of associative rings (Amer. J. Math. vol. 67 (1945) pp. 300–320) to alternative rings is continued. An alternative ring A is called *special* if for every $a, b, c\in A$, the equation $a(bc)-(ab)c=x+ax$ has a solution $x\in A$. It is proved that *a special alternative ring has zero Jacobson radical if and only if it is a subdirect sum of special primitive alternative rings.* As a prelininary result it is shown that the Jacobson radical of a special alternative ring is the intersection of the family of regular maximal right ideals (cf. I. Kaplansky, Ann. of Math. (2) vol. 49 (1948) p. 692). Special alternative rings include all direct sums whose summands are associative rings, alternative division rings, or alternative radical rings. However, the Cayley-Dickson algebra, called a vector-matrix algebra by Zorn (Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1930) p. 144), is not a special alternative ring. (Received July 12, 1949.)

496. Alfred Tarski: *A fixpoint theorem for lattices and its applications.* Preliminary report.

Let L be a complete lattice with inclusion relation $\leqq$, and $F(L)$ the family of all increasing functions on L to L. Theorem I. (i) *Given* $f\in F(L)$, *the set* S *of all* $x\in L$ *with* $f(x)=x$ *is nonempty and forms a complete lattice under* $\leqq$. (ii) *The conclusion still holds if,* G *being a commutative subfamily of* $F(L)$ ($fg=gf$ *for all* $f, g\in G$), S *is the set of all* $x\in L$ *with* $f(x)=x$ *for every* $f\in G$. Taking for L a closed interval of real numbers, I(i) is improved: Theorem II. *If* f, g *are real functions on* $[a, b]$, f *increasing,* g *con-*

tinuous, $f(a)\to g(a)$, *and* $f(b)\leqq g(b)$, *then the set* S *of all* $x\in[a, b]$ *with* $f(x)=g(x)$ *is nonempty, continuously ordered, and has a minimum and a maximum.* Taking for L a complete Boolean algebra, I(i) leads to: Theorem III. *Given* a, $b\in L$ *and* f, $g\in F(L)$, *there are* c, $d\in L$ *with* $f(a-c)=d$ *and* $g(b-d)=c$. Several theorems on equality of set power (for example, Cantor-Bernstein theorem, mean value theorem) reduce directly to III. I, III are applicable to topological functions (closure, derivative, and so forth); for example, I gives the familiar theorem: every closed set has a largest perfect subset. Theorems I–III were found in 1939; I(i), III improve earlier results of Knaster-Tarski (Annales de la Société Polonaise de Mathématique vol. 6, p. 133). (Received June 27, 1949.)

497. R. M. Thrall: *On projectivities in a finite-dimensional modular lattice.* Preliminary report.

It is well known that any non-distributive modular lattice L contains at least one sublattice isomorphic to the five-element lattice $\{r, s, t, u, w\}$ where $r=s\cap t=s\cap u=t\cap u$ and $w=s\cup t=s\cup u=t\cup u$. Any such sublattice is called a *projective root* in L; if, in particular, s covers r, the sublattice is called a *prime objective root* (p.p.r.). The relationship between the set of p.p.r.'s and the set of projective prime quotients in L is studied, and a canonical form for projectivities is obtained. A Galois correspondence is set up between sublattices L of the lattice N of all subspaces of an n-dimensional vector space V and subalgebras A of the algebra T of all linear transformations on V. It is shown that the closed elements L under this correspondence possess a property called *projective closure* which is defined in terms of linear transformations induced by projectivities in L. (Received July 23, 1949.)

498. Leonard Tornheim: *Lattice packings in the plane without crossing arcs.*

It is known that the non-overlapping lattice packings of an open set K, or its closure $\overline{K}$, are in one-to-one correspondence with the admissible lattices of $K-K$, the set of points which are the difference of two points in K. For sets J which are composed of a finite number of arcs and open sets (or their closures), conditions on the set $J-J$ are found so that a lattice arrangement of J be non-overlapping, have no arcs intersect open sets, and have no arcs cross. One of these conditions results from the fact that the set $c-d$, where c and d are two crossing arcs, contains a neighborhood of the origin. (Received July 20, 1949.)

499. L. R. Wilcox: *On a certain relational equation.*

Let S be a finite set and, for every binary relation X on $S\times S$, let X_t be the smallest transitive relation containing X. The following results are obtained concerning the equation $X_t=R$, where R is a given transitive relation. If R is a partial ordering, the solutions X are exactly those relations for which $R_0\subset X\subset R$, R_0 being the *prime* subrelation of R (aR_0b in case aRb and no c exists with aRc, cRb). If R is the universal relation $S\times S$, then every symmetric solution X is of the form $Y+Y^*$, where Y is the prime subrelation of a suitable partial ordering (Y^* being the transpose of Y). All nonsymmetric solutions are obtained in the form $Y+Z$ with $Z\subset Y^*$. A characterization is given for the relations Y and Z which yield solutions. Finally, if R is an equivalence relation, the problem is easily reduced to the preceding case (even if S is infinite). (Received July 18, 1949.)

ANALYSIS

500. Joshua Barlaz: *A generalization of the Toeplitz problem of summability.*

Let A and B be two linear sequence-to-sequence summability methods applied to a series $\sum a_n$. With certain conditions on A and B, the author proves that if $\sum a_n$ is A-summable, then the B transform of $\sum a_n$ is also A-summable. (In the Toeplitz problem, A is the identity transform.) The results of St. Mazur (Math. Zeit. vol. 28 (1928) p. 599), determining conditions for one summability method to be more powerful than another, are used. A systematic application of the results is made to Nörlund means and to some Riesz means. Special theorems where A is a sequence-to-function transform are also obtained. (Received July 14, 1949.)

501. E. F. Beckenbach: *A class of mean value functions.*

For a set of positive values $(a_1, a_2, \cdots, a_n)$, a mean value function is defined by $\mathfrak{N}_t(a) = \sum a_j^t / \sum a_k^{t-1}$, and its properties are investigated. It is shown in particular that $\mathfrak{N}_t(a)$ increases continuously from min (a) to max (a) as t increases from $-\infty$ to $+\infty$, and that $\mathfrak{N}_t(a)$ satisfies the triangle inequality for $1 \leqq t \leqq 2$, and satisfies the same inequality with sign reversed for $0 \leqq t \leqq 1$. (Received July 14, 1949.)

502*t*. Lipman Bers: *The expansion theorem for sigma-monogenic functions.*

A Σ-monogenic function $f = u(x, y) + iv(x, y)$ is a complex-valued function $f = u(x, y) + iv(x, y)$ such that u and v satisfy the equations (1) $\sigma_1(x)u_x = \tau_1(y)v_y$, $\sigma_2(x)u_y = -\tau_2(y)v_y$ $(\sigma_1\sigma_2\tau_1\tau_2 > 0)$. Special Σ-monogenic functions, called "formal powers," can be obtained by quadratures (L. Bers and A. Gelbart., Trans. Amer. Math. Soc. vol. 56 (1944) pp. 67–93). The expansion theorem states that in the neighborhood of a point of regularity any Σ-monogenic function can be expanded in a "formal power series." This has been proved previously (loc. cit.) under the hypothesis that $\sigma_i(x)$, $\tau_j(y)$ are analytic functions. In the present paper the proof is given for the case where these functions are twice continuously differentiable. The expansion theorem leads to very precise information on the regularity properties of solutions of elliptic partial differential equations. Example: if $f(y) \in C'$, then any solution of $\Delta u + f(y)u_y = 0$ is analytic in x. (Received July 18, 1949.)

503. H. D. Block: *An integral transformation.* Preliminary report.

Consider the family $(n = 0, 1, 2, \cdots)$ of integral transformations on a suitable function $g(y)$, defined by $f_n(x, k) = \int_{-\infty}^{\infty} |x-y|^n e^{-k|x-y|} g(y)dy = L_n[g(y)]$. Since $f_n(x+a, k) = L_n[g(y+a)]$, $D_x f_n(x, k) = L_n[D_y g(y)]$ if $g(y) \in C, \in C^1$ in sections. Corresponding formulas are given if $g(y), g'(y)$ have finite jumps, and so on. For a fixed value of k we have for the successive application $L_m(L_n) = \sum_{i=0}^{n} C_{n,i}((m+i)!/(2k)^{m+i+1}) \cdot L_{n-i} + \sum_{j=0}^{m} C_{m,j}((n+j)!/(2k)^{n+j+1}) L_{m-j} + (m!n!/(m+n+1)!) L_{m+n+1} = L_n(L_m)$. For computation of the iterated kernels this formula may be simplified. Note also that L_n can be written as a linear combination of the iterates of L_0. This shows that all the eigen-functions of L_0 are eigen-functions of L_n (continuous spectrum). From previous results on the corresponding integral equations the converse is not true. These results also give the inversion of the transform (k fixed). Allowing k to vary one has the inversion formula $\lim_{k\to\infty} (k^{n+1}/n!) L_n[g(y)] = g(x+) + g(x-)$, thus restricting the class of possible transform functions. Further properties of the transformation are being studied. (Received July 18, 1949.)

504t. A. P. Calderón: *On a theorem of Marcinkiewicz and Zygmund.*

Let $F(x_1, x_2, \cdots, x_n)$ be a function harmonic for $x_n>0$ and $x_1, x_2, \cdots, x_{n-1}$ arbitrary. Suppose that every point of a measurable set E of $x_n=0$ is the vertex of a cone, otherwise contained in $x_n>0$, where the function is bounded for x_n small. Then, for almost every point Q of E, the integral of $x_n^{-n+2} \operatorname{grad}^2 F$ extended over any non-tangential neighborhood of Q is finite. This generalizes a result of Marcinkiewicz and Zygmund concerning harmonic functions of two variables. (Received August 2, 1949.)

505t. A. P. Calderón: *On the behaviour of harmonic functions at the boundary.*

(1) Let $F(x, y, z)$ be a harmonic function for $z>0$ and x, y arbitrary. Suppose that for every point $(x, y, 0)$ belonging to a set E of positive two-dimensional measure there is a cone with vertex at that point and otherwise contained in the half-space $z>0$, and such that within the cone and near to the point F is bounded. Then at almost every point of E the function F has a non-tangential limit. This is an extension to three dimensions of a well known result of Priwaloff. The proof holds for any number of dimensions, and is extensible to functions harmonic in groups of variables and in particular to regular functions of several complex variables. (2) Let $F(z_1, z_2)$, where $z_1=x_1+iy_1$, $z_2=x_2+iy_2$, be regular for $y_1>0$, $y_2>0$. Then at almost every point (x_1, x_2) of the plane $y_1=y_2=0$, either the function has a finite non-tangential limit, or the range of the values of F at every non-tangential neighborhood of (x_1, x_2) is dense in the whole complex plane. Again this result is valid for regular functions of n complex variables. For $n=1$ it reduces to a well known result of Plessner. (Received August 2, 1949.)

506t. M. M. Day: *Means and ergodicity of semigroups.*

A bounded semi-group $\mathfrak{S}$ of linear operators from a Banach space B to itself is called ergodic if there exists a directed system $\mathcal{A}$ of averages A such that $\lim_A (AS-A) = \lim_A (SA-A)=0$ for every S in $\mathfrak{S}$; we have three strengths of ergodicity of $\mathfrak{S}$ according as uniform, strong, or weak convergence is used in the operator algebra. It is shown that weak ergodicity of every bounded representation of Σ is equivalent to weak ergodicity of the right and left representations of Σ by right and left translations on $m(\Sigma)$, and equivalent to Σ amenable (see abstract 507). Uniform ergodicity is similarly related to strong amenability of Σ. (Received July 5, 1949.)

507. M. M. Day: *Means on semigroups and groups.*

A mean μ on a semigroup Σ is a positive linear functional of norm one on the space $m(\Sigma)$ of bounded, real-valued functions on Σ. Σ is called amenable if there exists a μ-invariant under all right- and left-translations by elements of Σ. Theorem 1. If Σ is the union of an expanding directed system of amenable subsemigroups, then Σ is amenable. Theorem 2. If G is a group and there is a left-invariant mean, then G is amenable. Theorem 3. If G is a group and H a normal subgroup of G, and if H and G/H are amenable, so is G. Results like Theorems 2 and 3 also hold if amenability is replaced by strong amenability: G is strongly amenable if there exists a directed system

Γ of finite means γ such that $\lim_{\gamma \in \Gamma} \|T\gamma - \gamma\|_{l_1} = 0$ for every right- or left-translation T of $l_1(G)$. (Received July 5, 1949.)

508. Aryeh Dvoretzky: *Bounds for the coefficient of univalent functions.*

Let (1) $f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$ be regular and univalent in $|z| < 1$ and denote by W the domain in the w-plane onto which $|z| < 1$ is mapped through $w = f(z)$. It is well known that, as $n \to \infty$, always $a_n = O(n)$ while $a_n = o(n^{-1/2})$ whenever W is bounded. This and similar facts suggest that limitations of a geometrical nature on the "extension" of W affect strongly the growth of the coefficients of $f(t)$. However this "extension" has little to do with the area of W. In the present paper it is gauged by means of the function $A(R) = \max_{|w|=R} \min_{w' \notin W} |w' - w|$, that is, the radius of the largest circle with center on $|w| = R$ the whole interior of which is contained in W. $A(R)$ is continuous for all $0 \leqq R < \infty$ and satisfies $0 \leqq A(R) \leqq R + 1$. A general result giving bounds for a_n in terms of $A(R)$ is obtained. We quote here two very special applications: (1) There exists a finite positive contant K such that $\limsup_{n=\infty} |a_n|/n \leqq KL$ holds for all functions (1) satisfying $\limsup_{R=\infty} A(R)/R \leqq L$. (2) If W is such that it does not fully cover arbitrarily large circles, then $a_n = o(\log n)$ as $n \to \infty$. The methods extend to p-valent or mean p-valent functions and the results may be strengthened when special classes of univalent functions are considered. (Received July 15, 1959.)

509. W. F. Eberlein: *Banach-Hausdorff limits.* I.

A Banach-Hausdorff (B-H) functional is defined to be a linear functional $L(x)$ on the space m of bounded real sequences x which satisfies the four conditions for the ordinary Banach limit and, in addition, the fifth condition: $L(Hx) = L(x)$, where H is any regular Hausdorff transformation. The existence and structure of B-H functionals is determined, plus the domain of uniqueness—that is, the class of sequences x on which all B-H functionals coincide. (Received August 30, 1949.)

510. G. C. Evans: *An infinitely-valved harmonic function with branch curves of order two.* Preliminary report.

Given two closed curves in space, s_1, s_2, capped respectively by surface films D_1, D_2 of negligible surface tension, which are conductors, what positions will those surfaces assume if they are held respectively at potentials V_1 and V_2? The space is equivalent topologically to one in which the curves are distinct unlinked circles; the curves are of zero capacity. Let H be an infinitely leaved space of which the leaves H^j are pinned together in pairs alternately on s_1, s_2 as branch curves, H^{2k} with H^{2k+1} on s_1, and with H^{2k-1} on s_2, $-\infty < k < \infty$; and $v(M)$ be uniform and harmonic on H, bounded on $\{H^j\}$ for j bounded, with preassigned values at ∞, $v(M^{2k}) \to 2k(V_1 - V_2)$, $v(M^{2k+1}) \to 2V_1 + 2k(V_1 - V_2)$. Such functions exist with arbitrary values h_j on H^j at ∞ provided that the series $\sum |h_j| m^{|j|}$, from $-\infty$ to $+\infty$, converges for m a certain constant less than 1, and are uniquely determined in a class of functions which do not increase too rapidly with $|j|$. The D_1 and D_2 are respectively the loci $v(M^0) = v(M^1)$ and $v(M^0) = v(M^{-1})$ and are bounded unless V_1 or V_2 vanishes. The H degenerates into a finite number of leaves, in that case two, if and only if $V_1 = V_2$. (Received July 18, 1949.)

511*t*. L. Fejér and Gabor Szegö: *On special conformal mappings.*

1. Two proofs are given for the following theorem. Let $f(z) = u(r, \theta) + iv(r, \theta)$ be

regular for $r \leqq R$, real for real z, and $\partial u(r, \theta)/\partial\theta < 0$ for $r = R$, $0 < \theta < \pi$. Then for the same r and θ we have $v(r, \theta) > 0$. 2. Let $\sum_1^\infty b_n r^n \sin n\theta > 0$ for $r = 1$, $0 < \theta < \pi$. Then the same is true for $r < 1$, $0 < \theta < \pi$. This assertion is generalized by replacing r^n by λ_n where the sequence $\{\lambda_n\}$ is monotonic of order 4. 3. Let $S_n^{(\lambda)}(z)$ be the Cesàro sum of order k and degree n of the geometric series $\sum_0^\infty z^n$. Writing $S_n^{(\lambda)}(e^{i\theta}) = u_n(\theta) + iv_n(\theta)$, one has $(u_n(\theta) - u_n(0))v_n'(\theta) - u_n'(\theta)v_n(\theta) = (n+1)(1 - \cos\ \theta)(P_{n-1}^{(2)}(\cos\ (\theta/2)))^2/2$. Moreover writing $S_n^{(2)}(e^{i\theta}) = u_n(\theta) + iv_n(\theta)$ one has $u_n'(\theta)v_n''(\theta) - u_n'(\theta)v_n'(\theta) = (n+2)(P_{n-1}^{(2)}(\cos\ (\theta/2)))^2/3$. Here $P_{n-1}^{(\lambda)}$ denotes the ultraspherical polynomials, that is, the coefficient of w^{n-1} in the expansion of $(1 - 2\cos\ (\theta/2) \cdot w + w^2)^{-\lambda}$. These results are refinements of certain results of Egerváry (Math. Zeit. vol. 42 (1937)). 4. A power series $f(z)$ is constructed with totally monotonic coefficients such that the mapping $w = f(z)$ of the unit circle is not convex. (Received July 27, 1949.)

512. D. B. Goodner: *Projections in normed linear spaces.*

A normed linear space X has property P_s, $s \geqq 1$, if and only if for every normed linear space Y containing X there is a projection T, $\|T\| \leqq s$, of Y onto X. *Theorem* 1. If a normed linear space X has property P_1 and an extreme point on its unit sphere C, then X is equivalent to the space of real-valued continuous functions on some extremally disconnected compact Hausdorff space H. *Theorem* 2. If an abstract (M)-space has property P_1, then it has a unit element. (Received July 14, 1949.)

513*t*. L. M. Graves: *On the definition of generalized curves.*

Let $I = [0 \leqq t \leqq 1]$, E be a Euclidean space, and S the unit sphere in E. Let $H = $ [all $F(y, r)$ continuous for $y \in E$, $r \in S$]. A representation ρ consists of a point $y(0)$ in E and a real function $v(t, F)$ on $I \times H$, such that: 1) $v(t, F)$ is A.C. on I; 2) $v(t, F)$ is linear on H; 3) if $F \geqq 0$, $v(t, F)$ is nondecreasing in t; 4) $v(0, F) = 0$; 5) if $F_i(r) \equiv r^i$, $y^i(t) \equiv y^i(0) + v(t, F_i)$, and $F(y(t), r) = 0$ for $r \in S$, $t_1 \leqq t \leqq t_2$, then $v(t, F)$ is constant on $t_1 \leqq t \leqq t_2$. Let π consist of all $\theta(t)$ which are increasing, quasilinear, and map I into itself. Let H_1 consist of all F in H which satisfy a Lipschitz condition with constant one and $|F(y, r)| \leqq 1$. For two representations $\rho_1 = (y_1(0), v_1)$, $\rho_2 = (y_2(0), v_2)$, let $\beta(v_1, v_2, \theta) = \text{l.u.b.}\ |v_1(t, F) - v_2(\theta(t), F)|$ for $t \in I$, $F \in H_1$, and let the distance $\delta(\rho_1, \rho_2) = \|y_1(0) - y_2(0)\| + \text{g.l.b.}\ \beta(v_1, v_2, \theta)$ for all $\theta \in \pi$. A generalized curve is a class of representations whose mutual distances are zero. In the space of generalized curves, the desired compactness theorem now follows at once from the theorem of Ascoli for mappings between compact metric spaces. There is a correspondence between the representations introduced here and those used by McShane (Duke Math. J. vol. 6 (1940) pp. 513–536). The approach outlined here gives a simpler aspect to the basic part of the theory. (Received July 18, 1949.)

514. William Gustin: *On decomposition of an interval into finitely many congruent sets.*

Let G be an additive subgroup of the real numbers; and let I be an interval of G. It is shown that if I can be decomposed into a finite number (>1) of congruent disjoint subsets, then I can also be decomposed into a finite number (>1) of congruent disjoint intervals. As a consequence no open or closed interval of real numbers can be decomposed into finitely many disjoint congruent sets. (Received July 14, 1949.)

515. Fritz Herzog and George Piranian: *Sets of convergence of Taylor series.*

Let $\sum_n a_n z^n$ be a Taylor series of radius of convergence one. The totality of those points on the unit circle C at which the Taylor series converges will be called its *set of convergence*. It was shown by Neder that every closed arc on C is the set of convergence of some Taylor series. Mazurkiewicz extended this result by showing that every closed set on C, as well as every open set on C, is the set of convergence of some Taylor series. In the present paper the authors obtain the result that every set of type F_σ on C is the set of convergence of some Taylor series. It is known, on the other hand, that every set of convergence is necessarily of type $F_{\sigma\delta}$. Another result obtained in the paper states that if M is any closed set on C, there exists a Taylor series which converges *uniformly* on M and diverges on $C-M$. This is the best result in the sense that if a Taylor series converges uniformly on its set of convergence then this set of convergence must necessarily be closed. (Received February 23, 1949.)

516*t*. Edwin Hewitt and H. S. Zuckerman: *Certain rings related to locally compact groups*. Preliminary report.

Let G be a locally compact topological group. Let $\mathfrak{C}_{00}(G)$ be the linear space of all complex continuous functions on G each of which vanishes outside of some compact set. Let $x \to U_x$ be an irreducible weakly continuous representation of G by unitary operators U_x on a Hilbert space $\mathfrak{H}$ having an orthonormal basis $\{\phi_\gamma\}$, $\gamma \in \Gamma$. Let $\mathfrak{A}(G)$ be the set of all linear combinations of functions $(U_x\phi_\gamma, \phi_\delta)$, for all possible representations $x \to U_x$ of the type described. Let $\mathfrak{R}_u(G)$ be the set of all complex linear functionals on $\mathfrak{C}_{00}(G)$ which are continuous in the uniform topology for $\mathfrak{C}_{00}(G)$; let $\mathfrak{S}_u(G)$ be the set of all complex linear functionals on $\mathfrak{A}(G)$ continuous in the uniform topology for $\mathfrak{A}(G)$; and let $\mathfrak{S}_a(G)$ be the set of all complex linear functionals on $\mathfrak{A}(G)$ continuous in the norm $|||\sum_{j=1}^{n}\alpha_j f_j||| = \sum_{j=1}^{n}|\alpha_j|$. Each of these sets of functionals becomes a ring under the usual definition of sum and under the definition $MN(f) = \iint_{G\times G} f(xy)\,d\mu(x)\,d\nu(y)$, where μ and ν are the measures corresponding to M and N respectively. All three rings are extensions of the L_1-algebra of G, containing this algebra as a closed two-sided ideal. All three rings contain isomorphs of G, are commutative if and only if G is Abelian, and have identity elements. If all the representations $x \to U_x$ described above are finite-dimensional, then $\mathfrak{R}_u(G)$ and $\mathfrak{S}_u(G)$ have the property that the intersection of all maximal two-sided ideals is the nil-ideal. $\mathfrak{R}_u(G)$, $\mathfrak{S}_u(G)$, and $\mathfrak{S}_a(G)$ all have nil Jacobson radical. The complete structure of maximal ideals is described for certain special cases. (Received September 8, 1949.)

517. Edwin Hewitt and H. S. Zuckerman: *Certain rings related to locally compact rings*. Preliminary report.

Let G be a locally compact topological group. Let $\mathfrak{C}_{00}(G)$ be the linear space of all complex continuous functions on G each of which vanishes outside of some compact set. Let $x \to U_x$ be an irreducible weakly continuous representation of G by unitary operators U_x on a Hilbert space $\mathfrak{H}$ having an orthonormal basis $\{\phi_\gamma\}_{\gamma\in\Gamma}$. Let $\mathfrak{A}(G)$ be the set of all linear combinations of functions $(U_x\phi_{\gamma_1}, \phi_{\gamma_2})$, for all possible representations $x \to U_x$ of the kind described. Let $\mathfrak{R}(G)$ be the set of all complex linear functionals on $\mathfrak{C}_{00}(G)$ which are linear combinations of non-negative functionals; let $\mathfrak{R}_u(G)$ be the set of all functions in $\mathfrak{R}(G)$ continuous in the uniform topology for $\mathfrak{C}_{00}(G)$; let $\mathfrak{S}_u(G)$ be the set of all complex linear functionals on $\mathfrak{A}(G)$ which are continuous in the uniform topology for $\mathfrak{A}(G)$; and let $\mathfrak{S}_a(G)$ be the set of all complex linear functionals on $\mathfrak{A}(G)$ continuous in the norm $|||\sum_{j=1}^{n}\alpha_j f_j||| = \sum_{j=1}^{n}|\alpha_j|$. Each of these sets of functionals becomes a ring under the definitions $MN(f) = \iint_{G\times G} f(xy)\,d\mu(x)\,d\nu(y)$

where μ and ν are the measures corresponding to M and N respectively, and $(M+N)(f) = M(f)+N(f)$. If G is compact, then $\mathcal{R}(G)=\mathcal{R}_u(G)=\mathfrak{S}_u(G)$, while $\mathfrak{S}_a(G)$ is in general a super-ring of the others. All four rings are extensions of the L_1-algebra $\mathcal{L}_1(G)$; in the uniform topologies for $\mathcal{R}_u(G)$, $\mathfrak{S}_u(G)$, and $\mathfrak{S}_a(G)$, $\mathcal{L}_1(G)$ is closed. All four of the rings under discussion contain isomorphs of G, are commutative if and only if G is Abelian, and have identity elements. $\mathcal{R}_u(G)$, $\mathfrak{S}_u(G)$, and $\mathfrak{S}_a(G)$ have zero Jacobson radical. The complete structure of maximal ideals is described for particular cases. (Received July 8, 1949.)

518*t*. Einar Hille: *On the differentiability of semi-group operators.*

Let $T(\xi)$ be a semi-group operator such that (i) $T(\xi_1+\xi_2)=T(\xi_1)T(\xi_2)$, (ii) $\|T(\xi)\| \leqq 1$, (iii) $\lim_{\xi\to 0}(\xi)x = x$ for all x in the operand space and let A be its infinitesimal generator with resolvent $R(\lambda; A)$ existing at least for $\mathcal{R}(\lambda)<0$. $T'(\xi)=AT(\xi)$ is ordinarily unbounded; this note contains a study of what conditions must hold in order that $T'(\xi)$ shall exist as a bounded operator for each $\xi<0$. If $T'(\xi_0)$ is bounded, so is $T^{(n)}(\xi)$ for $\xi \geqq n\xi_0$, $n=1, 2, 3, \cdots$. Lower limits may be found for the rate of growth of $g(\xi)=\|T'(\xi)\|$ as $\xi\to 0$ but no upper ones. Knowing $g(\xi)$ we can estimate the distance $\delta(\tau)$ of the spectrum of A from the points of the line $\mathcal{R}(\lambda)=1$. In particular, if $\log g(\xi)\in L(0, 1)$, then $R(\tau)=\|R(1+i\tau; A)\| (1+|\tau|)^{-1}\in L(-\infty, \infty)$. Conversely, if $R(\tau)\in L(-\infty, \infty)$ and $\delta(\tau)$ is fairly regular, then $T(\xi)$ has bounded derivatives of all orders for $\xi<0$. (Received August 16, 1949.)

519*t*. Einar Hille: *On the integration problem for Fokker-Planck's equation in the theory of stochastic processes.*

A temporally homogeneous one-parameter stochastic process leads to the partial differential equation $[b(x)U]_{xx}-[a(x)U]_x=U_t$ where $b(x)>0$. The integration problem consists in finding for what functions $a(x)$ and $b(x)$ the equation has a unique solution $U(x, t)$, $t>0$, in $L(-\infty, \infty)$ which converges in the mean of order one to a preassigned function $g(x)$ in $L(-\infty, \infty)$ when $t\to 0$. K. Yosida has recently shown that this problem is equivalent to finding when the differential operator $A[f]=[b(x) f(x)]''-[a(x)f(x)]'$ generates a semi-group of translation operators in $L(-\infty, \infty)$. The present paper gives a simplified discussion of the problem and leads to less restrictive conditions than those of Yosida. $A[f]$ generates contraction operators if merely $b''(x)-a'(x)$ is bounded above. If in addition, for instance, $|b'(x)-a(x)| \leqq K(|x|+1)$ and the integral of $t/b(t)$ diverges over the ranges $(-\infty, 0)$ and $(0, \infty)$, then the equation defines a stochastic process. (Received August 16, 1949.)

520*t*. Shizuo Kakutani: *Concrete representation of Brownian motions.*

Let $(\Omega, \mathfrak{B}, \mu)$ be a probability space (that is, a measure space with $\mu(\Omega)=1$), and $(S, \mathfrak{M}, m)$ a measure space with $m(S)\leqq\infty$. Let $\mathfrak{M}_0=\{M \mid M\in\mathfrak{M}, 0<m(M)<\infty\}$. A real-valued function $x(M, \omega)(M\in\mathfrak{M}_0, \omega\in\Omega)$ is a generalized Brownian motion if (i) for any fixed $M\in\mathfrak{M}_0$, $x_M(\omega)=x(M, \omega)$ is a Gauss function (that is, measurable in ω and has a Gaussian distribution with mean 0) with variance $m(M)$, (ii) if M_k, $k=1, \cdots, n$, are disjoint, then $x(M_k, \omega)$, $k=1, \cdots, n$, are independent and $x(\bigcup_{k=1}^{n} M_k, \omega)=\sum_{k=1}^{n} x(M_k, \omega)$ a.e. on Ω. If $(S, \mathfrak{M}, m)$ is the ordinary Lebesgue measure space on the infinite real line $S=\{s \mid -\infty<s<\infty\}$, and if we consider only finite intervals $I=\{s \mid a<s\leqq b\}$ as $M\in\mathfrak{M}_0$, then $x(M, \omega)$ is a Brownian motion in the sense of N. Weiner. It is shown that every generalized Brownian motion determines a linear

and isometric embedding $f(s) \to x_f(\omega)$ of $L^2(S)$ into $L^2(\Omega)$ as a Gauss subspace (that is, a closed linear subspace of $L^2(\Omega)$ consisting only of Gauss functions) such that $f_M(s) \to x(M, \omega)$, where $f_M(s)$ is the characteristic function of M. Conversely, every such embedding of $L^2(S)$ into $L^2(\Omega)$ determines a generalized Brownian motion. Properties of Brownian motions are discussed by using this concrete representation. (Received July 14, 1949.)

521*t*. Shizuo Kakutani: *Determination of the spectral type of the flow of Brownian motion.*

Let $x(M, \omega)$ be a generalized Brownian motion as defined in the preceding abstract. Assume that the ring generated by $\{x_M(\omega) \mid M \in \mathfrak{M}_0\}$ is dense in $L^2(\Omega)$. Then, for any flow (=one-parameter group of measure preserving transformations) $\{\psi_t(s)\}$ on $(S, \mathfrak{M}, m)$, there corresponds a flow $\{\phi_t(\omega)\}$ on $(\Omega, \mathfrak{B}, \mu)$ such that $x(\psi_t(M), \omega) = x(M, \phi_t(\omega))$ a.e. on Ω for any $M \in \mathfrak{M}_0$ and for any t. If $(S, \mathfrak{M}, m)$ is the ordinary Lebesgue measure space on the infinite real line $S = \{s \mid -\infty < s < \infty\}$ and if $\psi_t(s) = s + t$, then the corresponding flow $\{\phi_t\}$ is the flow of Brownian motion discussed by N. Wiener. The main purpose of this paper is to determine the spectral type of the one-parameter group of unitary transformations $\{U_t\}$ defined on $L^2(\Omega)$ by $U_t x(\omega) = x(\phi_t(\omega))$. It is shown that $L^2(\Omega)$ is decomposed into a direct sum of a countable number of closed linear subspaces $\mathfrak{M}_n$, called the homogeneous chaos of order n, each $\mathfrak{M}_n$ invariant under U_t, in such a way that $\{U_t\}$ on $\mathfrak{M}_n$ is spectrally isomorphic with $\{V_t^{(n)}\}$ on $L_s^2(S^{(n)})$, where $L_s^2(S^{(n)})$ is the L^2-space of all real-valued measurable functions $f(s_1, \cdots, s_n)$ defined on $S^{(n)} = S \times \cdots \times S$ (n times) and symmetric in its n variables, such that $\int\int_{S \times \cdots \times S} |f(s_1, \cdots, s_n)|^2 ds_1 \cdots ds_n < \infty$, and $\{V_t^{(n)}\}$ on $L_s^2(S^{(n)})$ is defined by $V_t^{(n)} f(s_1, \cdots, s_n) = f(\psi_t(s_1), \cdots, \psi_t(s_n))$. This determines completely the spectrum of $\{U_t\}$. The construction of the homogeneous chaos $\mathfrak{M}_n$ is obtained by using the Hermite polynomials $H_n(u, \sigma) = (n!)^{-1}(-1)^n(\sigma^n e^{u^2/2\sigma}) d^n(e^{-u^2/2\sigma})/du^n$ and an essential use is made of the following functional relation: $H_n(u+v, \sigma+\tau) = \sum_{k=0}^{n} H_k(u, \sigma) H_{n-k}(v, \tau)$. (Received July 14, 1949.)

522. R. E. Lane: *Foundations of the theory of continued fractions.* I.

The author lays the foundations for a general theory of the continued fraction with bounded sequence $\{f_n\}_0^\infty$ of approximants by establishing the existence of a nest R_{n+1} of finite circular regions such that (i) for each positive integer, n, R_n is the smallest circular region containing f_{n-1} and R_{n+1} and (ii) the region R common to all of the R_n is the smallest circular region which contains all sequential limit points of $\{f_n\}_0^\infty$. This fundamental result implies, for example, that every convergent continued fraction is positive definite or can be transformed, by a contraction and an equivalence transformation, into a positive definite continued fraction. The theorem provides a sequence of inequalities, each involving a partial numerator and two partial denominators of the continued fraction, necessary and sufficient for the sequence $\{f_n\}_0^\infty$ to be bounded. In addition, it provides a relatively simple algebraic condition which is necessary and sufficient for convergence. (Received July 18, 1949.)

523. C. E. Langenhop and A. B. Farnell: *The existence of forced periodic solutions of second order differential equations near certain equilibrium points of the unforced equation.*

The equation $\ddot{x} + a\dot{x} + x + x^2 = k \cos \omega t$, $a > 0$, $k > 0$, is discussed in detail. This equation is replaced by a system ($\dot{x}=y$, $\dot{y}=f(x, y, t)$) the solutions of which define a mapping T of the (x, y) plane into itself. This mapping is obtained by replacing t by $t+2\pi/\omega$ in every solution of the system. For k sufficiently small a closed convex region Ω can be constructed from the solutions of certain associated linear systems so that Ω is mapped into itself under T. Brouwer's fixed point theorem is then utilized to assert the existence of at least one periodic solution of the system in Ω. Another region Φ surrounding Ω is obtained and its boundary is shown to have the index zero with respect to the vector field $[P_0P_1]^{\rightarrow}(P_1=TP_0)$, which implies the existence of another periodic solution in $\Phi-\Omega$. The method of obtaining Ω above is then extended to prove that for k sufficiently small there are periodic solutions of the equation $\ddot{x}+f(x)\dot{x}+g(x)=ke(t)$ in the neighborhood of points $(x={}_0x, \dot{x}=0)$ such that $g(x_0)=0$, $g'(x_0)>0$, $f(x_0)\neq 0$, if $f(x)$, $g(x)$, $e(t)$ are analytic and $e(t)$ is periodic. (Received June 27, 1949.)

524t. Joseph Lehner: *Note on power series with integral coefficients.*

H. Graetzer has proved (J. London Math. Soc. (1947) pp. 90–92) that any complex number x_0 with $|x_0|<1$ is a root of a power series with rational integral coefficients and radius of convergence unity. The author proves the following generalization: Let E be a definite or countable set of complex numbers all lying in the interior of the unit circle; $\overline{E}$ is the set of complex conjugates. E shall have no limit points interior to the unit circle. Then there is a power series, with rational integral coefficients and radius of convergence unity, which vanishes on E and $\overline{E}$, and nowhere else inside the unit circle. (Received July 11, 1949.)

525t. Walter Leighton: *On the detection of the oscillation of solutions of a second order linear differential equation.*

The principal result of this paper is the following: Let (1) $(r(x)y')'+p(x)y=0$ be a differential equation in which r, r', and p are continuous and $r>0$ on $0<x<\infty$. If $p(x)>0$ near $x=+\infty$ and if $\int_1^\infty r^{-1}dx=\int_1^\infty p\,dx=+\infty$, every solution of (1) has an infinity of zeros on the interval $1<x<\infty$. Similarly, if $p>0$ near $x=0+$ and if $\int_0^1 r^{-1}\,dx=\int_0^1 p\,dx=+\infty$, every solution of (1) has an infinity of zeros on the interval $0<x<1$. The paper will be published in the Duke Mathematical Journal. (Received July 15, 1949.)

526. C. B. Morrey: *Quasi-convexity and the lower semicontinuity of double integrals.*

Assume $f(u, v, \mathbf{x}, p, q)(\mathbf{x}=x, \cdots, x^N)$ continuous, D a domain, $I(\mathbf{x}, D)$ the double integral of f. $\phi(p, q)$ is called *quasi-convex* if linear functions minimize its integral. The following are proved: (1) A necessary and sufficient condition that $I(\mathbf{x}, D)\leqq\liminf I(\mathbf{x}_n, D)$ whenever $\mathbf{x}_n\rightrightarrows\mathbf{x}$ with $|\mathbf{x}_{nu}|$ and $|\mathbf{x}_{nv}|$ uniformly bounded is that f be quasi-convex in (p, q). (2) A necessary condition that f be quasi-convex in (p, q) is that $f(u_0, v_0, \mathbf{x}_0, p_0+\lambda\xi, q+\mu\xi)$ be convex in ξ and in (λ, μ); if f is of class C', this is the Weierstrass condition. (3) A sufficient condition that $\phi(p, q)$ be quasi-convex in (p, q) is that, for each (p_0, q_0), there are constants A_i, B_i, and $C_{ij}(C_{ji}=-C_{ij})$ such that $\phi(p_0+\xi, q_0+\mathbf{n})\geqq\phi(p_0, q_0)+A_\alpha\xi^\alpha+B_\alpha\eta^\alpha+C_{\alpha\beta}\xi^\alpha\eta^\beta$. (4) If f is quasi-convex, of class C', and homogeneous of degree 2 in (p, q) with $m(p^2+q^2)\leqq f\leqq M(p^2+q^2)(m>0)$ then $I(x, D)$ is lower-semi-continuous with respect to weak convergence in $\mathfrak{P}_2$. (5) With the hypotheses in (4), there is a continuous minimizing vector with given boundary values. (1) and (2) generalize to k-tuple integrals. (Received July 18, 1949.)

527t. Helen K. Nickerson: *On overconvergence of Dirichlet series.*

Let $f(s)$, where $s=\sigma+it$, denote the function defined by the Dirichlet series $\sum_{n=1}^{\infty} a_n \exp(-\lambda_n s)$ (assumed to have finite abscissa of ordinary convergence) and its analytic extension. For each integer $r>1$ an absolutely convergent integral formula is derived, representing the partial sums (remainders) of the series in the half-plane $\sigma \leqq \eta_k (\sigma > \eta_k)$, where $f(s)$ is regular and $O(|t|^k)$ in $\sigma > \eta_k$ and $r>k$. Abscissae η_k surely exist for $k \geqq 1$. These formulae are used to prove the following theorem: If $l = \limsup_{p\to\infty} (1/\lambda_{n_p}) \log [1/(\lambda_{n_{p+1}} - \lambda_{n_p})[< +\infty$, then the partial sums corresponding to the indices $\{n_p\}$ converge to $f(s)$ in the half-plane $\sigma > \eta_k + kl$, for each value of k, uniformly on any closed bounded subset. Thus, if $l=0$ for a suitable subsequence of indices, $f(s)$ can be represented by a sequence of partial sums in the entire half-plane in which it is regular and of finite order; overconvergence due to Ostrowski gaps in the sequence of exponents $\{\lambda_n\}$ extends to this half-plane (and not merely to a suitable neighborhood of each point of the line of convergence in which the limit function is regular). (Received July 18, 1949.)

528t. Helen K. Nickerson: *On overconvergence of general series.*

The method of harmonic majorants, used by Bourion in the study of overconvergence for Taylor's series and by Walsh for general sequences with the same order of convergence as geometric series, is extended to more general series for which the convergence is weaker, as indicated by Bourion for Dirichlet series. In the general case the following results hold: overconvergence, when it occurs, is not local but extends to all points of the boundary of the region of convergence in which the limit function is regular (with exceptions possible in the case that the domain of convergence consists of two or more regions); the Ostrowski "gap" hypothesis, or a suitable analogue, implies overconvergence at all regular points of the boundary of the region of convergence (analogue of Hadamard gap theorem as corollary); non-gap overconvergence must, if the subsequence is not too sparse, occur at all points of the boundary of the region of convergence. The hypotheses used can be verified not only for Dirichlet series and factorial series, but also for Hermitian series, series of Laguerre polynomials, generalized factorial series and Newton series, a class of series studied by Carmichael, and the "generalized" Dirichlet series studied by Greenwood. (Received July 18, 1949.)

529t. Ruth E. O'Donnell: *A note on the location of the zeros of polynomials.*

Let $p(z) = \sum_{k=0}^{n} a_k z^k$ and form the ratios $-a_k/a_{k+1}$ $(k=0, \cdots, n-1)$. If no coefficient of $p(z)$ vanishes, then every half-plane containing the origin and at least one of the ratios also contains at least one zero of $p(z)$. If one or more of the coefficients vanish, then there is at least one zero of $p(z)$ in every half-plane containing the origin. If all the ratios lie in $\theta - \pi/h \leqq \arg z \leqq \theta + \pi/h$ $(h \geqq n)$, then all the zeros of $p(z)$ lie in $\theta - \pi[(n-1)/n + 1/h] \leqq \arg z \leqq \theta + \pi[(n-1)/n + 1/h]$ and all the k-fold zeros of $p(z)$ lie in $\theta - \pi[(n-k)/(n+k+1) + 1/h] \leqq \arg z \leqq \theta + \pi[(n-k)/(n-k+1) + 1/h]$. (Received August 7, 1949.)

530. A. M. Ostrowski: *On Theodorsen and Garrick's method of conformal mapping.*

A discussion of convergence features of the method for conformal mapping developed by Theodorsen and Garrick is given. The estimates given by Warschawski in the case of a specific initial approximation are further improved and developed under general assumptions about the initial approximation and the given contour. The corresponding convergence and existence theorems are given for a more general integral equation. (Received June 29, 1949.)

531. George Piranian and Barbara Piranian: *Analytic theory of Nörlund transformations*. Preliminary report.

The convergence field of the Nörlund transformation (N, p) is studied in terms of the function represented by the Taylor series (1) $\sum p_n z_n$. For the cases where (1) converges absolutely on the unit circle, a more or less explicit description of the convergence field is obtained. In particular, this description supplies a negative answer to the following question by G. M. Wing: If (N, p) is a regular Nörlund transformation stronger than convergence, does there necessarily exist another regular Nörlund transformation which is stronger than convergence but weaker than (N, p)? (Received July 16, 1949.)

532*t*. Tibor Rado and P. V. Reichelderfer: *On generalized Jacobians.*

Let T be a continuous mapping from a simply connected Jordan region R into Euclidean three-space E_3 which represents a surface having finite Lebesgue area. If F is a Cartesian frame of reference in E_3, then T gives rise to a series of rectangle functions, each of which possesses a derivative almost everywhere in the interior R_0 of R [T. Radó, *Length and area*, Amer. Math. Soc. Colloquium Publications, vol. 30, New York, 1948]. It is known that these derivatives satisfy certain relations at all points in R^0 except those belonging to a set $e(T, F)$ of measure zero and depending upon both T and F. The main purpose of this paper is to show that in a number of important cases, the set $e(T, F)$ may be replaced by a set $e(T)$ of measure zero and independent of F. These results are used to strengthen various theorems in the theory of surface area. (Received May 5, 1949.)

533*t*. L. L. Rauch: *Oscillation of a third order nonlinear autonomous system.*

In-the-large properties of the solutions of the system $\dot{x}=K_1[f(x)-(1+K_4)x-z]$, $\dot{y}=-K_2[f(x)-x-z]$, $\dot{z}=-K_3(y+z)$ are considered. K_1, K_2, K_3, and K_4 are independent positive constants and the nonlinearity is introduced by the function $f(x)$ which is assumed to be continuous and bounded above and below. In addition it is assumed that $xf(x)\geqq 0$ and that $f'(0)$ exists. When written as a single third-order equation in x the system becomes $k_1\dddot{x}+[k_2+k_3g(x)]\ddot{x}+k_3g'(x)\dot{x}^2+g(x)\dot{x}+x=0$ which is a generalization to the third order of Liénard's equation $\ddot{x}+g(x)\dot{x}+x=0$. By an application of Brouwer's fixed point theorem in the phase space (coordinates x, y, and z) it is proved that a sufficient condition for the existence of at least one period solution is $f'(0)>1+K_4+(1/2K_1)[K_2+K_3-((K_2-K_3)^2+4K_2K_4)^{1/2}]$ and $K_4>5.0K_1+9.7K_2+2.4K_1K_4+4.6\sup_{-\infty\leqq x\leqq\infty}f(x)/x$. It is also shown under this condition that after a long enough time every solution, with the exception of two singular cases, becomes a nondecaying oscillation which is not too far from the periodic solution or solutions in the phase space. It is established that the existence proof still holds when the system is generalized to $k_1\dddot{x}+[k_2+k_3g(x)]\ddot{x}+k_3g'(x)\dot{x}^2+g(x)\dot{x}+h(x)=0$. (Received July 14, 1949.)

534*t*. Francis Regan and Charles Rust: *On natural boundaries of a generalized Lambert series.*

In this paper the authors are interested in determining conditions of the coefficients of the series $F(z) = \sum_{n=1}^{\infty}(a_n b_n z^n)/(1-a_n z^n)$ introduced in 1932 by Feld, so that the function represented by the series will have a natural boundary. Special cases of this series have natural boundaries. If $a_n = a^n$, and if $b_n \equiv 1$ or if $0 < b_n \leqq B < \infty$, the series represents a function which has the circle $|z| = 1/|a|$ as a natural boundary. If $a_n = a$ and $|a| \leqq 1$ and if $b_n \equiv 1$, the series has the unit circle as a natural boundary. If a_n is such that $|a_n| \leqq 1$ and $\text{LUB}[|a_n|^{1/n}] = 1$, this series and the Lambert series $L(z) = \sum_1^{\infty}(c_n z^n)/(1-z^n)$ represent the same function within the unit circle, provided $|c_n| \leqq C < \infty$, where $a_n b_n = \sum_{d|n} S(n/d) \sum_{d_1|d} c_{d_1}$ [$S(d)$ is a generalized Moebius function; Doyle, Ann. of Math. vol. 40, p. 357]. With these conditions a theorem for the Feld series analogous to Knopp's extension of the Franel theorem for the Lambert series is established; and we can show that if the sequences a_n and b_n are real and $0 < c_n \leqq C < \infty$, the unit circle is a natural boundary for the function represented by the Feld series. (Received July 15, 1949.)

535. W. T. Reid: *Self-adjoint boundary value problems.*

This paper presents an analysis of self-adjoint boundary value problems consisting of a system of first-order linear ordinary differential equations and two-point boundary conditions, and which are definite, according to either the definition of Bliss [Trans. Amer. Math. Soc. vol. 44 (1938) pp. 413–428] or that of the author [Trans. Amer. Math. Soc. vol. 52 (1942) pp. 381–425]. In particular, there is given a canonical form for such boundary problems. (Received July 18, 1949.)

536. L. G. Riggs and W. T. Scott: *Partially bounded J-fractions.*

Let Y_p be a constant such that $F_p(x) = \sum_1^p \beta_n x_n^2 - 2\sum_1^{p-1} \alpha_n x_n x_{n+1} \geqq -Y_p \sum_1^p x_n^2$, for all values of x_n, where α_n, β_n, x_n are real. The J-forms $F_p(x)$ are called partially bounded if lim g.l.b. $Y_p = Y$ exists. A sufficient condition for partial boundedness of the $F_p(x)$ is $|\alpha_p| \leqq A$, $\beta_p \geqq B$. Application to the J-fraction $-\mathrm{K}_1^{\infty}(-a_{p-1}^2/(z+b_p))$ of the method of Wall (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 686–693) establishes the existence of a uniquely determined convex region corresponding to the J-fraction. The J-fraction converges uniformly in any closed region at a positive distance from the convex set. (Received July 18, 1949.)

537. E. K. Ritter: *Second differentials of functions in exterior ballistics.*

The equations of motion of the mass-center of a projectile ordinarily are integrated under simplifying "standard" assumptions, yielding "normal" trajectories as solutions. Calculation of actual effects of "disturbances" (departures from standard conditions) is tedious and costly; linear approximations thereto (called "differential effects" and based on a theory developed by G. A. Bliss during World War I) are more easily computed. In an attempt to meet a need for improved estimates, the present paper develops a mathematical theory of *second* differential effects of disturbances. It includes: (1) a proof of the existence of a second differential of the mapping defined by the equations of motion; (2) a demonstration that this second differential is expressible in terms of the second variations of the mass-center coordinates; and (3) a derivation of the differential system satisfied by these second variations. (Received July 22, 1949.)

538*t*. W. W. Rogosinski and Gabor Szegö: *Extremum problems on non-negative trigonometric polynomials.*

Let $\sum_1^n b_k \sin k\theta$ be a sine-polynomial of degree n which is not identically zero and non-negative in $0 \leqq \theta \leqq \pi$. Let $b_1 = 1$. Exact bounds are obtained for the coefficients b_n, b_{n-1}, b_2, b_3. General methods based on the theory of orthogonal polynomials are discussed in order to obtain such bounds in the case of any b_k, and these methods are illustrated by the cases b_4 and b_5. (Received July 27, 1949.)

539*t*. Herman Rubin: *An axiomatic approach to integration.*

Let $\mathcal{E}$ be a linear space of real-valued functions on a set X, E a positive linear functional (called elementary integral) on $\mathcal{E}$. Two cases are considered, according to whether for a sequence of positive functions whose sum is greater than some function, the sum of their integrals is necessarily greater than the integral of that function, or not. $\sum$ denotes a countable sum if the first case (countably additive) is assumed, a finite sum (finitely additive) otherwise. An $f \in \mathcal{E}$ is called an ϵ-approximant of g if there exist positive h_1 such that $|f-g| \leqq \sum h_1$ and $\sum E(h_1) < \epsilon$. A function g is termed integrable if for every positive ϵ, g has an ϵ-approximant. Let $\mathcal{E}^*$ be the space of integrable functions. Then $\mathcal{E}^*$ is a lattice if for every $f \in \mathcal{E}$, $|f|$ has a positive ϵ-approximant for all ϵ. If $(X, \mathcal{E}, E)$ and $(Y, \mathcal{F}, F)$ are two such systems, then $(X \times Y, \mathcal{E} \times \mathcal{F}, E \times F)$ is also, and countable additivity and the lattice property are preserved. This leads to an elementary proof of Fubini's Theorem in both cases. (Received July 16, 1949.)

540*t*. Herman Rubin: *Measures and axiomatically defined integrals.*

Let $(X, \mathcal{E}, E)$ be a set, a linear space of functions on X, and an elementary integral on $\mathcal{E}$. Let $\mathcal{E}^*$ be a lattice. Then a function $\mathcal{F}$ is measurable if for any positive g in $\mathcal{E}^*$, mid $(-g, \mathcal{F}, g)$ is in $\mathcal{E}^*$. If 1 is measurable, a measure μ can be defined such that the integral (in the finitely additive case, of bounded functions vanishing outside a set of finite measure) corresponds to μ. A somewhat analogous situation prevails if there is an everywhere positive measurable function. The measurable sets correspond to the open and closed sets of a space (in the sense of A. D. Alexandroff). An unsolved question is whether or not there always exists an everywhere positive measurable function. (Received July 16, 1949.)

541*t*. Walter Rudin: *Integral representation of continuous functions.*

Let P denote a point in the finite plane domain D. Put $\Delta F(P) = \lim_{r \to 0} 4(m(F; P, r) - F(P))/r^2$, where $m(F; P, r)$ is the mean of F on the circle of radius r about P. $\Delta^* F$, $\Delta_* F$ are defined likewise, with lim sup, lim inf in place of lim. Let Z be a compact set of capacity zero. Theorem I. Let $F(P)$ be continuous in D. Suppose (i) $\Delta^* F(P) > -\infty$, $\Delta_* F(P) < +\infty$ on $D - D \cdot Z$; (ii) there is a function $y(P)$, $y \in L$ on every closed subset of D, such that $y(P) \leqq \Delta^* F(P)$ in D. Then $\Delta F \in L$ on every closed subset of D, and, for almost all P in D, $F(P) = -(1/2\pi)\iint_R \Delta F(Q)\, g(P, Q)\, dQ + H(P)$, where R is a bounded domain whose closure lies in D, and which contains P, $g(P, Q)$ is Green's function for R, and H is harmonic in R and assumes the boundary values of F. As a consequence, we obtain: Theorem II. Let $u(P)$ be continuous and subharmonic in D. Suppose $\Delta_* u(P) < +\infty$ in $D - D \cdot Z$. Then, at all points P of D, $u(P) = -(1/2\pi)\iint_R \Delta u(Q) g(P, Q) dQ + H(P)$, where H is the least harmonic majorant of u in R. In comparison with Riesz's theorem (Acta Math. vol. 54 (1930) p. 350), the significance of Theorem II lies in the appearance of a Lebesgue integral in place of a Stieltjes integral. Analogous results hold in spaces of higher dimension. (Received July 18, 1949.)

542. A. C. Schaeffer: *Fourier transforms of approximating functions.* Preliminary report.

Let $\phi_0(t)=c$, $\phi_1(t)$, $\phi_2(t)$, $\cdots$ denote a set of functions with continuous first order derivatives in a finite interval $0\leqq t\leqq a$ such that every continuous function is the limit in the mean in this interval of finite sums $\sum a_\nu\phi_\nu(t)$. If $p(t)$ belongs to L_2 in $0\leqq t\leqq a$ and its Fourier cosine transformation has exclusively real zeros, then there are finite sums $\sum a_\nu\phi_\nu(t)$ which converge in mean to $p(t)$ and whose Fourier cosine transformations have only real zeros. The smoothness conditions on the given sequence can be relaxed. The result is first proved in the special case in which the set ϕ_0, ϕ_1, ϕ_2, $\cdots$ is the set of functions $\cos(\nu\pi t/a)$, $\nu=0, 1, 2, \cdots$, and then in the general case using well known properties of entire functions of exponential type. (Received September 3, 1949.)

543*t*. H. M. Schaerf: *Determination of non-atomic measures.*

The family of all sets on which a measure m assumes a fixed value is called a *level family* of m. A class L of level families of m is termed *bounded* if there is a level family of m whose sets contain every set belonging to any family in L. A countably additive σ-finite measure m is called *non-atomic* if every set A of positive m-measure contains a measurable set B with $m(B)\cdot m(A-B)>0$. *Theorem: A non-atomic measure is uniquely determined (to within a multiplicative constant) by every infinite and bounded class of its level families.* The proof of this theorem relies on the Uniqueness Theorem in the abstract 544, *General theorems on the uniqueness and structure of invariant measures.* (Received July 14, 1949.)

544*t*. H. M. Schaerf: *General theorems on the uniqueness and structure of invariant measures.*

A measure m is called *invariant* under a relation R between some ordered pairs of measurable sets if ARB implies $m(A)=m(B)$. It is said to have the *intersection property* if, for every two sets A, B of positive measure, there are alike sets $A'\subset A$, $B'\subset B$ with $A'RB'$. R is called *regular* if it is an equivalence relation preserving countable disjoint partitions. *Uniqueness Theorem: In order that a σ-finite measure invariant under R be unique (to within a multiplicative constant) it is sufficient, and if R is regular also necessary, that every such measure have the intersection property. Structure Theorem: If a σ-finite measure m invariant under a regular relation R is unique, then any two sets of equal m-measure are almost congruent by countable partition. Moreover, then the values of m either are multiples of a positive number or fill a closed interval with left end point* 0. (Received July 14, 1949.)

545*t*. H. M. Schaerf: *Generalization of a theorem of Kakutani and Kodaira.*

This paper contains the proof of the following theorem: *Let G be an arbitrary locally compact topological group. Let S be the smallest μ-field of subsets of G which contains all closed compact G_δ-sets. Then the σ-field obtained by completing S with respect to any left-invariant measure defined in S and finite on some open set contains all closed compact subsets of G.* This theorem generalizes a result of S. Kakutani and K. Kodaira

(*Über das Haarsche Mass in der lokal bikompakten Gruppe,* Proc. Imp. Acad. Tokyo vol. 20 (1944) pp. 444–450). (Received July 14, 1949.)

546*t*. H. M. Schaerf: *On the uniqueness of Haar's measure.*

This paper removes all restrictions under which the uniqueness of *Haar's* measure is proved in the literature (like separability of the space or restriction to *Radon* measures). More generally, it contains the proof of the following theorem: *Let G be an arbitrary locally compact topological group. Let T be the smallest σ-field of subsets of G which contains all closed compact subsets of G. Let S be an arbitrary σ-field contained in T and containing a complete system of open neighborhoods of every point in G. Then a σ-finite left-invariant measure defined in S and finite on some open set is unique (to within a multiplicative constant).* (Received July 14, 1949.)

547*t*. H. M. Schaerf: *Uniqueness of an invariant measure in an abstract group.*

This paper contains a short and simple proof of the uniqueness of an invariant measure in an abstract group under more general assumptions than the ones made in A. Weil, *L'Intégration dans les groupes topologiques,* pp. 148–149. The proof is based on the Uniqueness Theorem of Abstract 544. (Received July 14, 1949.)

548*t*. Robert Schatten: *The space of completely continuous operators on a Hilbert space.*

In a Banach space whose elements are (some, not necessarily all) linear bounded operators on a Hilbert space, the norm of an element is in general different from its bound. In this connection a direct proof of the following theorem is given: Let $\mathfrak{C}$ denote the Banach space of all completely continuous (linear bounded) operators on a Hilbert space $\mathfrak{H}$, where the norm of an operator is given by its bound. Then, its first adjoint space $\mathfrak{C}^*$ may be interpreted as the trace-class. The trace-class is the Banach space of all linear bounded operators A on $\mathfrak{H}$, for which $\sum_i((A^*A)^{1/2}\phi_i, \phi_i) < +\infty$ for a complete orthonormal set (ϕ_i), the last sum which is independent on the chosen complete orthonormal set represents the norm of A in $\mathfrak{C}^*$. Moreover, the second adjoint space $\mathfrak{C}^{**}$ of $\mathfrak{C}$ may be interpreted as the Banach space of all linear bounded operators on $\mathfrak{H}$ where again the bound of an operator represents its norm. (Received June 30, 1949.)

549*t*. Annette Sinclair: *Generalization of Runge's theorem to approximation by analytic functions.*

Runge's Theorem on approximation of a function analytic on a closed set by a rational function is generalized to approximation on a set S whose components are closed and whose sequential limit points are in the complement of S. (A sequential limit—s.l.—point is a limit point of some set consisting of one point of each component of S.) The approximating function is analytic except possibly on a preassigned set C consisting of (1) the s.l. points and (2) one point of each component of the complement of S which contains no s.l. point. If the restriction is placed on the function to be approximated that it be nonvanishing on S and that its logarithm be single-valued on S, the approximating function can be required to have not only its singularities but also its zeros in C. The approximation obtained is not only uniform on S but is such that the closeness of approximation can be preassigned independently for each component. (Received July 5, 1949.)

550. H. L. Smith: *Relative contingent and n-hedra to paths.*

If P_0 is a limit point of a set E in a Euclidean space and M is a class of vectors in that space, then a vector ξ_0 is a semi-tangent to E at P_0 relative to M if there is in E a sequence of points $\{P_n\}$ which converges to P_0 and is such that sig $\omega(P_n - P_0, M)$ converges to ξ_0, where $\omega(P_n - P_0, M)$ denotes the component of the vector $P_n - P_0$ which is orthogonal to M, and sig $\omega(P_n - P_0, M)$ is $\omega(P_n - P_0, M)$ divided by its length if $\omega(P_n - P_0, M) \neq 0$ and is 0 otherwise. The contingent to E at P_0 relative to M is the set of all its semi-tangents at P_0 relative to M. This concept is used to give an intrinsic definition of n-hedron to a path. Conditions necessary and conditions sufficient for the existence of such an n-hedron are derived. (Received July 20, 1949.)

551. E. J. Specht: *Estimates on the mapping function and its derivatives in conformal mapping of nearly circular regions.*

Let C denote a simple, closed curve with polar equation $\rho = \rho(\phi)$, $0 \leqq \phi \leqq 2\pi$, where $\rho(\phi)$ is positive, continuous, and periodic of period 2π. Let the function $w = f(z)$ map the circle $|z| < 1$ conformally onto the interior of C in such a manner that $f(0) = 0$ and $f'(0) > 0$. If, for some positive ϵ, $1 \leqq \rho(\phi) \leqq 1 + \epsilon$ and if there is a positive number M such that $|\rho(\phi + \delta) - \rho(\phi)| \leqq M\epsilon\delta$ where δ is any real number, then there exists a constant K which depends only on M such that $|f(z) - z| \leqq K\epsilon$. In fact $K \leqq 1 + M(2 \log 2 + 1)$ if $\epsilon \leqq 1$. (This result was stated by A. R. Marchenko in 1935 without proof and without a numerical estimate of K.) By imposing further restrictions on the curve C (one of which involves essentially the tangent angle or the curvature), analogous numerical estimates of $|(d/d\theta) \arg f(e^{i\theta}) - 1|$, $|f'(z) - 1|$, $|(d^2/d\theta^2) \arg f(e^{i\theta})|$, and $|f''(z)|$ are obtained and the existence of bounds for $|(d^n/d\theta^n) \arg f(e^{i\theta})|$ and $|f^{(n)}(z)|$ $(n \geqq 3)$ is established. The principal tools used in the proofs are the Poisson integral representation of a harmonic function in the unit circle and the well known formula for the conjugate $\tilde{g}(\theta)$ of an integrable function $g(\theta)$. (Received July 18, 1949.)

552. Otto Szasz: *On the Gibbs' phenomenon for Euler means.*

The classical Gibbs' phenomenon for the partial sums of certain Fourier series at a point of discontinuity is here generalized to Euler means. It is shown that the sequence of Euler's means presents essentially the same Gibbs' phenomenon. Similar results hold for certain Hausdorff means and for Borel's summability method. (Received July 12, 1949.)

553. H. P. Thielman: *On functional equations and means.*

Some of the results of this paper are generalizations of the results of Aczél (Comment. Math. Helv. vol. 21, pp. 247–252). The method for deriving the results consists in reducing the problems to the solution of the well known Cauchy functional equations. Functional equations of the form $F(ax + by + c) = \alpha G(x) + \beta H(y) + \gamma$ are solved and the results are used to derive results for generalized means. Solutions to sets of simultaneous functional equations are also found. These functional equations are broad generalizations of such functional equations as $g(xy) = y^\beta g(x) + x^\beta g(y)$, which appeared recently in the mathematical literature (Amer. Math. Monthly vol. 56 (1949) p. 414). (Received July 15, 1949.)

554. J. L. Ullman: *A problem of Tchebychef.* Preliminary report.

Tchebychef proposed, as a method of approximating an integral $\int_0^1 f(t)dt$, the formula $(1/n)\sum_1^n f(t_{ni})$, where, for each n, $n=1, 2, \cdots$, the n numbers $t_{n1}, \cdots, t_{nn}$ satisfy the condition that $\int_0^1 P_n(t)dt=(1/n)\sum_1^n P_n(t_{ni})$, where $P_n(t)$ represents an arbitrary polynomial of degree n. S. Bernstein showed that for sufficiently large n, the numbers t_{ni} would not be real, so that this method would not be applicable if $f(t)$ were defined for real values only. This suggests the following problem. Let $\alpha(t)$ be a positive, continuous function on the open interval $(-1<t<1)$ satisfying $\int_{-1}^1 \alpha(t)dt=1$. As an approximation formula for the integral $\int_{-1}^1 f(t)\alpha(t)dt$, consider the expression $(1/n)\sum_1^n f(t_{ni})$, where for each n, the numbers $t_{n1}, \cdots, t_{nn}$ satisfy the condition that $\int_{-1}^1 P_n(t)\alpha(t)dt=(1/n)\sum_1^n P_n(t_{ni})$, where $P_n(t)$ is an arbitrary polynomial of degree n. For which functions $\alpha(t)$ will the limit points of the set t_{ni} $(i=1, \cdots, n; n=1, 2, \cdots)$ all lie on the interval $(-1\leqq t\leqq 1)$? For those $\alpha(t)$ for which this is true, the integral can be approximated by the sum for any $f(t)$ analytic in a domain containing the interval $(-1\leqq t\leqq 1)$. It is a well known result, that if $\alpha(t)$ is given by $(1/\pi)(1-t^2)^{-1}$, then the t_{ni} will be the roots of the Tchebychef polynomials of the interval $(-1\leqq t\leqq 1)$ and will all lie in this interval. The more general result obtained, using theorems on the derived set of zeros of Faber polynomials, is that if $\alpha(t)=(1/\pi)(1-t^2)^{-1/2}\operatorname{Re} zF'(z)/F(z)$, $z=t+(t^2-1)^{1/2}$, where $F(z)$ is the normalized exterior mapping function of a simple analytic curve, symmetric to the real axis, and contained in the circle $|z|\leqq 1$, then the limit points of the t_{ni} will all lie on the interval $(-1\leqq t\leqq 1)$. (Received July 18, 1949.)

555*t*. Antoni Zygmund: *A remark on functions of several complex variables.*

Let $f(z_1, z_2, \cdots, z_k)$ be a function of complex variables $z_1, \cdots, z_k$ regular for $|z_1|^2+\cdots+|z_k|^2<1$, and such that the integral $\int_{\sigma_r}\log^+|f|d\sigma_r$ is bounded for $r<1$-where σ_r denotes the sphere $z_1^2+\cdots+z_k^2=r^2$ and $d\sigma_r$ the element of the ((2$k-1$), dimensional) volume of the sphere. Then f has a finite non-tangential limit at almost every point of the sphere $|z_1|^2+\cdots+|z_k|^2=1$. As a corollary of this and of a recent result of Rauch, one obtains that if $\int_{\sigma_r}|f|^p d\sigma_r$ is bounded for some $p>0$, then $\int_{\sigma_r}|f(z_1, \cdots, z_k)-f(rz_1, \cdots, rz_k)|^p d\sigma_r\to 0$ as $r\to 1$. (Received August 2, 1949.)

APPLIED MATHEMATICS

556. Nathaniel Coburn: *"Characteristic directions" in three-dimensional supersonic flows.*

It has been shown (*Supersonic flow and shock waves*, R. Courant and K. O. Friedrichs, Interscience, New York, 1948, pp. 71–75; *Nonlinear hyperbolic differential equations for functions of two independent variables*, K. O. Friedrichs, Amer. J. Math. vol. 70 (1948) pp. 555–589) that a system of n-totally hyperbolic quasilinear first order partial differential equations can be replaced by another system (consisting of linear combinations of the original equations) with the property that each of the latter equations involves differentiation in only one direction. Such directions are called "characteristic directions." In the present paper, it is shown that no such "characteristic directions" exist for non-isentropic or isentropic, irrotational, steady, three-dimensional flows. If "characteristic pairs of directions" are defined by the condition that the linear combinations of the original equations involve differentiations with respect to two directions, then such directions always exist for isentropic, irrotational flows. In fact, any two directions lying in the planes which envelope the bicharacteristic cone are permissible. (Received February 11, 1949.)

557. K. L. Conrad: *Stress distribution due to hydrostatic pressure on a parabolic boundary.*

The boundary condition for the stress distribution due to hydrostatic pressure on a parabolic boundary has been expressed in parabolic coordinates as a single integral by means of the Fourier integral theorem. A biharmonic solution, X, can be expressed in the terms of two complex functions $\phi(z)$ and $f(z)$ by means of the expression $X=\bar{z}\phi(z)+z\bar{\phi}(\bar{z})+f(z)+\bar{f}(\bar{z})$. The two relationships of the stresses in parabolic coordinates, $J_{\xi\xi}+J_{\eta\eta}$ and $J_{\xi\xi}-J_{\eta\eta}+2iJ_{\xi\eta}$, are determined by the complex functions $\phi(z)$ and $f(z)$. A finite form has been obtained for those complex functions from which the stresses and displacements have been determined. (Received July 16, 1949.)

558. M. Dresden: *The Birkhoff "billiard ball problem" in quantum mechanics.*

Many of the problems of dynamics can be formulated in a natural fashion in quantum mechanics. In particular those problems dealing with the existence and explicit evaluation of time averages would appear to find a more appropriate formulation in terms of quantum theory. This reformulation has been performed for the Birkhoff "billiard ball problem." In some cases one can obtain explicit solutions to the quantum mechanical problem. Then one can utilize the connection between the classical time averages and the approximate (WKB) quantum averages to evaluate explicitly some classical averages. This general connection suggests two types of problems. The first group of problems concerns the relationship between the results in classical mechanics of Poincaré and Birkhoff and similar ones in quantum mechanics. The second group of problems refers to the fashion in which these classical results are contained in the quantum mechanical formalism. These general problems will be discussed using the "billiard ball problem" as an example. (Received July 15, 1949.)

559. R. N. Goss: *Center of flexure of beams of triangular section.*

The center of flexure is that point of loading in a section of a beam for which the local twist vanishes at the centroid of the section. For a beam of right-triangular section with vertices at $(0, 0)$, $(0, c)$, $(-c/m, 0)$ in the plane $z=z_1$, when the load acts parallel to the hypotenuse, the center of flexure is found to lie on the line $y-mx=[c(3+4\sigma)/5(1+\sigma)]-[4\sigma Dm(m^2+1)/\mu c^3(1+\sigma)]$ in this plane, where σ is Poisson's ratio, μ is the modulus of rigidity, and D is the torsional rigidity. In the case of the general triangular section with sides $y=d$, $y=m_1x$, $y=m_2x$ and any direction of load, the coordinates of the center of flexure have been found for $\sigma=1/2$. The results agree with those obtained previously for the isosceles-triangular section (Bull. Amer. Math. Soc. Abstract 55-7-412). (Received July 14, 1949.)

560. Eberhard Hopf: *Integration of Burgers' simplest space-time model for free turbulence.*

As a simplified model for turbulent fluid flow not activated by external forces (free turbulence), J. M. Burgers considered the partial differential equation $u_t+uu_x=\nu u_{xx}$ (see his article in *Advances in Applied Mechanics*, edited by v. Mises and v. Kármán, New York, 1948, pp. 171–196, in particular pp. 182–192). The present author supplements Burgers' approximative study of his equation by a rigorous and explicit solution of the problem. (Received July 14, 1949.)

561*t*. Robert Kahal: *The realization of the transfer function of a four-terminal network.*

The transfer function, $\theta(p)$, of a four-terminal network is defined as the ratio of the output voltage to the input voltage. The following theorems are proved. (I) The necessary and sufficient conditions that a function $\theta_L(p)$ shall be the transfer function of an L network are: (1) $\theta_L(p)$ has no poles or zeros in Re $(p) > 0$. Poles on the boundary are simple. (2) $\theta_L(p)$ is real when p is real, and $0 < \theta_L(p) < 1$ when p is real and positive. (3) $|\arg \theta_L(p)| \leq \pi$ when $|\arg p| = \pi/2$. (II) The necessary and sufficient conditions that $\theta(p)$ shall be, to within a constant factor, the transfer function of a ladder network are: (1) $\theta(p)$ has no poles or zeros in $\mathrm{Re}(p) > 0$. Poles on the boundary are simple. (2) $\theta(p)$ is real when p is real, and $0 < \theta(p) < 1$ when p is real and positive. (III) The necessary and sufficient conditions that a function $\psi(p)$ shall be the product of two positive real functions are: (1) $\psi(p)$ has no poles or zeros in $\mathrm{Re}(p) > 0$. (2) $|\arg \psi(p)| \leq \pi$ when $|\arg p| = \pi/2$. (3) $\psi(p)$ is real when p is real. A method for the synthesis of the L and ladder networks is given. (Received July 16, 1949.)

562. M. H. Martin: *Steady, plane, rotational Prandtl-Meyer flows of a polytropic gas.*

The theory of Prandtl-Meyer flows, first discovered among the irrotational flows of a polytropic gas, is extended to rotational flows of a polytropic gas. If the streamlines ($\psi =$const.) and the isobars ($p =$const.) are taken as curvalinear coordinates, the general problem of determining steady plane flows can be reduced to finding two functions, $q = q(p, \psi)$, $\theta = \theta(p, \psi)$ (that is, the speed and direction of flow) so that they satisfy the equation $q((q_{pp} - q\theta_p^2)/\theta_\psi)_\psi + (q^2\theta_p)_p = 0$. If $q = q(p, \psi)$ is preassigned, the problem of determining Prandtl-Meyer flows reduces to the determination of solutions of this equation having the form $\theta = \theta(q)$, in which case the equation reduces to an ordinary differential equation of Bernoulli type. The properties of rotational Prandtl-Meyer flows are investigated with the aid of the solutions of this ordinary differential equation. (Received July 8, 1949.)

563. P. F. Nemenyi and A. H. Van Tuyl: *Two-dimensional plastic stress systems with isometric principal stress trajectories.*

Carathéodory and Schmidt, in their paper on the maximum shear trajectories in plane plastic stress with the yield condition $\tau = (\sigma_1 - \sigma_2)/2 =$constant, discussed the possibility of these trajectories forming an isometric net. They found that the only such isometric nets are characterized by the complex function $Z(s)$ satisfying the equation $s + a_1 + ia_2 = \int \exp [aZ^2 + (b_1 + ib_2)Z + c_1 + ic_2]dZ$, where a, a_1, a_2, b_1, b_2, c_1, and c_2 are real constants. The same nets are also possible principal stress trajectories. The authors generalize the inquiry of Carathéodory and Schmidt to an arbitrary yield function $\tau = f(\sigma)$ $(2\sigma = \sigma_1 + \sigma_2)$. When the principal stress trajectories form an isometric net corresponding to the function $Z = \phi + i\psi$, the Lamé equations of equilibrium give the relations $2 \log |dZ/ds| = \log f(\sigma) + F(\phi) - G(\psi)$ and $\int d\sigma/f(\sigma) = F(\phi)$ $G(\psi)$. A complete discussion of these equations shows that only for four distinct families of yield conditions, depending on from two to five parameters, do there exist isometric nets satisfying the conditions of the problem beyond the trivial nets consisting of concentric circles and concurrent straight lines. The functions Z belonging to each family are given either explicitly or in the form of quadratures. It is found that any linear yield function is consistent with the same family of isometric nets as the yield function $\tau =$ constant. (Received September 12, 1949.)

564. L. E. Payne: *Torsion and flexure of composite sections.* I. *Torsion.*

The Saint Venant torsion problem for composite cylinders of various types of cross sections has been considered. The sections investigated fall into two general groups. In the first group are sections made up of two different isotropic materials. The corresponding solutions are obtained for the cases of eccentric circles, confocal and concentric ellipses, as well as rectangular sections. In the second group the sections are partly isotropic and partly orthotropic. Here, solutions are obtained for the concentric circular and confocal elliptic sections and the rectangular section. It is shown that in the first group the rigidity of the composite section may be obtained by adding together the rigidities of the two different isotropic parts only if the lines dividing them are lines of shearing stress. The rigidities in the various cases are compared with those corresponding to the completely isotropic and completely orthotropic sections. The flexure problem is currently being investigated. (Received July 12, 1949.)

565. B. R. Seth: *Finite elasto-plastic torsion.*

In the customary treatment of elasto-plastic torsion the displacements and the strains are both assumed to be small. In a number of plastic problems both these assumptions need not hold good. We propose to discuss the torsion of a circular cylinder by taking finite components of displacements and strains. The problem is examined from the point of view of both the rival theories of plasticity—the theory of plastic flow and the theory of plastic deformation. For a first approximation in which axial stresses are neglected both the theories give the same results. A second approximation has been carried out and the corresponding torsional couple and stresses have been calculated. (Received July 11, 1949.)

566. Domina E. Spencer: *Separability conditions for some equations of mathematical physics.*

Though considerable research has been done on separability conditions for the equations of mathematical physics, no comprehensive treatment of the subject is available. In this paper, the method of Robertson (Math. Ann. vol. 98 (1927) p. 749), which was originally applied to simple separability of the Schrödinger equation, is extended to obtain necessary and sufficient conditions for both simple separability, $\phi = U^1(u^1)U^2(u^2)U^3(u^3)$, and R-separability, $\phi = U^1(u^1)U^2(u^2)U^3(u^3)/R(u^1, u^2, u^3)$, of the Laplace, Helmholtz, diffusion, wave, and Schrodinger equations. The conditions are expressed in terms of Stäckel matrices. (Received July 11, 1946.)

567*t*. John Todd: *The condition of certain matrices.*

Two measures (the N-condition number and the M-condition number) for the condition of a matrix A have been suggested by A. M. Turing (Quarterly Journal of Mechanics and Applied Mathematics vol. 1 (1948) pp. 287–308). A third measure is suggested by the results of John von Neumann and H. H. Goldstine (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1021–1099); we call $|\lambda|/|\mu|$ the P-condition number of A if $|\lambda|$, $|\mu|$ are the greatest and least of the absolute values of its characteristic roots. These measures (or reasonable appraisals of them) have been calculated for the matrices which arise in the approximate solution of ordinary and partial differential equations. Elementary methods, Rayleigh's Principle, and results of D. E. Rutherford (Proc. Royal Soc. Edinburgh vol. A52 (1947) pp. 229–236), among others, are employed. The results obtained show that these measures are in substantial agreement among themselves and with the experience of the practical computer (L. Fox, Proc. Cambridge Philos. Soc. vol. 45 (1949) pp. 50–68). Among the results is confirmation of the observed fact that the matrices occurring in the approximate solution of a partial differential equation, for example, $z_{xx}+z_{yy}=kz$, are in better condition than those arising from the corresponding ordinary differential equation, for example, $y''=ky$. (Received July 14, 1949.)

568. C. A. Truesdell: *On finite strain of an elastic body.*

There have been numerous recent investigations of the general theory of elasticity. For the case when the work of deformation is completely stored as strain energy, however, the general theory has been complete for a long time. Both Eulerian and Lagrangian strain measures were defined by Cauchy (1823, 1827, 1841). Green (1839) introduced the strain energy in full generality, and Kirchhoff (1852) in deriving a second order theory gave formal apparatus and methods from which the fully general theory follows easily, although it was not actually worked out until later by Kelvin (1863) and Boussinesq (1872). Finger (1894) discovered that for isotropic bodies the strain energy may be regarded as a function of Eulerian strain measures only, and worked out the resulting simplified form of the stress-strain relations. Apparently one reason for so many recent re-discoveries of classical results is the wilderness of long and awkward notations still in vogue. Rivlin (1948) pointed out that for incompressible materials the stresses are determined by the energy equation only up to an arbitrary Eulerian hydrostatic pressure, and proposed a simple one modulus nonlinear theory. In the present note Rivlin's approximate theory is compared with that of Kirchhoff. It is shown that Rivlin's theory results from discarding all terms retained by Kirchhoff but retaining the first order term, representing an initial Lagrangian pressure, which Kirchhoff put equal to zero. Since it is the arbitrary Eulerian pressure resulting from the condition of incompressibility which makes the resulting theory nontrivial, a corresponding approximation for compressible materials could not lead to useful results. The Rivlin theory can thus be put in a symmetrical form: the stress is always the sum of a Lagrangian pressure and an Eulerian pressure. (Received July 16, 1949.)

GEOMETRY

569. N. A. Court: *On the polar circles of the faces of a tetrahedron.*

Let H_a be the orthocenter of the face BCD of a tetrahedron $(T)=ABCD$, (H_a) the sphere having for great circle the polar circle of the triangle BCD, and (P_a) the orthoptic sphere of (H_a). Analogous elements are attached to the other faces of (T). Let (H'') denote the tetrahedron having for vertices the orthocenters H_a, H_b, $\cdots$. a. The sum of the squares of the distances of the circumcenter of (T) to the orthocenters of its faces is equal to twice the sum of the squares of the radii of the polar circles of the faces of (T) increased by the square of the circumdiameter of (T). b. The orthogonal sphere of the four spheres (H_a), (H_b), $\cdots$ is coaxal with the circumspheres of the tetrahedrons (T) and (H''). c. The circumsphere of (T) is orthogonal to the six spheres of similitude of the four spheres (H_a), (H_b), $\cdots$ taken in pairs. d. The orthogonal sphere of the four spheres (P_a), (P_b), $\cdots$ coincides with the circumsphere of the tetrahedron (T). (Received July 15, 1949.)

570. V. G. Grove: *A note on isothermal nets.*

This paper gives a new characterization of isothermally conjugate nets using a notion we have called the planar property of a family of congruences associated with a curve of the net. The net is isothermally conjugate if and only if any one of these congruences has the planar property. An extension is made to isothermally orthogonal non-conjugate nets. A by-product of the paper is a method of generating all classical canonical lines. (Received June 15, 1949.)

571. W. R. Hutcherson: *A cyclic involution of order eleven.*

In two previous papers (Bull. Amer. Math. Soc. vol. 37 (1931) pp. 759–765; and vol. 40 (1934) pp. 143–151) cyclic involutions of orders three, five, and seven were discussed. This paper considers the transformation T, $x_1':x_2':x_3':x_4' = x_1:\epsilon x_2:\epsilon^2 x_3:\epsilon^3 x_4$ where $\epsilon^{11}=1$. C is defined as any curve on the quartic surface, F_4, $ax_2x_3^3+bx_1x_2x_4^2+cx_1x_3^2x_4+dx_2^2x_3x_4=0$, passing through $P_3(0010)$, a point of coincidence (invariant point). The image of C under T is another curve on F_4 passing through P_3. This point P_3 is called "a perfect point of coincidence" if C and every image of C touches the same line at P_3 for every tangent to F_4 at P_3. However, P_3 is found to be a non-perfect point. Three other non-perfect points are $P_1(1000)$, $P_2(0100)$, and $P_4(0001)$. It was known that, for an involution of order three, points in the first order neighborhood of P_3 along the two invariant directions are perfect. For I_5 (involution of order five) one may have second order neighborhood perfect points as well as first order; for I_7, third order as well as second order. This paper found that P_3, which is simple on F_4, has a perfect point in the third order neighborhood along the $x_1=x_2=0$ direction while along the other invariant direction $x_2=x_4=0$, one must go out to the point of the fourth order neighborhood. The involution contained on F_4 was mapped upon a ϕ surface of order 44 in a space of thirty-two dimensions. (Received July 14, 1949.)

572*t*. B. E. Mitchell: *The complex complete quadrangle and quadrilateral.*

This paper is presented in the interest of constructions in the complex plane. The complete quadrangle, tetrastigm (Lachlan), is a configuration consisting of seven points and nine lines. The complete quadrilateral, tetragram, is made up of seven lines and nine points. The two configurations are dual. Interest centers in the distribution of the real and the imaginary among the sixteen elements of each configuration as step by step the base elements, four points in the one case and four lines in the other, become imaginary. The case in which all the base points (lines) become imaginary and conjugate in pairs receives special attention. In this case half the elements are real and half are imaginary—ideal complexity. All imaginary elements, both points and lines, are given real representation. (Received July 10, 1949.)

573*t*. T. S. Motzkin: *The lines and planes connecting the points of a finite set.*

The author's conjecture (Dissertation, Basel, Jerusalem 1936, p. 31) that, in d-dimensional projective space over an arbitrary coördinate field, n points that are not on one hyerplane determine at least n hyperplanes is proved by use of a combinatorial lemma and an idea of de Bruijn and Erdös, Indagationes Mathematicae vol. 10 (1948) p. 422, who proved the case $d=2$. For the real plane the theorem follows also from Sylvester's conjecture, proved by Gallai, that if n points are not on one straight line there exists an ordinary line, that is, a straight line containing only two of the points. The question of de Bruijn and Erdős whether the number of ordinary lines tends to infinity with n is answered in the affirmative. It is further shown that n points in real 3-space that are not on one plane determine at least four ordinary planes, that is, planes on which all of the given points but one are collinear. For general coördinate fields and $n=7$ or $n>8$ no ordinary line need exist. (Received July 22, 1949.)

LOGIC AND FOUNDATIONS

574*t*. A. R. Schweitzer: *An examination of Bergson's "Creative evolution."*

Bergson's *Creative evolution* is examined with regard to: I. doctrine of opposites, II. main working hypothesis, III. theory of relations, IV. references to orientation. The author interprets Bergson's treatise as a theory of knowledge, primarily of biological reference, elaborating a discrimination between "intuition" and "intellect." For intuition is associated by Bergson with instinct, life, the vital order, creation, invention, non-mathematical time, duration; and intellect is associated with intelligence, matter, the geometrical order, logic, space. The preceding two opposed sets of terms permit transition to one of Bergson's central themes, namely, his working hypothesis of divergent lines of evolution mutually complementary (inverse) and motivated by a common impulse (impetus). An analogy is found between Bergson's working hypothesis and the author's "principle of furcation" reported in Bull. Amer. Math. Soc. vol. 21 (1915) pp. 376–377, abstract 5; vol. 22 (1916) p. 294, abstract 25; see also ibid. vol. 20 (1914) p. 454, abstract 8. Relations are associated by Bergson with intelligence whose function it is to establish them; relation is essentially comparison. Bergson's references to orientation include ascent, descent, motion (movement) organization, order, inversion, complementariness, oppositeness. (Received July 18, 1949.)

575*t*. A. R. Schweitzer: *On a relation of mathematics to biology.*

This paper aims to effect a gradual transition from mathematics to biology based on the author's foundations of geometry in terms of "right" and "left" (sameness and difference of orientation) in Amer. J. Math., 1909, 1913 with the monograph of W. Ludwig, *Das Rechts-Links-Problem im Tierreich und beim Menschen* (Berlin, 1932) and Kant's philosophy as connecting bonds. The author cites Ludwig, loc. cit., p. 442, paragraphs 2, 3. Ludwig asserts "right" and "left" to have been the source of Kant's philosophy and, more generally, to have been associated of old with pairs of opposites (for example, the author notes, by the Pythagorean school of philosophy). The author remarks that "right" and "left" do not seem to occur explicitly in Kant's *Critique of pure reason*; however, the latter treatise abounds in pairs of opposites. Reference is made to the statement of Sir James Jeans (*Physics and philosophy*, Cambridge and New York, 1946, p. 38) that Kant was one of the earliest of biological evolutionists. (Received July 18, 1949.)

576. D. M. Studley: *Identity, equivalence, and distinction.*

The three elements of conception, identity, equivalence, and distinction, are considered with attention centered upon relations among them. In order to obtain full grasp on the relations, the system of method of the author's Lawrence paper (*The mathematics of language*) is used. Individual point or element identity is separated in thought from identity of form or class. Equivalence which is contained in identity is shown to be intersection of form. Further, equivalence is defined as the space between identity and distinction. The third element, distinction, is seen as the sum of two identities modulo two. Corporeal distinction is separated in thought from distinction of form. The author's concept of abstract isomorphism (Mathematics Magazine vol. 22 (1949) pp. 191–193) is classed as equivalence. A clarification of orientation results from this study. (Received June 2, 1949.)

577. Wanda Szmielew and Alfred Tarski: *Theorems common to all complete and axiomatizable theories*. Preliminary report.

All the theories considered are formalized within lower predicate calculus; they are supposed, for simplicity, to contain one non-logical constant—operation symbol $+$. Consistent, complete, axiomatizable (that is, with a finite system of non-logical axioms) theories of this kind will be called *CA-theories*. Many different examples of *CA*-theories are known; for example, the theories of (i) groups of a given prime order, (ii) atomless Boolean algebras, (iii) simply and densely ordered lattices with zero and unit. Nevertheless, *there are non-tautological statements derivable from axioms of every CA-theory*; they will be called *CA-statements*. For example, the negated conjunction of all non-logical axioms of any axiomatizable, essentially undecidable theory (see Journal of Symbolic Logic vol. 14, p. 75) is a *CA*-statement. Results in Bull. Amer. Math. Soc. Abstract 55-1-79 lead to another class of *CA*-statements. It consists of the statements which sometimes fail when applied to group operations in infinite commutative groups, but hold for any other operation $+$; for example: *If, for all* x, y, z, $x+(y+z)=(x+z)+y$, *and* $x=y+(u+u)$ *for some* u, *then, for no* x, $(x+x)+x=x \neq x+x$ (in other words, if $+$ is a commutative group operation with all elements divisible by 2, then no element is of order 2). Finally notice: there are consistent, complete, non-axiomatizable theories in which all the *CA*-statements hold. (Received June 27, 1949.)

STATISTICS AND PROBABILITY

578*t*. Maria Castellani: *Theorems on convergence of compound distributions with symmetric components*.

The theorems deal with operations of convolution in R_1 concerning specific families of distributions. Thus, $K_\alpha(x)=F(x)^*G_\alpha(x)$ is obtained by combining any d.f. $F(x)$ with any d.f. $G_\alpha(x)$ which depends on a parameter under the restriction of symmetry when $G_\alpha(x+h)+G_\alpha(x-h)=1$ for any $h>0$. A generalization of Cantelli's inequalities will enable us to prove that a given d.f. $F(x)$ in R_1 can by association with a convenient symmetric d.f. $G_\alpha(x)$ of continuous type give a continuous d.f. $K_\alpha(x)$, which for $\alpha\rightarrow\infty$ converges asymptotically almost everywhere to the given $F(x)$. Therefore at any continuity bordered interval for a given $F(x)$ there exist convenient, uniformly convergent series of continuous functions which asymptotically approach the given d.f. $F(x)$. (Received July 14, 1949.)

579*t*. Herman Chernoff: *Remarks on a rational selection of a decision function*.

Problems in the theory of statistics seem to reduce to the selection of a strategy (decision function) from several available strategies. The problem of the selection of such a strategy is analogous to that in the two person zero sum game where nature is the statistician's opponent. A major difference between these two problems lies in the fact that it does not seem proper to regard the unknown state of nature as a strategy of an opponent. Several criteria previously mentioned for determining a strategy are discussed and seem to have inadequacies. An axiom system is constructed for a criterion to be considered "rational." When applied to the set of problems where there are only two states of nature, the results are that the only criterion satisfying the axiom system is equivalent to assuming an a priori probability of 1/2 for each state of nature. (Received July 14, 1949.)

580*t*. K. L. Chung and Paul Erdös: *Probability limit theorems assuming only the first moment*. Preliminary report.

We consider independent random variables $X_1, X_2, \cdots$, having the same distribution $F(x)$; $S_n = \sum_{k=1}^{n} X_k$. The novelty consists in that about this distribution we assume only $\int_{-\infty}^{\infty} x dF(x) = 0$ or both $\int_0^{\infty} x dF(x)$ and $\int_{-\infty}^{0} x dF(x)$ diverge. Analytical and combinatorial methods are developed for treating this case, the latter being more successful at present. For simplicity we state results for the case where each X assumes only integral values and such that if a is any integer, $P(S_n = a) > 0$ for sufficiently large n. Estimates of $P(S_n = a)$ are obtained; in particular we prove that $\lim P(S_n = a)/P(S_n = b) = 1$. Further let $N_n(a)$ and $N_n(b)$ denote the number of a's and b's respectively among the sequence $S_1, \cdots, S_n$, then $P(\lim N_n(a)/N_n(b) = 1) = 1$. Extensions to continuous distributions will be considered. (Received July 18, 1949.)

581. C. L. Dolph and M. A. Woodbury: *Optimal linear prediction of stochastic processes whose covariances are Green's functions.*

A method of unbiased, minimal variance, linear prediction is developed for problems similar to those of prediction and filtering treated by Wiener. It differs from these in that the unbiased condition is imposed, only a finite part of the past is employed, and no stationary assumption is used. It is shown that the special stationary case discussed by Cunningham and Hund, *Random processes in problems of air warfare* (Supplement to the Journal of the Royal Statistical Society, 1946) succeeds because the correlation function $e^{-\lambda|t-s|}$, well known to that of the process defined by the Langevian equation, is the Green's function of the homogeneous differential equation formed by letting the adjoint differential operator of the Langevian equation operate on the operator of this equation. This relationship is shown to persist for any physically stable linear differential equation driven by "white noise." The well known equivalence between integral and differential equations is then extended by use of Stieltjes integrals and used to effect the solutions of the integral equations of the first kind which yield the "optimum" linear prediction. The nonstationary example consisting of purely random motion about a mean linear path in the presence of radar type errors is treated in detail. (Received June 27, 1949.)

582. H. H. Germond: *The integral of the Gaussian distribution over the area bounded by an ellipse.*

This paper describes the preparation of tables from which to obtain the integral of a bivariate Gaussian distribution over the area of an ellipse. The center of the ellipse need not coincide with the mean of the Gaussian distribution, nor need the axes of the ellipse have any special orientation with respect to the Gaussian distribution. (Received July 15, 1949.)

583. P. G. Hoel; *On the problem of optimum classification.*

Let f_i, $i = 1, 2, \cdots, k$, be the probability density function of population i and let p_i be the probability that population i will be sampled. Assume certain differentiability conditions and moment properties. Then, for known parameters, the probability of a correct classification will be maximized by choosing the region M_i, which corresponds to classifying into population i, as that part of variable space where $p_i f_i \geq p_j f_j$, $j=1, 2, \cdots, k$. If the parameters are unknown, an asymptotically optimum set of estimates will be given by the set that minimizes a certain form in the covariances. Among uncorrelated estimates, maximum likelihood estimates are seen to be asymptotically optimum. If weight functions, W_{ij}, are introduced and the expected value of the loss is minimized, the same methods of proof show that the region M_i becomes that part of variable space where $\sum_{r=1}^{k} p_r f_r (W_{rj} - W_{ri}) \geq 0, j=1, 2, \cdots, k$, and that the criterion for an asymptotically optimum set of estimates is of the same form as the preceding criterion. (Received July 11, 1949.)

584. Leo Katz: *On the relative efficiencies of BAN estimates.*

J. Neyman, in the *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability*, 1949, proved that χ^2 minimum estimates with either of two alternative definitions of χ^2 are efficient, as also are the maximum likelihood estimates. He also raised the question whether some of these estimates were better than others. This paper bears on that question. In making χ^2 minimum estimates, it is often necessary to avoid small frequencies by grouping together at least one tail of the distribution. It is with respect to the parameters of these *modified* distributions that the χ^2 estimates are efficient. Define *relative efficiency* in these circumstances as the ratio of the variance of an efficient estimator in the unmodified case to that of one in the modified case. It is shown that, except for a rectangular probability law, the relative efficiency <1 and, further, it decreases as the tail grouping is made wider. Formulae are given for the relative efficiencies of χ^2 minimum estimators for binomial and Poisson probability laws and some representative values computed to exhibit these effects. (Received July 18, 1949.)

585*t*. Julius Lieblein: *Partial sums of the negative binomial in terms of the incomplete beta-function.*

In acceptance sampling a certain size sample is taken at random from a lot of items and the lot is accepted if the number of defective items do not exceed a predetermined number characteristic of the sampling plan. The Statistical Engineering Laboratory has been studying the probabilities that a decision to accept or reject can be made before the sample is completely inspected. Such probabilities are found to involve certain sums apparently not previously treated. In this note the author proves a simple identity connecting these sums which greatly facilitates their computation and shows how they may be written in terms of the well known incomplete beta-function of Karl Pearson, for which extensive tables are available. (Received July 14, 1949.)

586. Jack Sherman and Winifred J. Morrison: *Adjustment of an inverse matrix corresponding to changes in the elements of a given column or a given row of the original matrix.*

A simple computational procedure is derived for obtaining the elements b'_{ij} of an nth order matrix (B') which is the inverse of (A'), directly from the elements b_{ij} of a matrix (B) which is the inverse of (A), when (A') differs from (A) only in the elements of one column, say the Sth column. The equations which form the basis of the computation are: $b'_{Sj} = b_{Sj} / \sum_{r=1}^{n} b_{Sr} a'_{rS}, j=1, 2, \cdots n \cdot b'_{ij} = b_{ij} - b'_{Sj} \sum_{r=1}^{n} b_{ir} a'_{rS}$, $i=1, 2, \cdots, S-1, S+1, \cdots, n; j=1, 2, \cdots, n$. Analogous equations are derived for the case that A and A' differ in the elements of a given row rather than a column. (Received July 5, 1949.)

587. D. M. Studley: *Structure of statistical elements.*

Research in logical semantics and in practical elementation has set forth the proposition that all words and ideas have set form. As a consequence of this universal proposition all notions and conceptions in statistics should be accessible to set-theoretic analysis and interpretation. This paper explains the results of a preliminary analysis performed on statistical notions and conceptions with a view to a proper organization of definitions and conceptions which will, it is hoped, make possible a better and simpler construction of statistics from a system of basic notions. (Received July 14, 1949.)

588t. J. E. Walsh: *Large sample tests and confidence intervals for mortality rates.*

In computing mortality rates from insurance data, the unit of measurement used is frequently based on number of policies or amount of insurance rather than on lives. Then the death of one person may result in several units of "death" with respect to the investigation; moreover, the number of units per individual may vary noticeably. Thus the usual large sample methods of obtaining significance tests and confidence intervals for the true value of the mortality rate are not applicable to these situations. If the number of units associated with each person in the investigation were known, accurate large sample results could be obtained; however, determination of the number of units associated with each individual would require an extremely large amount of work. This article presents some valid large sample tests and confidence intervals for the mortality rate which do not require much work and are reasonably efficient. The procedure followed consists in first dividing the risks into twenty-six subgroups on the basis of the first letter of the last name of the person insured. Some of the groups are then combined until 10 to 15 subgroups yielding approximately the same number of units are obtained. The fraction consisting of the total number of units paid divided by the total number of units exposed is computed for each subgroup. Asymptotically the resulting observations represent independent observations from continuous symmetrical populations with common median equal to the true value of the rate of mortality. Tests and confidence intervals for the rate of mortality are obtained by applying the results of the paper *Some significance tests for the median which are valid under very general conditions* (Annals of Mathematical Statistics vol. 20 (1949) pp. 64–81) to these observations. (Received July 2, 1949.)

TOPOLOGY

589. R. D. Anderson: *A nonhomeomorphic monotone interior mapping of the plane onto itself.*

The author shows that there exists a monotone interior transformation of the plane onto itself which is not a homeomorphism by construction of a continuous collection G of mutually exclusive compact nondegenerate continua filling up the plane such that with respect to its elements as points G is topologically equivalent to the plane. (Received July 18, 1949.)

590t. R. H. Bing: *A normal space which is not collectionwise normal.*

A space is collectionwise normal if for each collection K of mutually exclusive closed sets such that each subcollection of K has a closed sum there is a collection W of mutually exclusive open sets such that each element of K is covered by an element of W and no element of W covers two elements of K. An example is given of a normal Hausdorff space which is not collectionwise normal. (Received July 18, 1949.)

591. R. P. Dilworth: *The completion of the lattice of continuous functions.*

Let S be a topological space and let $C(S)$ denote the set of all bounded, real, continuous functions on S. In previous papers (Bull. Amer. Math. Soc. Abstracts 54-11-535 and 55-3-207) it has been shown that the normal completion of $C(S)$, as a lattice, is isomorphic to the lattice of all bounded continuous functions on a suitable compact Hausdorff space T_S. It is proved here that if S is completely regular, then T_S is the Boolean space associated with the complete Boolean algebra of regular open sets of S. (Received July 11, 1949.)

592. A. M. Gleason: *On the existence of arcs in locally compact connected groups.*

Theorem. Every locally compact connected group with more than one element contains an arc. The proof rests on the following lemma. Let G be a locally compact connected group with more than one element which has no compact connected subgroups except e. Then G contains an increasing family F_t of compact sets with non-negative real indices such that (i) $F_t \neq e$ for $t>0$, (ii) $F_0 = \bigcap_{t>0} F_t = \{e\}$, and (iii) $F_t F_u = F_{t+u}$. Since a compact connected group with more than one element contains not only an arc but even a one-parameter subgroup, the theorem is clearly true for groups with a nontrivial compact connected subgroup. On the other hand it is easy to construct an arc having given the family F_t described in the lemma. The lemma is proved by considering the family of all compact subsets of G as a semigroup. A locally compact topology compatible with the semigroup operation can be introduced and with its aid the construction of the one-parameter subsemigroup F_t can be effected. (Received July 14, 1949.)

593*t*. W. H. Gottschalk: *Asymptotic relations in topological groups.*

Let G be an additive, abelian, locally compact group, let μ be Haar measure in G, let E be a totally bounded measurable subset of G such that some translate of int E contains 0 and generates G, and let $x \in G$. That $\lim_{n\to\infty} \mu((nE)\cap(nE+x))/\mu(nE) = \lim_{n\to\infty} \mu((n+1)E)/\mu(nE) = 1$ is proved. (Received July 14, 1949.)

594*t*. Meyer Jerison: *The space of bounded maps into a Banach space.*

Let B^X be the Banach space of continuous maps from a compact space X into a real Banach space B. It is not true, in general, that (*) if B^X is equivalent to B^Y, then X is homeomorphic to Y. By generalizing Eilenberg's proof for real-valued functions (Ann. of Math. vol. 43 (1942) pp. 568–579), (*) is proved for a space B with the property that if two maximal convex subsets of the surface of the unit sphere intersect, there is a third one that meets neither of them. A characterization of B^X with strictly convex B is given, using methods of Myers (Ann. of Math. vol. 49 (1948)) and of Arens and Kelley (Trans. Amer. Math. Soc. vol. 62 (1947)). If B is strictly convex and T is an equivalence of B^Y onto B^X, then $(T\beta)(x) = \omega(x)[\beta(h(x))]$

for all $\beta\in B^Y$, $x\in X$, where h is a homeomorphic mapping of X onto Y and ω is a continuous mapping of X into the space of equivalences of B onto itself. (Received July 15, 1949.)

595t. W. S. Massey: *Homotopy groups of triads.* II. Preliminary report.

Let $X^*=X\cup E^n$ be a space obtained by adjoining an n-cell E^n to a topological space X, and let $\dot{E}^n=X\cap E^n$ denote the boundary of the cell E^n. In a recent paper (*The homotopy groups of a triad*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 322–328) the following theorem was stated: If the pair $(X, \dot{E}^n)$ is m-connected, then the triad $(X^*; X, E^n)$ is $(m+n-1)$-connected. In this paper the problem of determining the first non-vanishing homotopy group of the triad $(X^*; X, E^n)$ is solved under fairly general conditions as follows. Assume that: (1) X^* is a topological space which can be subdivided into a finite simplicial complex in such a way that X and E^n are subcomplexes. (2) The boundary E^n of the cell E^n is an $(n-1)$-sphere. (3) The pair $(X, \dot{E}^n)$ is m-connected $(m>0)$. Then the first non-vanishing homotopy group of the triad $(X^*; E^n, X)$, that is, $\pi_{m+n}(X^*; E^n, X)$, is isomorphic to the group $\pi_{m+1}(X, \dot{E}^n)$. The functional cup product, recently introduced by Steenrod, is used in setting up the isomorphism. This theorem includes as a special case the Freudenthal "Einhängung" theorems in the critical dimensions. (Received July 14, 1949.)

596. S. B. Myers: *Functional uniformities.*

Let G be a family of sets of real continuous functions on a completely regular space X, such that each $g\in G$ is equicontinuous and G separates X (in the sense that for each $P\in X$ and open set $N(P)$ there is a $g\in G$ and $\delta>0$ such that if $|f(Q)-f(P)|<\delta$ for all $f\in g$, then $Q\in N(P)$). Such a family G determines a uniform structure ("functional uniformity") V compatible with the topology of X as follows: for each $g\in G$ and $\delta>0$ let $V_\alpha=\{P, Q\in X \mid |f(Q)-f(P)|<\delta \text{ for all } f\in g\}$. A functional uniformity is *bounded* if all members of every $g\in G$ are bounded, *weak* if every g is a finite set. Then it is shown that every uniform structure on X is isomorphic to a bounded functional uniformity; every precompact uniform structure is isomorphic to a weak bounded functional uniformity, and conversely every weak bounded functional uniformity is precompact. Weak functional uniformities are investigated, as well as the case that each $g\in G$ is countable. (Received July 18, 1949.)

597. O. M. Nikodým: *On linear functionals.*

Relationships between pseudotopologies and integral representations of linear functionals are studied. (Received July 12, 1949.)

598t. O. M. Nikodým: *On "pseudotopology" and integration related to it.*

A modification, termed "pseudotopology," of the ordinary open-set topology for abstract sets is axiomatized and properties of corresponding pseudo-continuous functions examined. Relationships between pseudotopologies and more general families of functions are considered. A general notion of integral and summability related to a pseudotopology is defined and examined. (Received July 12, 1949.)

599. L. T. Ratner: *Semi-continuous multi-valued transformations.*

This paper considers upper and lower semi-continuous multi-valued transformations of one weakly separable Hausdorff space into another such space. Basic properties of semi-continuous multi-valued transformations are explored and extended. Sufficient conditions for preserving connectedness and compactness are given. Results are established giving information as to the character of limits of various types of sequencies of multi-valued transformations. Finally, consideration of semi-continuity in metric spaces leads to analogues of a number of the standard theorems of real variables and of topology. (Received July 18, 1949.)

600. C. N. Reynolds: *An analysis of recent progress on the problem of coloring maps in four colors.*

This is a brief analysis, particularly of the work of S. M. de Backer, implying that those maps which are irreducible with respect to the four-color problem have at least 84 regions. (Received August 31, 1949.)

601*t*. W. R. Utz: *Unstable homeomorphisms.*

If X is a metric (d) space, the homeomorphism $f(X)=X$ is called unstable provided there exists a number $\delta(f, X)>0$ such that corresponding to each pair of distinct points x_1, x_2 of X, there is an integer $n(x_1, x_2)$ for which $d(f^n(x_1), f^n(x_2))>\delta$; the orbits of the points $x_1, x_2 \in X$ are called positively (negatively) asymptotic if for each $\epsilon>0$ there exists an integer N such that $i>N(i<N)$ implies $d(f^i(x_1), f^i(x_2))<\epsilon$. It is shown that if X is compact, dense-in-itself, and metric, and $f(X)=X$ is unstable, then some pair of distinct points of X have orbits asymptotic in at least one sense. Some properties of unstable homeomorphisms are established and examples are cited. (Received July 18, 1949.)

602. P. A. White: *On the union of two generalized manifolds.*

A closed subset M of the compact Hausdorff space S whose boundary relative to S is F is a generalized n-manifold with boundary relative to S if (a) dim $M=n$, (b) M is i-colocally-connected $(0\leqq i\leqq n-1)$, (c) M is i-locally connected $(0\leqq i\leqq n)$, (d) for each point of M the local n-betti number relative to F is 1, (e) for each point in F the local n-betti number is 0. (The local n-betti number of $p\in A\subset S$ relative to $B\subset A$ is the smallest positive integer k such that to any open set P of S containing p, there corresponds an open set Q with $p\in Q\subset P$ such that any $(k+1)$ n-dimensional Čech cycles on A mod $(S-P)\cup B$ are linearly independent on A mod $(S-Q)\cup B$.) An n-manifold M is closed if $F=0$, and is orientable if the n-betti number of M mod F is 1 irreducibly. It is shown that if M_1 and M_2 are generalized n-manifolds with a common boundary M_{12} relative to S such that $M_1\cap M_2=M_{12}$, and if M_{12} is a generalized $(n-1)$-manifold with boundary relative to $\overline{(S-M)}\cup M_{12}$, then $M=M_1\cup M_2$ is a generalized n-manifold with boundary relative to S. Furthermore if M_1, M_2, and M_{12} are orientable, so is M; if M_{12} is a closed manifold, so is M. (Received May 20, 1949.)

J. W. GREEN,
Associate Secretary

TWO STATEMENTS CONCERNING THE ARTICLE ON G. H. HARDY[1]

The undersigned consider that some points in the lively and interesting article of Professor Norbert Wiener on G. H. Hardy need correction. They offer no comment on what is matter of opinion, but wish to mention the following facts.

(1) Hardy was not a conscientious objector. In the First World War, he volunteered for military service under the Derby scheme.

(2) Bohr and Landau cannot correctly be described as students of Hardy. The work of all three overlapped.

(3) To say that "Hardy chose as his field the analytic theory of numbers" is to leave out of account his important work on Tauberian Theorems, on Diophantine approximation, and on Fourier series.

J. E. LITTLEWOOD, G. PÓLYA, L. J. MORDELL,
E. C. TITCHMARSH, H. DAVENPORT

The first thing I ought to say is that I had been definitely misinformed concerning Hardy's attitude in the first World War. I regret this as well as any harm that may have been done.

While Bohr and Landau cannot be described as students of Hardy, they are certainly mathematicians who have received a maximum influence from his work so that I consider that my statement although false in detail is correct in implications.

Finally, I may be quite wrong, but I feel that Hardy's very great work on Tauberian theorems, Diophantine approximation, and Fourier series is very largely an outgrowth of his interest in analytical number theory and even tools for its development. This is in no way to minimize the importance of these pieces of work for themselves, but it is to give them the orientation which I think proper in describing Hardy's career.

NORBERT WIENER

---

[1] Norbert Wiener, *Godfrey Harold Hardy*, 1877–1947, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 72–77.

BOOK REVIEWS

Analytic theory of continued fractions. By H. S. Wall. New York, Van Nostrand, 1948. 13+433 pp. $6.50.

This is the first volume to appear in "The University Series in Higher Mathematics" which is planned to be a collection of "advanced text and reference books in pure and applied mathematics."

In order to make the book suitable as a text book the author gives detailed proofs and includes material which might be unfamiliar to "a student of rather modest preparation." Into this category fall such topics as: the Stieltjes-Vitali theorem, Schwarz' inequalities, matrix calculus, elementary properties of the Stieltjes integral, and basic concepts and formulae of the theory of continued fractions. To increase its usefulness as a text the book is provided with 131 exercises, grouped together at the end of each chapter. The material covered in this book is of such a nature that it or parts of it would make very attractive subject matter for graduate seminars. It is to be hoped that the book will thus contribute to spreading knowledge of and interest in the analytic theory of continued fractions among a larger group of people.

Numerous and important additions have been made to the analytic theory of continued fractions since the appearance of Perron's *Die Lehre von den Kettenbrüchen* in 1913. Only a few of these results were incorporated into the second edition of Perron's book in 1929. It is thus clear that the publication of the present book which brings so much, though unfortunately not all, of this material together in a single volume was welcomed by workers in this and in related fields.

The book consists of twenty chapters. Chapters I–VIII form the first part entitled "Convergence theory." The second part is called "Function theory." Classical as well as more modern results are treated in this book. That Wall himself has been during the last twenty years the foremost contributor to the advancement of the theory is apparent throughout this book. Not only are there many new theorems due to him and his collaborators but there are also numerous new proofs of previously known results.

Wall's most significant original additions to the theory are to be found in the first part. The outstanding result is probably the parabola theorem (§14) due to Scott and Wall. This in turn led Paydon and Wall and independently Leighton and the reviewer (a fact which is not mentioned in this book) to the discovery of a family of parabolic convergence regions (§32).

In the second part of the book Wall's original contributions consist largely of generalizations and extensions of previously known results and methods. This is not the case in Chap. X in which a new approach due to Wall and Frank to the problem of locating the roots of polynomial equations is presented. With every polynomial a "test fraction" is associated. The desired information is then obtained from the expansion of the test fraction into a J-fraction. A J-fraction (the term is due to Wall who has investigated J-fractions extensively) is a continued fraction of the form $K(-c_n^2/(b_n+z))$. The arrangement in tabular form of the computation for the expansions of functions into J-fractions, a device which is used a number of times in the book, seems to be due to Frank. This is not acknowledged. One also misses a reference to Frank's paper *The location of the zeros of polynomials with complex coefficients* (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 890–898) in §49 as well as in the bibliography. Chap. XV is devoted to a characterization of continued fraction expansions of holomorphic functions with certain prescribed domains of definition and of values. It contains among others results of Schur on functions bounded in the unit circle. Chapters XVIII and XIX contain a large collection of expansions of functions into continued fractions. The book ends with a bibliography of 150 entries which lists all papers cited in the text as well as a number of other recent papers dealing with continued fractions.

It is the reviewer's opinion that an author who writes a book which has the appearance of being a reference book for a certain field has the obligation to include all significant results clearly belonging to this field. If complete coverage is not intended then that fact should be explicitly stated in the title of the book. The title of this book, together with the fact that it appears in a series which proclaims to be devoted to reference books, would lead a reader unfamiliar with the subject to the erroneous conclusion that this is indeed such a reference book. However, a comparison with the second part of Perron's book shows that this is not the case. In view of the fact that Perron's book is at present unavailable these omissions are regrettable. Not only classical material has been omitted however. In addition to the approaches to the convergence problem discussed in this book there is another one due to Leighton and the reviewer which led to a number of interesting results (see 7a, 56, 101a, 102, 103, 103a; the numbers refer to the bibliography at the end of Wall's book).

The author points out that a continued fraction can be considered as a sequence of linear fractional transformations and demonstrates in many places the usefulness and importance of this approach. In view of this it is surprising that he is reluctant to introduce and use iterated fractions (that is, sequences of general linear fractional transformations). It is true that in most cases iterated fractions can be transformed into continued fractions. However, even when this is possible, the resulting continued fraction may be considerably more involved than the iterated fraction from which it was derived. This is the case, for example, in Schur's expansion of functions bounded in the unit circle into iterated fractions. Wall uses continued fractions instead. Another instance where use of iterated fractions would have led not only to a somewhat more general result but also to a more elegant proof is in the discussion of the convergence of periodic continued fractions (§8). This could have been accomplished by the use of Schwerdtfeger's [84a] proof instead of Lane's. Finally, use of iterated fractions would have made inclusion of a discussion of the Pick-Nevanlinna interpolation problem extremely natural.

W. J. THRON

Substitutional analysis. By D. E. Rutherford. (Edinburgh University Publications, Science and Mathematics, No. 1.) Edinburgh, University Press, 1948. 11+103 pp. 25 s.

The subject matter of this book, except for the last chapter, is Young's representation theory of the symmetric group. As Young developed the methods described here, he was always thinking of the elements of the symmetric group as substitutional operators applicable, in particular, to the theory of invariants. This fact explains the title.

In the introduction the author gives a brief account of Young's life as a country clergyman whose avocation remained the development of the mathematical ideas which interested him as a student at Cambridge. Those who knew him were always impressed by his sincerity and his modesty but above all by the originality and power with which he manipulated his own complicated machinery. The present book gives a connected account of Young's methods which has long been needed. The material was scattered throughout a long series of papers and, as is not surprising, the original presentation was sometimes involved. D. E. Rutherford has simplified it materially in places, and the reader can see the significance of the various steps taken.

As indicated above the theory here described was almost incidental in Young's work; it appeared as part of a larger plan, and in this light he always considered it. Young's originality was to some extent a handicap, for it prevented him from incorporating the work of others into his scheme of things; Schur's elegant demonstration of the connection between the full linear group and the symmetric group is a case in point. This leads to perhaps the chief criticism which can be levelled against the volume under review. No attempt is made to orient Young's work either as regards its original background of invariant theory, and this is understandable, or as regards the theory of groups generally. From this latter point of view the representation theory of the symmetric group plays a unique role in that it is explicit to a degree which cannot be approached by the general theory. Having such powerful tools available, one might hope that this theory of Young's should be able to throw some light on the general theory, and this stage seems to be approaching (Nakayama, Jap. J. Math. (1940); Brauer and Robinson, Transactions of the Royal Society of Canada (1947)).

The treatment includes chapters on the calculus of permutations, the calculus of tableaux, the semi-normal representation, the orthogonal and natural representations, group characters, substitutional equations. Chapter I gives an elementary introduction to the theory of the Symmetric group S_n. In Chapter II, Young's *tableau* or *diagram* $[\alpha] \equiv [\alpha_1, \alpha_2, \cdots, \alpha_h]$ with $\alpha_i \geqq \alpha_{i+1}$ and $\alpha_1+\alpha_2+\cdots+\alpha_h=n$ is introduced and a clear account of the properties of P^α and N^α is given. Writing $E^\alpha_{rs}=\sigma_{rs}P^\alpha_s N_s$, where σ_{rs} is a substitution which transforms $[\alpha]_r$ into $[\alpha]_s$, the author obtains "Young's formula" $E^\alpha_r E^\alpha_r = \theta^\alpha E^\alpha_r$ and the remaining relations between the E's in an elegant manner.

In Chapter III the fundamental notion of a *standard* diagram in which the symbols follow an agreed order in both row and column is introduced and this leads directly to the semi-normal representation, following the method of Thrall (Duke Math. J. (1941)). The passage to the orthogonal representation is clearly explained in Chapter IV. In Young's original work the natural representation came first, and the orthogonal representation came as an impressive conclusion to a difficult piece of reading. Here some of the difficulties have been smoothed away, but the natural representation appears as an anticlimax. Though reference to it had to be included, this reviewer would have preferred that it be in an appendix. The material of §§28–31 has historical and actual value, but it serves to obscure the magnitude of Young's real achievement, the orthogonal representation.

One can sympathize with the author in the writing of §§32–36 of Chapter V. How much of the general theory of group characters should be assumed? He has chosen to assume nothing and deduces the necessary theory for the symmetric group. The sections are well written, but the uninformed reader may not realize the generality of the methods used. To the remaining paragraphs of the chapter this reviewer feels obliged to take exception. They have to do with the calculation of the characters via what has been called the Murnaghan-Nakayama recursion formula. As originally obtained by Murnaghan (Amer. J. Math. (1937)) there was little reference to Young diagrams; the irreducible components appeared with multiplicites 0, ±1, and it remained for Nakayama (loc. cit.) to show the connection by introducing the notion of a *hook*. After the removal of the symbols $1, 2, \cdots, m$ from a standard *right* diagram $[\alpha]$, what is left can be designated by the symbol $[\alpha]-[\beta]$ and this is called a *distorted* or *skew* diagram. Just as an irreducible representation is associated with a right diagram, so a reducible representation is associated with a skew diagram. By restricting attention to the subgroup $S_m \times S_{n-m}$, the representation $[\alpha]$ of S_n breaks up into the direct sum of a number of Kronecker products of irreducible representations $[\beta]$ of S_m and reducible representation $[\alpha]-[\beta]$ of S_{n-m}. Thought of from this point of view the multiplicites 0, ±1 are characters of cycles of length $n-m$ in the skew representation $[\alpha]-[\beta]$ (Robinson, Amer. J. Math. (1947) and (1948)). In the last chapter the "substitutional" aspect of the theory is illustrated. Some words of explanation here would have helped the reader to see the usefulness of the results obtained.

To sum up: the book provides a long overdue account of Young's representation theory of the symmetric group. Many of Young's complicated proofs have been simplified through a free use of mathematical induction. The printing is excellent, and no errors have been detected. The Edinburgh University Press is to be congratulated on such a handsome first volume.

G. DE B. ROBINSON

NOTES

Applications before February 15, 1950, are invited for the Harry Bateman Research Fellowships at the California Institute of Technology. These are post-doctorate fellowships carrying a stipend of $3600 for the academic year.

The School of Mathematics of the Institute for Advanced Study will allocate a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year 1950–1951. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, N. J., and are returnable by February 1, 1950.

The University of Maryland and the Naval Ordnance Laboratory have established an Institute of Fluid Dynamics and Applied Mathematics, to be located at the University of Maryland and to be directed by Professors Joseph Kampé de Fériet of the University of Lille and Alexander Weinstein of the University of Maryland and Naval Ordnance Laboratory.

Professor Emil Artin of Princeton University has been appointed to a visiting professorship at the Institute for Mathematics and Mechanics, New York University.

Assistant Professor R. C. Buck of Brown University has been appointed to an associate professorship at the University of Wisconsin.

Assistant Professor F. L. Celauro of Newark College of Engineering has been appointed to an associate professorship at the Western State College of Colorado.

Mr. D. G. Chapman of the University of California has been appointed to an assistant professorship at the University of Washington.

Associate Professor Harold Chatland of the Ohio State University has been appointed to a professorship at Montana State University.

Mrs. Joan R. Clark of Brown Instrument Company has accepted a position as research assistant with the Institute for Cancer Research, Philadelphia, Pa.

Miss Helen F. Cullen of the University of Michigan has been appointed to an assistant professorship at the University of Massachusetts.

Associate Professor J. C. Currie of Alabama Polytechnic Institute has been appointed to an associate professorship at the Georgia Institute of Technology.

Mr. B. J. Eisenstadt of the University of Michigan has been appointed to an assistant professorship at Wayne University.

Dr. A. B. Farnell of Princeton University has been appointed to an assistant professorship at the University of Colorado.

Dr. Walter Fleming of Fort Hays State College has been appointed to an associate professorship at State Teachers College, Mankato, Minn.

Dr. H. L. Garabedian of the Atomic Energy Commission has accepted a position as senior scientist with the Atomic Power Division of Westinghouse Electric Corp., Pittsburgh, Pa.

Mr. I. C. Gentry of Duke University has been appointed to an assistant professorship at Wake Forest College, Wake Forest, N. C.

Associate Professor H. A. Giddings of the University of New Hampshire has been appointed to a professorship in the College of Engineering, New York University.

Mr. D. B. Goodner of the University of Illinois has been appointed to an assistant professorship at Florida State University.

Associate Professor Fritz Herzog of Michigan State College has been appointed to a visiting associate professorship at the University of Michigan for the first term of the current academic year.

Professor A. J. Kempner of the University of Colorado has retired and has been appointed to a visiting professorship at Pomona College.

Mr. L. L. Layton of the University of Illinois has been appointed to an assistant professorship at James Millikin University.

Mr. S. C. Lowell of New York University has accepted a position as scientific liaison officer with the Office of Naval Research.

Mr. H. F. Mathis of Northwestern University has been appointed to an associate professorship in electrical engineering at the Univerity of Oklahoma.

Professor F. S. Nowlan of the University of British Columbia has been appointed to a professorship at the Undergraduate Division of the University of Illinois, Chicago, Ill.

Mr. H. C. Parrish of Ohio State University has been appointed to an assistant professorship at the North Texas State Teachers College.

Associate Professor P. M. Pepper of the University of Notre Dame has been appointed to an associate professorship of industrial engineering at the Ohio State University.

Assistant Professor Mary Pettus of Lander College has been appointed to an assistant professorship at the University of Richmond.

Mr. D. W. Pounder of Illinois Institute of Technology has accepted a position as aerodynamicist with A. V. Roe Canada, Ltd., Toronto.

Professor Hans Rademacher of the University of Pennsylvania has been appointed to a visiting professorship at the Institute for Mathematics and Mechanics, New York University.

Mr. L. T. Ratner of the University of California at Los Angeles has been appointed to an assistant professorship at Vanderbilt University.

Professor J. B. Rosser of Cornell University has accepted a position as chief of the Institute for Numerical Analysis of the National Bureau of Standards.

Mr. Herman Rubin has been appointed to an assistant professorship in statistics at Stanford University.

Mr. R. H. Scherer of the University of Minnesota has been appointed to an assistant professorship at Macalester College, St. Paul, Minn.

Associate Professor Ernst Snapper of the University of Southern California has been appointed to a visiting associate professorship at Princeton University.

Mr. Herbert Solomon of the Strategic Vulnerability Branch of the Air Forces has accepted a position as mathematical statistician in the Office of Naval Research.

Assistant Professor R. D. Specht of the University of Wisconsin has accepted a position as associate mathematician with the Rand Corporation, Santa Monica, Calif.

Dr. Olga Taussky has accepted a position as mathematician with the National Bureau of Standards.

Mr. H. E. Tinnappel of the Ohio State University has been appointed to an assistant professorship at Bowling Green State University.

Dr. L. V. Toralballa of Fordham University has been appointed to an associate professorship at Marquette University.

Dr. W. R. Transue of the Institute for Advanced Study has been appointed to a professorship at Kenyon College.

Dr. Jean B. Walton of the University of Pennsylvania has been appointed Dean of Women, Pomona College.

Professor C. E. Wilder of Dartmouth College has retired.

Mr. J. F. Williams of the University of Tennessee has been appointed to an assistant professorship at Memphis State College.

The following promotions are announced:

B. A. Amirà, Hebrew University, to an associate professorship.

L. W. Akers, Central Missouri College, to an associate professorship.

Alfredo Baños, Physics Department, University of California at Los Angeles, to a professorship.

R. E. Basye, Agricultural and Mechanical College of Texas, to an associate professorship.

I. L. Battin, Drew University, to an associate professorship.

M. G. Boyce, Vanderbilt University, to a professorship.

O. J. Farrell, Union College, to a professorship.

A. H. Fox, Union College, to a professorship.

Bernard Friedman, Institute for Mathematics and Mechanics, New York University, to an associate professorship.

C. M. Fulton, University of California at Davis, to an assistant professorship.

R. C. James, University of California, to an assistant professorship.

P. A. Lagerstrom, California Institute of Technology, to an associate professorship of aeronautics.

M. V. Morkovin, University of Michigan, to an associate professorship.

Max Morris, Case Institute of Technology, to a professorship.

B. J. Pettis, Tulane University of Louisiana, to a professorship.

R. M. Robinson, University of California, to a professorship.

Fariebee P. Self, Centenary College, to an assistant professorship.

E. A. Sturley, Allegheny College, to an assistant professorship.

Leonard Tornheim, University of Michigan, to an assistant professorship.

Bertram Yood, Cornell University, to an assistant professorship.

The following appointments to instructorships are announced: Anitoch College: Mr. D. J. Myatt; Case Institute of Technology: Dr. F. C. Leone; University of Chicago: Dr. A. F. Strehler; Columbia University, Barnard College: Mr. A. J. Hoffman; University of Connecticut: Dr. D. M. Adelman; University of Connecticut, Hartford Branch: Mr. D. K. Pease; Florida State University: Mr. R. T. Gregory; University of Illinois: Mr. Meyer Jerison; Massachusetts Institute of Technology: Dr. T. M. Apostol; University of Michigan: Mr. Imanuel Marx; University of North Carolina: Mr. W. R. Mann; Oklahoma Agricultural and Mechanical College: Mr. Arthur Zeichner; University of Oregon: Mr. F. H. Young; Rensselaer Polytechnic Institute: Mr. J. C. Gibson; University of Tennessee: Mr. E. L. Eagle; Tulane University of Louisiana: Mr. H. J. Cohen; Union College: Mr. M. R. Bates; Washington University: Dr. H.

Margaret Elliott; University of Wisconsin: Mr. L. H. Kanter.

Professor Zbigniew Lepecki of the University of Paraná died in May, 1949 at the age of forty-seven years.

Reverend J. E. Case of St. Louis University died on August 5, 1949 at the age of fifty-three years. He had been a member of the Society for sixteen years.

Professor R. E. Gleason of Temple University died on July 7, 1949. He had been a member of the Society for twenty-eight years.

Associate Professor R. K. Luneburg of the University of Southern California died August 19, 1949 at the age of forty-six years.

NEW PUBLICATIONS

Blakey, J. *University mathematics.* London, Blackie and Son, 1949. 8+527 pp. 25s.

Bochner, S., and Chandrasekharan, K. *Fourier transforms.* (Annals of Mathematics Studies, no. 19.) Princeton University Press, 1949. 10+219 pp. $3.50.

Chandasekharan, K. See Bochner, S.

Darbon, A. *La philosophie des mathématiques: étude sur la logistique de Russell.* Paris, University Press, 1949. 203 pp. 400 francs.

Enriques, F. *Le superficie algebriche.* Bologna, Zanichelli, 1949. 16+464 pp. 3000 lire.

Goldman, S. *Transformation calculus and electrical transients.* New York, Prentice-Hall, 1949. 14+439 pp. $6.25.

Hildebrand, F. B. *Advanced calculus for engineers.* New York, Prentice-Hall, 1949. 14+594 pp. $6.00.

Marden, M. *The geometry of the zeros of a polynomial in a complex variable.* (Mathematical Surveys, no. 3.) New York, American Mathematical Society, 1949. 10+183 pp. $5.00.

Roth, L. See Semple, J. G.

Schmeidler, W. *Vorträge über Determinanten und Matrizen mit Anwendungen in Physik und Technik.* Berlin, Akademie Verlag, 1949. 8+155 pp. 10 DM.

Semple, J. G., and Roth, L. *Introduction to algebraic geometry.* Oxford University Press, 1949. 16+446 pp. $7.50.

Wiener, N. *Extrapolation, interpolation, and smoothing of stationary time series.* New York, Wiley, 1949. 10+163 pp. $4.00.

Yates, F. *Sampling methods for censuses and surveys.* New York, Hafner, 1949. 14+318 pp. $6.00.

CLASSIFICATION OF 2-MANIFOLDS WITH SINGULAR POINTS

CHIEN-KE LU

1. **Introduction.** By a *closed 2-manifold*, or simply a 2-manifold, we mean here a two-dimensional connected finite simplicial complex every point of which has a neighborhood homeomorphic to a circular disk, that is, the interior of a circle. If there is a point not having the latter property, we call it a *singular point* of it.

In this paper, we shall give a complete classification (§2) and some properties (§3) of 2-manifolds with a single singular point. Obviously one may get one such geometrical figure by identifying certain points of a 2-manifold or several ones together. Conversely, we shall show that every such figure may be obtained in such a manner (see (2.3)).

In §4, we generalize these results to 2-manifolds with any number of singularities.

2. **The classification.** The classification lies in the investigation of the nature of the neighborhoods of the singular point. Let[1] $\cdot\mathfrak{M}^2$ have its singular point at 0. We first establish the following lemma.

LEMMA (2.1). *Any neighborhood of* 0 *is homeomorphic to a finite number, say p, of circular disks with all their centers identified. We call it a p-bundle and call* 0 *its center; and the boundaries of these p disks are simply said to be the boundary of the p-bundle.*

PROOF. Consider a simplicial subdivision $\mathfrak{K}^2$ of $\cdot\mathfrak{M}^2$. We first note that 0 must be a vertex of $\mathfrak{K}^2$. For, if 0 were an inner point of a 2-simplex, then 0 could not belong to any other simplex and hence would be an ordinary point; and if it were an inner point of a 1-simplex, then all the points of this 1-simplex would be singular points for the same reason. It is also evident that 0 cannot be a vertex of a 1-simplex unless it is a vertex of a 2-simplex.

Let 0 be a vertex of a 2-simplex $\mathfrak{N}^2$. Then there must be many 2-simplexes including $\mathfrak{N}^2$ forming a circular disk surrounding 0, as otherwise there would be two edges of singularities. Besides, 0 must be a vertex of another 2-simplex, say $'\mathfrak{N}^2$, by noting that 0 is a singular point. Hence we get another circular disk consisting of

---

Received by the editors September 15, 1948.

[1] We use the notation $\cdot\mathfrak{M}$ instead of a dot directly over $\mathfrak{M}$ for typographical convenience.

2-simplexes surrounding 0. In such a way, finally, we obtain p such circular disks (since $\cdot\mathfrak{M}^2$ is finite), and the lemma is proved.

We say $\cdot\mathfrak{M}^2$ has a singularity of *order* p at 0. $\cdot\mathfrak{M}^2-(0)$ in general is not connected and consists of n components $\mathfrak{L}_i^2$ $(i=1,\cdots,n)$. We name $\mathfrak{L}_i^2+(0)$ a *sheet* of $\cdot\mathfrak{M}^2$. Then $\cdot\mathfrak{M}^2$ is the sum of these sheets with the identification of 0. Moreover, each circular disk of 0 belongs wholly to one and only one sheet. Let us denote these sheets by $\cdot\mathfrak{M}_i^2=\mathfrak{L}_i^2+(0)$ $(i=1,\cdots,n)$, then

$$\cdot\mathfrak{M}^2=\sum_{i=1}^{n}\cdot\mathfrak{M}_i^2, \tag{1}$$

$$p=\sum_{i=1}^{n}p_i, \tag{2}$$

where p_i is the order of 0 in $\cdot\mathfrak{M}_i^2$ ($p_i=1$ in case $\cdot\mathfrak{M}_i^2$ itself is a 2-manifold).

Hence it is sufficient for us to consider $\cdot\mathfrak{M}_i^2$ separately. But the structure of $\cdot\mathfrak{M}_i^2$ is quite clear; for, if we take away the p_i circular disks surrounding 0, the rest is a bounded 2-manifold with p_i holes, the classification of which is already well known.[3] Therefore we get:

THEOREM (2.2). *Any 2-manifold with one singularity may be decomposed into the form* (1), *where* $\cdot\mathfrak{M}_i^2$ *are sheets, that is, bounded 2-manifolds with* p_i *holes adjoined with a* p_i*-bundle having its boundary identified with the boundaries of these holes; and all the centers of these bundles are to be identified.*

We may, however, consider the p-bundle separately as p circular disks and identify each of their circumferences with each of the boundaries of the holes. Thus we get 2-manifolds, and then identify the p centers. Hence we obtain:

THEOREM (2.3). *Every 2-manifold with one singular point is the sum of a finite number of 2-manifolds each with some points identified all together.*

The preceding two theorems lead us to obtain a 2-manifold with a single singular point from bounded and closed 2-manifolds respectively. In practice the latter is much more useful than the former.

3. Simple properties. The most evident property of 2-manifolds with one singularity is contained in the following theorem.

THEOREM (3.1). $\cdot\mathfrak{M}^2$ *and* $\cdot\mathfrak{M}^{*2}$ *are homeomorphic if and only if they*

---

[3] Cf., for example, Seifert-Threlfall, *Lehrbuch der Topologie*, 1934, §40.

have the same structures, hence necessarily, after suitably arranging their sheets,

$$(p_i) = (p_i^*) \tag{3}$$

(*consequently* $n = n^*$, $p = p^*$).

But we should notice that (3) is not a sufficient condition for $\cdot\mathfrak{M}^2$ and $\cdot\mathfrak{M}^{*2}$ to be homeomorphic, since the corresponding 2-manifolds constituting them may not be homeomorphic. We also note that the property of orientability is preserved:

THEOREM (3.2). *A 2-manifold with one singularity is orientable if and only if all the 2-manifolds constituting it are orientable.*

Now we come to prove the important theorem:

THEOREM (3.3). *The (integral) homology group $\mathfrak{h}$ of any dimension of a 2-manifold with one singularity is the direct sum of those of its sheets. The homology group of a sheet $\cdot\mathfrak{M}^2_i$ is the same as that of the 2-manifold $\mathfrak{M}^2_i$ except for dimension 1, where $\mathfrak{M}^2_i$ is the 2-manifold which is the same as $\cdot\mathfrak{M}^2_i$ but without those p_i points to be identified in $\cdot\mathfrak{M}^2_i$ being identified in $\mathfrak{M}^2_i$. For dimension 1, if $\cdot\mathfrak{M}^2_i$ and $\mathfrak{M}^2_i$ have the homology groups $\cdot\mathfrak{h}^1_i$ and $\mathfrak{h}^1_i$ respectively, then*

$$\cdot\mathfrak{h}^1_i = \mathfrak{h}^1_i + (p_i - 1)\mathfrak{g} \tag{4}$$

where $k\mathfrak{g}$ represents the direct sum of k free cyclic groups $\mathfrak{g}$.

PROOF. Let us consider the last statement only as the others are obvious. In case $p_i = 1$, (4) is trivial.

Suppose $0_1, \cdots, 0_{p_i}$ are the points of $\mathfrak{M}^2_i$ which are to be identified in $\cdot\mathfrak{M}^2_i$. Take a sufficiently small simplicial subdivision of $\mathfrak{M}^2_i$ such that all these points are vertices, and it *induces* a simplicial subdivision on $\cdot\mathfrak{M}^2_i$. Then any 1-cycle Z^1 on $\cdot\mathfrak{M}^2_i$ is either a 1-cycle on $\mathfrak{M}^2_i$ or a broken line joining two 0's, say 0_j and 0_k. In the latter case Z^1 is neither homologous nor division homologous to zero. For if $mZ^1 \sim 0$ $(m \neq 0)$, then there would exist a 2-chain C^2 whose boundary $\partial C^2 = mZ^1$ and thence

$$\partial\partial C^2 = m\partial Z^1 = m(\pm 0_j \pm 0_k) \neq 0,$$

which is a contradiction. Hence Z^1 as an element of $\cdot\mathfrak{h}^1_i$ generates a free cyclic group.

We then join 0_1 to the other 0's and get $p_i - 1$ broken lines, each of which generates a free cyclic group since no two of them are homologous or division homologous to zero by the same reason.

Any broken line between 0_j and 0_k may be replaced by an algebraic sum of two broken lines starting from 0_1 and ending in 0_j, 0_k respectively, and a 1-cycle through these three points on $\mathfrak{M}_i^2$ will be discussed below.

A 1-cycle on $\mathfrak{M}_i^2$ not passing through any 0 is not influenced in constructing $\cdot\mathfrak{M}_i^2$, while one passing any 0, say 0_k, may be modified by omitting the two edges through it and adding the third edge of the 2-simplex that is incident with 0_k as well as the two edges through 0_k (see (2.1)). Hence the homology classes made by the 1-cycles on $\mathfrak{M}_i^2$ are unchanged on $\cdot\mathfrak{M}_i^2$. Thus (3.3) is established.

COROLLARY (3.4).

$$\cdot\mathfrak{h}^1 = \sum_{i=1}^{n} \cdot\mathfrak{h}_i^1 + (p - n)\mathfrak{g}.$$

This shows us a method for constructing a 2-complex with any preassigned Betti number whatever.

Analogously, we have the following theorem.

THEOREM (3.5). *The fundamental group of a 2-manifold with one singularity is the free product of those of its sheets, and the fundamental group* $\cdot\mathfrak{f}_i$ *of a sheet* $\cdot\mathfrak{M}_i^2$ *is the free product of the fundamental group* $\mathfrak{f}_i$ *of* $\mathfrak{M}_i^2$ *and* $p_i - 1$ *free cyclic groups.*[3]

4. **Generalizations.** By the finiteness of a 2-manifold it is evident that the number of singular points on it, if any, is finite. Let $'0, ''0, \cdots, {}^{(m)}0$ be the only singular points on a 2-manifold $\cdot\mathfrak{M}^2$. Lemma (2.1) is valid for each ${}^{(j)}0$ $(j=1, \cdots, m)$ and we may speak of the *order* at ${}^{(j)}0$, say ${}^{(j)}p$. The generalized Theorems (2.2) and (2.3) have their natural forms, the latter of which we state as follows.

THEOREM (4.1). $\cdot\mathfrak{M}^2$ *is the sum of a finite number of 2-manifolds* $\cdot\mathfrak{M}_i^2$ $(i=1, \cdots, n)$ *on which* ${}^{(j)}p_i$ *points are identified to the point* ${}^{(j)}0$ $(j=1, \cdots, m)$.

Moreover,

$$(5) \qquad p_i = \sum_{j=1}^{m} {}^{(j)}p_i, \qquad {}^{(j)}p = \sum_{i=1}^{n} {}^{(j)}p_i, \qquad p = \sum_{i=1}^{n} p_i = \sum_{j=1}^{m} {}^{(j)}p,$$

where p_i and p are defined as the *orders* of $\cdot\mathfrak{M}_i^2$ and $\cdot\mathfrak{M}^2$ respectively.

---

[3] We may first prove (3.5) and so (3.3) follows immediately by a relation between the homology group and the fundamental group, cf. ibid. p. 173.

Theorem (3.1) now takes the form:

THEOREM (4.2). *$\cdot\mathfrak{M}^2$ and $\cdot\mathfrak{M}^{*2}$ are homeomorphic if and only if they have the same structures, hence necessarily, after suitably arranging their sheets and the order of their singular points,*

$$(6) \qquad ({}^{(j)}p_i) = ({}^{(j)}p_i^*)$$

(consequently $m=m^*$, $n=n^*$, $p=p^*$).

Theorem (3.2) is true in its original form. We establish now the following theorem.

THEOREM (4.3). *The homology group $\cdot\mathfrak{h}_i^1$ of $\cdot\mathfrak{M}_i^2$ is given by*

$$(7) \qquad \cdot\mathfrak{h}_i^1 = \mathfrak{h}_i^1 + (p_i - m)\mathfrak{g},$$

where $\mathfrak{h}_i^1$ is the homology group of $\mathfrak{M}_i^2$, the 2-manifold which is the same as $\cdot\mathfrak{M}_i^2$ but without any points being identified.

PROOF. For each j we consider the ${}^{(j)}p_i$ points to be identified to ${}^{(j)}0$ as in the proof of (3.3), that is, join broken lines from one of them to all the others. They are 1-cycles on $\cdot\mathfrak{M}_i^2$, each of which generates a free cyclic group in $\cdot\mathfrak{h}_i^1$ and any two of which are neither homologous nor division homologous to zero on $\cdot\mathfrak{M}_i^2$. Any 1-cycle on $\cdot\mathfrak{M}_i^2$ may be replaced by an algebraic sum of these broken lines and a 1-cycle on $\mathfrak{M}_i^2$. Hence by the same reason as in (3.3), from the first equation of (5) we have (7), and thus the theorem is proved.

In order to get the homology group $\cdot\mathfrak{h}^1$ of $\cdot\mathfrak{M}^2$, we again introduce a lemma which may be readily proved.

LEMMA (4.4). *If $\mathfrak{K}$ is a connected simplicial complex of any dimension and $P_1, \cdots, P_k$ are k arbitrary distinct points on it, and $\mathfrak{K}^*$ is the complex made by $\mathfrak{K}$ in addition with the 1-simplexes (P_1P_2), (P_1P_3), $\cdots$ (P_1P_k) (not in $\mathfrak{K}$); then*

$$(8) \qquad \mathfrak{h}^{*1} = \mathfrak{h}^1 + (k-1)\mathfrak{g},$$

or

$$(8)' \qquad \mathfrak{h}^1 = \mathfrak{h}^{*1} - (k-1)\mathfrak{g},$$

where $\mathfrak{h}^1$ and $\mathfrak{h}^{*1}$ are homology groups of $\mathfrak{K}$ and $\mathfrak{K}^*$ respectively and the minus sign indicates a difference group.

Eventually, we have the following theorem.

THEOREM (4.5). *The homology group $\cdot\mathfrak{h}^1$ of $\cdot\mathfrak{M}^2$ may be written as*

$$(9)\qquad \cdot\mathfrak{h}^{1} = \sum_{i=1}^{n} \mathfrak{h}_i^{1} + (p - n - m + 1)\mathfrak{g}.$$

PROOF. In constructing 1-simplexes ('0''0), · · · , ('0$^{(m)}$0) (not belonging to $\cdot\mathfrak{M}^2$), we get $\cdot\mathfrak{M}^{*2}$, $\cdot\mathfrak{M}_1^{*2}$, · · · as in the lemma. By (7) and (8), we have

$$(10)\qquad \cdot\mathfrak{h}_i^{*1} = \mathfrak{h}_i^{1} + (p_i - 1)\mathfrak{g},$$

where $\cdot\mathfrak{h}_i^{*1}$ is the homology group of $\cdot\mathfrak{M}_i^{*2}$. The newly constructed simplexes form a connected 1-complex whose 1-dimensional homology group contains the identity only. Therefore from a famous theorem (cf. Seifert-Threfall, p. 179), by (5) we get

$$(11)\qquad \cdot\mathfrak{h}^{*1} = \textstyle\sum_{i=1}^{n} \mathfrak{h}_i^{1} + (p - n)\mathfrak{g},$$

where $\cdot\mathfrak{h}^{*1}$ is the homology group of $\cdot\mathfrak{M}^{*2}$. Therefore (9) is finally established in virtue of (11) and (8)'.

Theorem (3.5) may be extended analogously.

NATIONAL WUHAN UNIVERSITY

A NOTE ON EQUICONTINUITY

M. K. FORT, JR.

During a recent seminar discussion of his paper *Transitivity and equicontinuity* [1],[1] W. H. Gottschalk proposed the following question:

"Is the center of every algebraically transitive group of homeomorphisms on a compact metric space equicontinuous?"

An affirmative answer to the above question is given in this note.

1. **Definitions.** We let X and Y be compact metric spaces and let d be the metric for Y.

A set F of functions on X into X is *algebraically transitive* if corresponding to each pair p and q of points in X there exists $f \in F$ such that $f(p) = q$.

A sequence $[g_n]$ of functions on X into Y converges to a function

---

Presented to the Society, November 27, 1948; received by the editors August 10, 1948.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

g *uniformly at a point* $p \in X$ if $\epsilon > 0$ implies that there exists $N > 0$ and a neighborhood V of p such that $d(g_n(x), g(x)) < \epsilon$ whenever $x \in V$ and $n > N$.

We shall need to make use of the fact that if $[g_n]$ is a sequence of continuous functions on X into Y which converges pointwise to a function g on X, then the sequence converges uniformly at each point of a set residual in X. This fact has been proved by Kuratowski in [2]. Although the notation implies that Kuratowski is restricting himself to more special spaces than those with which we are dealing, the proof given in [2] is actually valid for any compact metric spaces X and Y.[2]

2. A more general theorem.

We shall now prove a theorem which yields as a corollary the answer to Gottschalk's question.

THEOREM. *Let F be a set of continuous functions on X into X and G a set of continuous functions on X into Y, such that to each $f \in F$ there corresponds a continuous function f^* on Y into Y such that $g = f^* g f$ for all $g \in G$. If F is algebraically transitive then G is equicontinuous.*

PROOF. It is well known that in order to prove G equicontinuous it is sufficient to prove that every sequence in G has a uniformly converging subsequence. This is the converse of Ascoli's theorem.

Let S be any sequence in G. Choose a subsequence $[g_n]$ of S which converges at some point $p \in X$. This is possible since Y is compact. We shall prove that $[g_n]$ converges uniformly on X.

We first prove that $[g_n]$ converges pointwise on X. Let x be any point of X. Since we are assuming that F is algebraically transitive, we may choose $f \in F$ such that $f(x) = p$. There exists, by hypothesis, a continuous function f^* on Y into Y such that $g = f^* g f$ for all $g \in G$. Since $f(x) = p$ and $[g_n]$ converges at p, we see that $[g_n f(x)]$ is a converging sequence in Y. Since f^* is continuous on Y, it follows that the sequence $[f^* g_n f(x)]$ converges. This sequence is the same, however, as $[g_n(x)]$.

Since we now know that $[g_n]$ converges pointwise on X, we may let g_0 be the limit of the sequence of functions. The sequence $[g_n]$ converges to g_0 uniformly at each point of a set residual in X, and since X is a compact metric space this residual set is non-empty. Let q be a point at which $[g_n]$ converges uniformly to g_0.

---

[2] The theorem is true for functions on any topological space X into a separable metric space Y. The author has a proof of this fact which will be included in a later paper on applications of semi-continuous set-valued functions.

We now prove that $[g_n]$ converges uniformly at each point of X. Let x be a point of X and choose $f \in F$ such that $f(x) = q$. There exists f^*, continuous on Y into Y, such that $g = f^*gf$ for all $g \in G$. The function g_0 may not belong to G, but since g_0 is the pointwise limit of a sequence of elements of G it is easy to verify that $g_0 = f^*g_0f$. Suppose $\epsilon > 0$. There exists $\delta > 0$ such that if u and v belong to Y and $d(u, v) < \delta$ then $d(f^*(u), f^*(v)) < \epsilon$. There exists $N > 0$ and a neighborhood U of q such that $d(g_n(y), g_0(y)) < \delta$ whenever $n > N$ and $y \in U$. There exists a neighborhood V of x such that $f(V) \subset U$. It is now easy to see that if $z \in V$ and $n > N$ then $d(f^*g_nf(z), f^*g_0f(z)) < \epsilon$. We thus obtain $d(g_n(z), g_0(z)) < \epsilon$ whenever $z \in V$ and $n > N$. This proves that the convergence is uniform at x.

If a sequence of functions converges uniformly at each point of a compact space, then the sequence converges uniformly on the entire space. Therefore $[g_n]$ converges uniformly to g_0 on X.

COROLLARY 1. *If F is an algebraically transitive group of homeomorphisms of X onto X and G is a group of homeomorphisms of X onto X such that $gf = fg$ whenever $f \in F$ and $g \in G$, then G is equicontinuous.*

COROLLARY 2. *If F is an algebraically transitive group of homeomorphisms of X onto X, then the center of F is equicontinuous.*

BIBLIOGRAPHY

1. W. H. Gottschalk, *Transitivity and equicontinuity*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 982–984.

2. C. Kuratowski, *Sur les fonctions représentables analytiquement et les ensembles de première catégorie*, Fund. Math. vol. 5 (1924) pp. 75–91.

UNIVERSITY OF VIRGINIA

PARTITIONING A SET

R. H. BING

1. Introduction. Suppose space has a metric $D(x, y)$.

Under certain conditions in calculus we define $\int_b^a f(x)dx$ to be equal to $\lim \sum f(\xi_i)\Delta x_i$. The dividing of the interval from a to b into a finite number of pieces with lengths Δx_i is a partitioning of the interval from a to b. Similarly, partitioning may be used in integrating over an arbitrary set. The partitioning may provide a basis for a measure on the range of the function which is being integrated. We shall define partitioning on sets for which there is no accepted unit of measure such as length, area, or volume.

The notion of partitioning may be used to assign a convex metric to certain sets. See Theorems 7, 8, 9, and 10. Theorem 8 provides an answer to a question raised by Menger in [3].[1] See [1] for a discussion of and references to partial solutions of this problem. Further applications of partitioning are treated in my paper *Complements of continuous curves*, appearing in Fund. Math.

Most of the results of this paper (including Theorem 7 but not Theorems 1 and 8) were contained in an address I gave before the Mathematics Club at the University of Chicago in May, 1948. After finishing Theorems 1 and 8 in August, I found that Moise, working independently, had obtained a solution [4] to the above mentioned problem by Menger six weeks earlier.

Partitioning. A set M can be partitioned if for each positive number ϵ there is a finite collection G of connected mutually exclusive open subsets of M such that each element of G is of diameter less than ϵ and the sum of the elements of G is dense in M. We shall say that G is an ϵ-partitioning of M.

Refinement. If G and H are two partitionings of M, we say that G is a refinement of H if each element of G is a subset of an element of H.

Property S. A set M has property S if for each positive number ϵ, M is the sum of a finite number of connected subsets such that the diameter of each of these subsets is less than ϵ. This property is discussed in [5, 6, 7, 8].

Convex metric. A metric E for a set M is convex if for each pair of points x, y of M there is a point z of M such that $E(x, z) = E(y, z)$

Presented to the Society, September 10, 1948; received by the editors September 2, 1948.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

$=E(x, y)/2$. The metric F for M is almost convex [2] if for each pair of points x, y of M and each positive number ϵ there is a point z of M such that

$$|F(x, z) - F(x, y)/2| + |F(y, z) - F(x, y)/2| < \epsilon.$$

2. **Partitionable sets.** Theorem 1 characterizes sets that can be partitioned. First we consider two lemmas that will be used.

LEMMA 1. *For each connected set M with property S there is a compact locally connected continuum H and a homeomorphism T of M into a dense subset H' of H such that the diameter of each connected subset X of M is the same as the diameter of $T(X)$ and for each connected open subset R of H, $T^{-1}(R \cdot H')$ is connected.*

PROOF. The continuum H is the complete enclosure of the relative distance space for M. See [8; pp. 154–158].

LEMMA 2. *If M is a connected set with property* S *and H and K are subsets of M at a positive distance from each other, then there is a finite collection of connected, mutually exclusive open subsets of M such that the sum of this collection is dense in M, no element of this collection intersects both H and K, and each element of this collection that intersects K has property* S.

PROOF. We shall prove the lemma for the case where M is closed and compact. It will follow from Lemma 1 for the general case.

Let H' and K' be two sets at a positive distance from each other, containing H and K respectively, such that each point of M belongs to an arc of diameter less than 1/2 that intersects $H'+K'$. Denote by H_1 (or K_1) the set of all points of M that belong to a connected subset of M which intersects H' (or K') and is of diameter less than $D(H', K')/3$. We note that H_1 and K_1 have only a finite number of components.

Let W be a finite collection of points of M such that each point of M belongs to an arc in M of diameter less than 1/4 that intersects W. There is a finite collection A of arcs such that each element of A lies in $M-\bar{K}_1$ and intersects H_1 while A^*, the sum of the elements of A, contains each point of W which belongs to a component of $M-\bar{K}_1$ which intersects H_1. Let B be a finite collection of arcs such that A^*+B^* contains W and each element of B is of diameter less than 1/2, lies in $M-(\bar{H}_1+A^*)$, and intersects K_1.

Denote by H_2 (or K_2) the set of all points that belong to a connected subset of M which intersects H_1+A^* (or K_1+B^*) and is of diameter less than $D(H_1+A^*, K_1+B^*)/3$. We note that each point of K_2 belongs to a connected subset of K_2 which intersects K_1 and is of diameter less than $1/2+1/6$.

Similarly, there exist open subsets $H_3, K_3, H_4, K_4, \cdots$ of M such that (1) H_i is at a positive distance from K_i, (2) H_{i+1} and K_{i+1} contain $\bar{H}_i$ and $\bar{K}_i$ respectively, (3) each component of H_{i+1} contains a component of H_i, (4) each point of K_{i+1} belongs to a connected subset of K_{i+1} that intersects K_i and is of diameter less than $1/2^i+1/(3\cdot 2^i)$, and (5) each point of M belongs to an arc of diameter less than $1/2^i$ that intersects H_i+K_i. Now $\sum(H_i+K_i)$ is dense in M and has only a finite number of components while $\sum K_i$ has property S.

THEOREM 1. *A necessary and sufficient condition that a set M can be partitioned is that it have property* S.

PROOF OF SUFFICIENCY. We show that M can be partitioned by showing that each of its components C can be partitioned.

Let $(p_1, p_2, \cdots, p_n)$ be a finite collection of points of C such that each point of M is at a distance from $\sum p_i$ of less than $\epsilon/4$. Let H_i be the set of all points of C that are at a distance from p_i of less than $\epsilon/4$ and K_i be the set of those that are at a distance from p_i of more than $\epsilon/2$.

By Lemma 2 there are two mutually exclusive open subsets U_1 and V_1 of C such that U_1+V_1 is dense in C, U_1 has a finite number of components and contains H_1, and V_1 has property S and contains K_1. Also, there are two open subsets U_2 and V_2 of V_1 such that U_2+V_2 is dense in V_1, U_2 contains $V_1\cdot H_2$ and has a finite number of components, while V_2 contains $V_1\cdot K_2$ and has property S. Similarly, we define sets $U_3, V_3, U_4, V_4, \cdots, U_{n-1}, V_{n-1}$. The components of $U_1+U_2+\cdots+U_{n-1}+V_{n-1}$ are finite in number, each is of diameter less than ϵ, and their sum is dense in M.

3. **Refinements of partitionings.** We shall be interested in partitioning a set in such a way that each of the pieces into which it is partitioned can be partitioned. These pieces must have property S if this is to be done.

THEOREM 2. *If H and K are two connected subsets of the connected partitionable set M and ϵ is a positive number such that any connected subset of M intersecting both H and K is of diameter more than ϵ, then there are two connected, mutually exclusive open subsets U and V of M containing H and K respectively such that U+V has property* S *and is dense in M.*

PROOF. The general case will follow from Lemma 1 if we prove the case where M is a compact continuum. Hence, we suppose it is one.

For each integer i, let G_i be a $1/i$-partitioning of M. Let H' and K' be mutually exclusive subcontinua of M containing H and K respectively such that each element of G_2 intersects $H'+K'$. Let U_1 (or V_1) be the set of all points that belong to a connected subset of M of diameter less than $D(H', K')/3$ that intersects H' (or K').

Let A and B be finite collections of arcs such that (1) each element of A intersects U_1 while each element of B intersects V_1, (2) each element of $A+B$ is a subset of an element of G_2, (3) no element of A intersects either $\bar{V}_1$ or an element of B, while no element of B intersects $\bar{U}_1$, and (4) each element of G_4 contains a point of A^*+B^*.

Let U_2 (or V_2) be the set of all points that belong to a connected subset of M of diameter less than $D(U_1+A^*, V_1+B^*)/3$ that intersects U_1 (or V_1). We note that each point of U_2 (or V_2) belongs to a connected subset of U_2 (or V_2) of diameter less than $1/2+D(U_1+A^*, V_1+B^*)/3<1/2+1/6$ that intersects U_1 (or V_1).

Similarly, we obtain connected open subsets U_3, V_3, U_4, V_4, $\cdots$ such that (1) U_{i+1} contains $\bar{U}_i$ while V_{i+1} contains $\bar{V}_i$, (2) U_i+V_i intersects each element of G_{2^i}, and (3) each point of U_{i+1} (or V_{i+1}) belongs to a connected subset of U_{i+1} (or V_{i+1}) which is of diameter less than $1/2^i+1/(3\cdot 2^i)$ and intersects U_i (or V_i). Then $U=\sum U_i$ and $V=\sum V_i$ are the required sets.

Each of the following three theorems follows from repeated applications of the theorem preceding it.

THEOREM 3. *If H is a finite collection of connected subsets of the connected partitionable set M and ϵ is a positive number such that no connected subset of M of diameter less than ϵ intersects two elements of H, then there is a finite collection U of connected, mutually exclusive open subsets of M such that each element of U has property* S *and contains one and only one element of H, each element of H is contained in an element of U, and the sum of the elements of U is dense in M.*

THEOREM 4. *If M is partitionable, then for each positive number ϵ there is an ϵ-partitioning G of M such that each element of G has property* S.

THEOREM 5. *If M is a partitionable set, there is a sequence G_1, G_2, $\cdots$ such that G_i is a $1/i$-partitioning of M and G_{i+1} is a refinement of G_i.*

THEOREM 6. *Suppose space has a complete metric and R is a connected domain with property* S *and a nondegenerate boundary B. Then R contains a connected set E with property* S *such that $\bar{E}=E+B$, but no connected subset of E has this property. Also, E is topologically* *equivalent to a subset of a dendron, and any nondegenerate continuum in E is a dendron with a finite number of ends.*

PROOF. Let G_1, G_2, $\cdots$ be a sequence such that G_i is a $1/i$-partitioning of R and G_{i+1} is a refinement of G_i. Denote the collection of closures of elements of G_i by H_i. Let H_1' be a subset of H_1 irreducible with respect to covering B and H_{i+1}' be a subcollection of H_{i+1} such that each element of H_{i+1}' is a subset of an element of H_i', and H_{i+1}' is irreducible with respect to covering B.

Let m_1 be an integer so large that H_{m_1}' has more than one element. There is a dendron T_1 in R which intersects each element of H_{m_1}' but which contains no subcontinuum doing so. Let m_2 be an integer so large that $D(T_1, B) > 3/m_2$. There is a dendron T_2 such that (1) T_2 contains T_1, (2) T_2 intersects each element of H_{m_2}', (3) each component of $T_2 - T_1$ is a subset of an element of G_{m_1}, and (4) no subcontinuum of T_2 has these properties. Similarly, we define T_3, T_4, $\cdots$. Then $E = \sum T_i$.

4. **Convexification of partitionable sets.** We can assign a convex metric to certain partitionable sets.

THEOREM 7. *If M is a compact partitionable continuum, it can be assigned a convex metric.*

FIRST PROOF OF THEOREM 7. We shall define a sequence $E_1(x, y)$, $E_2(x, y)$, $\cdots$ of functions on the pairs of points of M such that $E(x, y) = \lim E_i(x, y)$ is a convex metric for M.

Let G_1, G_2, $\cdots$ be a sequence such that G_i is a $1/i$-partitioning of M and G_{i+1} is a refinement of G_i. There is a dendron T_1 in M such that T_1 intersects each element of G_1, and there is a convex metric F_1 for T_1 such that the diameter of T_1 under F_1 is less than $1/4$. Let U_1 be the set of all elements u such that u is a component of $g - g \cdot T_1$ for some element g of G_1. Denote by V_1 the set of all continua v such that v is either a point of T_1 or the closure of an element of U_1.

If v and w are two elements of V_1, we define $E_1(v, w)$ to be the greatest lower bound of all numbers of the sort $r/2 + s$, where some continuum is the sum of v, w, r other elements of V_1 and some arcs in T_1 whose total length under F_1 is s.

Let m_2 be an integer so large that $D(v, w) > 3/m_2$ if v and w are elements of V_1 such that $E_1(v, w) > 1/16$. In each element u_i of U_1 that contains an element of G_{m_2} there is a dendron $T_{2,i}$ which intersects each element of G_{m_2} that intersects u_i. It may be noted that there are only a finite number of such u_i's. Let $F_{2,i}$ be a convex metric for $T_{2,i}$ such that the diameter of $T_{2,i}$ under $F_{2,i}$ is less than $1/16$.

Let U_2 be the collection of all components of the sort $g-g\cdot(T_1+\sum T_{2,i})$, where g denotes an element of G_{m_2}. Let V_2 designate the set of all continua v such that v is either a point of $T_1+\sum T_{2,i}$ or the closure of an element of U_2.

If v and w are two elements of V_2, we define $E_2(v, w)$ to be the greatest lower bound of all numbers of the type $r/4+s$, where some continuum is the sum of v, w, r other elements of V_2 and a collection of arcs in $T_1+\sum T_{2,i}$ whose total length under their respective F's is s.

Similarly, we define m_3, G_{m_3}, $T_{3,j}$, $F_{3,j}$, U_3, V_3, $E_3(v, w)$, m_4, G_{m_4}, $T_{4,j}$, $F_{4,j}$, U_4, V_4, $E_4(v, w)$, $\cdots$. In general, m_{i+1} is an integer so large that

$$(1)\qquad D(v, w) > 3/m_{i+1} \quad \text{if} \quad E_i(v, w) > 1/4^{i+1} \qquad (v,\ w \text{ elements of } V_i).$$

Consider an element $u_{i,n}$ of U_i that contains an element of $G_{m_{i+1}}$. Let $K_{i,n}$ denote the collection of all continua k such that k is the closure of a component of the common part of $u_{i,n}$ and an element of $G_{m_{i+1}}$. Denote by $K'_{i,n}$ the finite collection of all elements k of $K_{i,n}$ such that k intersects an element of V_i which does not intersect any component of $T_1+\sum T_{2,j}+\cdots+\sum T_{i,j}$ which intersects k. There is a dendron $T_{i+1,n}$ in $u_{i,n}$ such that $T_{i+1,n}$ intersects each element of $K'_{i,n}$ and each element of $G_{m_{i+1}}$ which intersects $u_{i,n}$. Let $F_{i+1,n}$ be a convex metric for $T_{i+1,n}$ such that

$$(2)\qquad \text{diameter } T_{i+1,n} \text{ under } F_{i+1,n} < 1/4^{i+1}.$$

Denote by U_{i+1} the collection of all components of sets of the sort $g-g\cdot(T_1+\sum T_{2,j}+\cdots+\sum T_{i+1,j})$, where g denotes an arbitrary element of $G_{m_{i+1}}$. The elements of V_{i+1} are the points of $T_1+\sum T_{2,j}+\cdots+\sum T_{i+1,j}$ and the closures of the elements of U_{i+1}.

Let

$$E_{i+1}(v, w) = \text{greatest lower bound } (r/2^{i+1}+s) \quad (v,\ w \text{ elements of } V_{i+1}),$$

where some continuum is the sum of v, w, r other elements of V_{i+1} and a collection of arcs in $T_1+\sum T_{2,j}+\cdots+\sum T_{i+1,j}$ whose lengths under their respective F's total s. It may be noted that if v intersects w, $E_i(v, w)=0$.

If p and q are two points of M, we define $E_i(p, q)$ to be the greatest lower bound of all numbers $E_i(v, w)$, where v and w are elements of V_i containing p and q respectively. We shall show that $\lim E_i(p, q) = E(p, q)$ exists and $E(x, y)$ is a convex metric for M.

Using (1) we find that if some continuum is the sum of the elements v and w of V_i and n elements of V_{i+1}, then $E_i(v, w) < n/2^{i+1}$. From this and the fact that two elements of V_{i+1} are subsets of the same element of V_i if they intersect the same arc in $\sum T_{i+1,j}$, we find that if v and w are two elements of V_{i+1}, then there are elements v' and w' of V_i containing v and w respectively such that $E_i(v', w') \leqq E_{i+1}(v, w)$. Hence, $E_i(p, q)$ is a monotone nondecreasing function of i.

If a component C of $T_1 + \sum T_{2,j} + \cdots + \sum T_{i,j}$ intersects the sum of two elements v, w of V_i which intersect each other, it follows from (1) and (2) that under its respective metric

$$\text{diameter } C \cdot (v + w) < 1/4^i. \tag{3}$$

If v and w are two elements of V_{i+1} which are subsets of the same element of V_i, then

$$E_{i+1}(v, w) < 1/4^{i+1} + \delta(1/2^{i+1} + 1/4^i) \tag{4}$$

where δ is equal to 2, 1, or 0 according as neither, one, or both of the elements v, w are nondegenerate subsets of elements of $K'_{i,j}$ for some integer j.

Using (3) and (4) we find that if v and w are elements of V_{i+1} which are subsets of the sum of a coherent collection of n elements of V_i, then

$$E_{i+1}(v, w) < n(1/2^i + 1/4^i + 1/4^{i+1}) + 1/4^i < (n/2^i)(1 + 3/2^i). \tag{5}$$

Using (5) we find that

$$E_{i+1}(p, q) < (1 + 3/2^i)[E_i(p, q) + 2/2^i]. \tag{6}$$

Repeated applications of (6) give that

$$\begin{aligned} E_{i+k}(p, q) < (1 + 3/2^i)(1 + 3/2^{i+1}) \cdots (1 + 3/2^{i+k-1})[E_i(p, q) + 2/2^i \\ + 2/2^{i+1} + \cdots + 2/2^{i+k-1}]. \end{aligned}$$

Since

$$\begin{aligned} (1 + 3/2^i)(1 + 3/2^{i+1}) \cdots &< [1 + 3/(2^i - 6)][1 + 3/(2^{i+1} - 6)] \cdots \\ &= 2^i/(2^i - 6) \qquad (i > 3), \end{aligned}$$

then

$$E_i(p, q) \leqq E(p, q) < [2^i/(2^i - 6)][E_i(p, q) + 4/2^i] \qquad (i > 3). \tag{7}$$

That $\lim E_i(p, q) = E(p, q)$ exists follows from the fact that $E_1(p, q)$, $E_2(p, q)$, $\cdots$ is a bounded monotone sequence.

That $E(x, y)$ satisfies the triangle condition follows from the fact that for each triple of points p, q, r, $E_i(p, q) \leqq E_i(p, r) + E_i(r, q) + 1/2^i$. Since $E_i(x, y)$ is symmetric, $E(x, y)$ is also; that is, $E(x, y) = E(y, x)$. If R is an open subset of M containing the point p, there is an integer n so large that $D(p, M-R)>3/m_n$. Since $E_n(p, M-R)$ is positive, $E(p, M-R)$ is also. If p and q belong to the closure of the same element of G_{m_n+1}, then by (4), $E_n(p, q)<2/2^n$, and by (7), $E(p, q) <[2^n/(2^n-6)](2/2^n+4/2^n)$ if $n>3$. Hence $E(p, q_1)$, $E(p, q_2)$, $\cdots$ has zero as a limit if and only if $q_1, q_2, \cdots$ converges to p. Therefore $E(x, y)$ is a metric for M that preserves its topology.

That $E(x, y)$ is almost convex follows from the fact that for each pair of points p, q and each integer n there is a point r such that $|E_n(p, r)-E_n(p, q)/2|+|E_n(r, q)-E_n(p, q)/2|\leqq 1/2^n$. Since M is compact, $E(x, y)$ is a convex metric for M.

ALTERNATE PROOF OF THEOREM 7. The proof of Theorem 7 can be simplified by making use of Theorem 6 of this paper and Theorem 4 of [1] which states that *a compact locally connected continuum M has a convex metric if for each point p contained in an open subset R_1 of M, there is an open subset R_2 of R_1 containing p such that the boundary of R_2 with respect to M is a subset of a subcontinuum of M with a convex metric*. Hence, we need to show only that $\overline{E}$ of Theorem 6 has a convex metric.

Now E is the sum of a dendron T_1 and a finite collection U_1 of connected, mutually exclusive open subsets of E such that the common part of T_1 and the closure of an element of U_1 is an end point of T_1. There is a convex metric F_1 for T_1 such that the distance between any two of its end points is between 1/2 and 1. Let V_1 be the collection of all continua v such that v is either a point of T_1 or the closure of an element of U_1. If v and w are elements of V_1, define $E_1(v, w)$ to be the greatest lower bound of all numbers of the type $r/2+s$ where some continuum is the sum of v, w, r other elements of V_1 and arcs in T_1 whose lengths under F_1 total s.

Let ϵ_2 be a number so small that if v and w are two elements of V_1 which do not intersect, then $D(v, w)>3\epsilon_2$ and $D(T_1, \overline{E}-E)>3\epsilon_2$. Now E is the sum of a dendron T_2 and a finite collection U_2 of connected, mutually exclusive open subsets of E such that each element of U_2 is of diameter less than ϵ_2, and the common part of T_2 and the closure of an element of U_2 is an end point of T_2. Let F_2 be a convex metric for T_2 that preserves F_1 on T_1 and such that any arc in T_2 is of length between $1/2^2$ and 1/2 if the end points of this arc are also end points of the closure of a component of T_2-T_1. Denote by V_2 the collection of all elements v such that v is either a point of T_2 or the closure of an element of U_2. If v and w are elements of V_2, let $E_2(v, w)$ be the greatest lower bound of numbers of the type $r/4+s$, where some continuum is the sum of v, w, r other elements of V_2 and arcs in T_2 whose lengths under F_2 total s.

This process is continued to get a sequence of functions $E_3, E_4, \cdots$. In general, ϵ_{i+1} is a number so small that if v and w are any two elements of V_i that do not intersect each other, then $D(v, w) > 3\epsilon_{i+1}$ and $D(T_i, \bar{E}-E) > 3\epsilon_{i+1}$. There is a dendron T_{i+1} and a finite collection U_{i+1} of connected, mutually exclusive open subsets of E such that each element of U_{i+1} is of diameter less than ϵ_{i+1}, and the common part of T_{i+1} and the closure of an element of U_{i+1} is an end point of T_{i+1}. There is a convex metric F_{i+1} for T_{i+1} that preserves F_i on T_i such that any arc in T_{i+1} is of length, under F_{i+1}, of between $1/2^{i+1}$ and $1/2^i$ if the end points of this arc are end points of the closure of the same component of $T_{i+1}-T_i$. Use V_{i+1} to denote the collection of points of T_{i+1} and the closures of elements of U_{i+1}. For elements v and w of V_{i+1}, define $E_{i+1}(v, w)$ to be the greatest lower bound of $r/2^{i+1}+s$, where some continuum is the sum of v, w, r other elements of V_{i+1} and arcs in T_{i+1} whose lengths under F_{i+1} total s.

Define $E_i(p, q)$ to be the greatest lower bound of $E_i(v, w)$ where p and q are points of elements v and w respectively of V_i. The argument to show that $\lim E_i(p, q)$ exists and determines a convex metric for $\bar{E}$ is similar to and somewhat easier than that used in the first proof.

Applying Theorem 1 to Theorem 7 we obtain the following.

THEOREM 8. *Each compact, locally connected continuum has a convex metric.*

The application of Lemma 1 and Theorem 1 to Theorem 7 gives Theorems 9 and 10.

THEOREM 9. *Each connected partitionable set has an almost convex metric.*

THEOREM 10. *Each connected set with property* S *has an almost convex metric.*

REFERENCES

1. R. H. Bing, *A convex metric for a locally connected continuum*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 812–819.

2. ———, *Extending a metric*, Duke Math. J. vol. 14 (1947) pp. 511–519.

3. K. Menger, *Untersuchungen uber allgemeine Metrik*, Math. Ann. vol. 100 (1928) pp. 75–163.

4. E. E. Moise, *Grille decomposition and convexification theorems for compact locally connected continua*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1111–1121.

5. R. L. Moore, *Concerning connectedness im kleinen and a related property*, Fund. Math. vol. 3 (1922) pp. 232–237.

GRILLE DECOMPOSITION AND CONVEXIFICATION THEOREMS FOR COMPACT METRIC LOCALLY CONNECTED CONTINUA

EDWIN E. MOISE

A convex metric space is a metric space in which for each two points a and b there is a point c, different from a and b, such that $d(a, b) = d(a, c)+d(c, b)$, where d is the distance function. Menger has raised the question whether for every Peano space (that is, compact, metric, locally connected continuum) it is possible to define a distance function (preserving the original topology) with respect to which the space is convex.[1] In the present paper, this question is answered in the affirmative. For a general discussion of the problem and various partial solutions of it, see Menger[2] and Blumenthal.[3] After Blumenthal's book was written, Beer[4] and Bing[5] established convexification theorems for the one-dimensional and finite-dimensional cases respectively.

Only a short time after the present paper was written, Bing independently obtained a proof of the convexification theorem. His paper will be published in this Bulletin.

As Menger has shown[6] every compact convex metric space is locally connected. An affirmative answer to Menger's question therefore shows that for compact metric spaces, local connectedness and the existence of a convex metric are equivalent.

Theorems 1 and 2, our principal preliminary results, may be of interest independently of the use made of them here; they do not appear to be readily deducible from the convexification theorem, and their conclusions are considerably stronger than our present purposes require.

THEOREM 1. *Let S be a Peano space and let R, R', and D be open sets such that* (1) $R\subset R'\subset D$, (2) *the closures of* $\beta(R)-\beta(D)$ *and*

Presented to the Society, September 10, 1948; received by the editors November 26, 1948.

[1] K. Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. vol. 100 (1928) pp. 81 and 98.

[2] Menger, loc. cit. p. 96 ff.

[3] Leonard M. Blumenthal, *Distance geometries*, University of Missouri Studies, vol. 13, 1938, No. 2, p. 43.

[4] G. Beer, *Beweis des Satzes, dass jede im kleinen zusammenhängende Kurve konvex metrisiert werden kann*, Fund. Math. vol. 31 (1938) p. 281.

[5] R. Bing, *A convex metric for a locally connected continuum*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 812–819.

[6] Menger, loc. cit., p. 98.

$\beta(R')-\beta(D)$ do not intersect, and (3) D is locally connected at each point of $\beta(D)$.[7] *Then there is an open set R'' such that (1) $R\subset R''\subset R'$, and (2) R'' and $D-\overline{R}''$ are locally connected at each point of $\beta(R'')$.*[8]

PROOF. The proof of this theorem will involve some rather complicated definitions; and it may be well to explain their motivation by giving a very rough sketch of the proof. We wish to obtain a sequence $U_1, U_2, \cdots$ of open sets lying between R and R' (in the sense that $R\subset U_i\subset R'$) such that the sequence $U_1, U_3, \cdots$ is monotone ascending and $U_2, U_4, \cdots$ is monotone descending, and a sequence $G_1, G_2, \cdots$ of finite collections of open sets, each a refinement of the one before, with meshes[9] approaching 0, covering the annuli between successive terms of the U-sequence. It will turn out that R'' is the sum of the sets U_{2i+1} and that the closure of $\beta(R'')-\beta(D)$ is the common part of the sets G_i^*.[10] We shall want to use the elements of the collections G_i to establish local connectivity for R'' on $\beta(R'')$. We therefore want the sets $g\cdot U_{2i+1}$ (where g belongs to a term of the G-sequence) to be in some sense "as nearly connected as possible." We are forced to give rather complicated criteria of such "optimum connectness." Furthermore, at each stage we wish to define an element of the U-sequence so as to give ourselves a manageable situation for the next stage. Definition F gives a precise formulation of this condition. Finally, in Lemmas 4 and 5, we show that by doing our best at each stage, we do well enough.

DEFINITION A. By $U:V$ we mean that U and V satisfy the hypothesis of the theorem for R and R'.

DEFINITION B. If U is an open set lying in D, then by $B(U)$ we mean the closure of $\beta(U)-\beta(D)$.

DEFINITION C. If B is a closed set in $\overline{D}$, then an admissible covering of B is a finite collection G of open sets covering B such that for each g of G, $g\cdot D$ is connected.

---

[7] If A is a set, then $\beta(A)$ is the boundary of A. If A and B are sets, then $A-B$ denotes $A-A\cdot B$ whether B is a subset of A or not. A set M is locally connected at a point p (which may or may not belong to M) if for each open set U containing p, there is an open U' containing p and lying in U such that $U'\cdot M$ is connected. (See R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications, vol. 32.) An open subset of a compact space is uniformly locally connected if and only if it is locally connected at every point of its closure; the notion of local connectivity used in Theorem 1 therefore is equivalent to the more cumbersome notion of uniform local connectedness.

[8] This is a strengthened form of a theorem of R. L. Wilder, loc. cit. Theorem III 3.5; Wilder's theorem states that if R and R' are open subsets of a Peano space and $\overline{R}\subset R'$, then there is a uniformly locally connected open set R'' lying between R and R'.

[9] The mesh of a collection G of sets is the least upper bound of the diameters of the elements of G.

[10] If G is a collection of sets, then G^* is the set of all elements of elements of G.

DEFINITION D. Suppose that $U:V$ and that (1) G is an admissible covering of $B(U)+B(V)$ such that (2) $D\subset U+(S-\bar{V})+G^*$. Then G is *contractible with respect to U and V* if there is an admissible covering $T(G)$ (where T is one-to-one) having properties (1) and (2) such that if $g'=T(g)\in T(G)$, then $\bar{g}'\subset g$.[11] $T(G)$ is called a *contraction* of G. If we require merely that $g'\subset g$, then $T(G)$ is a *weak contraction* of G.

LEMMA 1. *If $U:V$ and G_I and G_0 satisfy conditions* (1) *and* (2) *of Definition* D, *and $G_I=T(G_0)$, then there is a G' which is a contraction of G_0 such that G_I is a contraction of G'.*

DEFINITION E. If $U:V$(or $V:U$), G is contractible with respect to U and V, G' is an admissible covering of $B(U)$, and $G'\gg G$,[12] then a *jumpless extension of V with respect to U, G, and G'* is an open set V' such that (1) $U:V':V$ (or $V:V':U$), (2) for each point p of $G^*\cdot(D-\bar{V}')$ (or of $G^*\cdot V'$) there is a g of G such that some component of $g\cdot(D-\bar{V}')$ (or of $g\cdot V'$) contains p and has a limit point in $B(V)$, and (3) G' is contractible with respect to U and V'.

DEFINITION F. Let $U:V$ (or $V:U$) and let Γ be contractible with respect to U and V. Let ϵ be a positive number. Then U is said to be *ϵ-approachable from V through* Γ if there is an open set V' and a collection G which is contractible with respect to U and V' such that (1) $U:V':V$, (or $V:V':U$), (2) $G\gg\Gamma$, (3) G has mesh less than ϵ, and (4) V' is a jumpless extension of V with respect to U, Γ, and G. Under these conditions, V' is an *ϵ-approach to U through* Γ.

LEMMA 2. *Let $U:V$, let G_I and G_0 be contractible with respect to U and V such that G_I is a contraction of G_0, and let ϵ be a positive number. Then there is a set U' and a collection Γ which is contractible with respect to U and V such that* (1) *Γ is a contraction of G_0,* (2) *G_I is a weak contraction of Γ,* (3) *if γ and g are corresponding elements of Γ and G_I under the appropriate weak contraction, then no point of $B(U)$ is a limit point of $\gamma-g$,* (4) *U' is a jumpless extension of U with respect to V, Γ, and Γ,*[13] *and* (5) *U' is ϵ-approachable from V through Γ.*

PROOF OF LEMMA. Let G' be an admissible covering of $B(U)$ and let W be an open set such that (1) $U:W:V$, (2) G' is contractible with respect to U and W, (3) $\bar{G}'^{*}\cdot B(V)$ is empty, (4) $G'\gg G_I$, and (5) the mesh of G' is less than ϵ. Let Z be the interior of the set of all points p of $D-\bar{U}$ for which there does not exist a g of G_I such that some component of $g\cdot(D-\bar{U})$ contains p and has a limit point in $B(V)$. Z is clearly open, and no point of $B(V)$ is a limit point of $U''=W+Z$. Let G'' be an admissible covering of $B(U'')$ and a closure refinement of G_I, of mesh such that $\bar{G}''^{*}\cdot B(V)$ is empty. Let G be $G'+G''$.

---

[11] For reasons of typographical convenience, the superior bar, indicating closure, has not been placed over primes or subscripts. For example, $\bar{g}'$, $\bar{G}^*$, and $\bar{U}_1$ are the closures of g', G^*, and U_1.

[12] $G'\gg G$ means that G' is a closure refinement of G; that is, the closure of each element of G' lies in some element of G.

[13] Here Γ plays the parts of both the G and the G' of Definition E.

It is now true that there is an open set U', and a collection Γ such that U', Γ, and G satisfy the conclusion of the lemma. Let X be an open set such that $U'':X:V$ and G is contractible with respect to U'' and X. Let Γ be a collection satisfying (1), (2), and (3) of the conclusion of the lemma, and having the further property that for each γ of Γ all but a finite number of components of $g\cdot(D-\bar{X})$ either lie in G^{*} or have a limit-point in $B(V)$. (Note that to obtain each γ of Γ from the corresponding g of G_I, we need only add to g the sum of a finite number of sufficiently small neighborhoods of points of the boundary of G^{*}.) If one of these exceptional components can be connected to $B(V)$ in $g\cdot(D-\bar{X})$ merely by making the set X smaller, we do so. The hypothesis for Z shows that by repeating this process a finite number of times, we obtain the desired U'.

Definition G. If U and g are sets, then $N(U, g)$ is the number of components of $U\cdot g$. If U is a set and G is a collection of sets, then $I(U, G)$ is $\sum_{g\in G} N(U, g)$.

Lemma 3. *If* $U:U_I:V_0:V$ *and* G_0 *and* $G_I=T(G_0)$ *are contractible with respect to* U *and* V, *then there are sets* U' *and* V', *and a collection* G *which is contractible with respect to* U *and* V *such that* (1) $U:U':U_I:V_0:V':V$, (2) G_I *and* G *are contractions of* G *and* G_0 *respectively, and* (3) $I(U', G)$ *and* $I(D-\bar{V}', G)$ *are both finite.*

Proof of lemma. Let g belong to G_I. Since $g\cdot D$ is connected, each component of $U\cdot g$ has a limit point in $\bar{g}\cdot B(U)$. Let H_g be a finite collection of open sets such that (1) for each h of H_g, $h\cdot D$ is connected, (2) if $h\in H_g$ and $g'\in G_I$, then $B(h)$ intersects $B(g')$ only if h intersects $B(g')$, (3) $\bar{h}\cdot B(U_I)$ is empty, (4) H_g covers $g\cdot B(U)$, (5) if g' and g'' belong to G_I and G_0 respectively and $\bar{g}'$ lies in g'', then no element of H_g contains a point of $B(G')$ and a point of $B(g'')$, and (6) there is a collection $Z(H_g)$ having properties (1), (2), (3), (4), and (5) such that for each h of H_g, $\bar{h}$ lies in $Z(h)$. (Here Z denotes a transformation throwing H_g onto $Z(H_g)$.)

Now let U_g be the sum of U and the elements g of H_g which intersect $g\cdot U$. Let $Y(g)$ be $g+U_g$. For each g' of G_I-g, let $Y(g')$ be the sum of g' and the elements of H_g that intersect $U\cdot g'$. It is now true that $N(U_g, Y(g))$ is finite and $I(U_g, Y(G_I-g)) \leqq I(U, G_I-g)$. Moreover, there is a collection $Y'(G_I)$, of which $Y(G_I)$ is a contraction, which has all the same properties, such that $Y'(G_I)$ is a contraction of G_I and G_0 is a contraction of $Y'(G_I)$, with respect to U and V, and also with respect to U_g and V.

Clearly, by a finite number of such operations, first for U and then for V, we obtain the sets U', V' and the collection G required in the conclusion of the lemma.

Definition H. Let $U_1 : U_2$ and let G be contractible with respect to U_1 and U_2. Then for each g of G, the index Ind (U_2, U_1, g) of U_2 with respect to U_1 and g is the number of components of $g\cdot U_2$ that have a limit point in $B(U_1)$; and the index Ind (U_2, U_1, G) of U_2 with respect to U_1 and G is $\sum_{g\in G}$ Ind (U_2, U_1, g). (Obviously Ind $(U_2, U_1, G) \leqq I(U_2, G)$.) An analogous definition of Ind (U_2, U_1, G) may be given for the case $U_2 : U_1$.[14]

With the help of the above lemmas, we may now define a sequence $U_1, U_2, \cdots$ of open sets and a triangular sequence $G_{i,j}$ of finite collection of open sets ($i=1, 2, \cdots; j=i, i+1, \cdots$) satisfying the following conditions:

(1) For each i, $U_i : U_{i+2} : U_{i+1}$ or $U_{i+1} : U_{i+2} : U_i$, according as i is odd or even;

(2) For each i, j, $G_{i,j}$ is contractible with respect to U_i and U_{i+1} and is of mesh less than $1/i$;

(3) $G_{i,j} \gg G_{i-1,i-1}$ (if the latter is defined) and is a weak contraction of $G_{i,j+1}$;

(4) For each i, U_i is a jumpless extension of U_{i-2} with respect to U_{i-1}, $G_{i-1,i-1}$, and $G_{i,i}$;

(5) If $i \leqq j$, $i \leqq k$, then Ind $(U_k, U_{k-1}, G_{i,j})$ is finite;

(6) If g belongs to $G_{i,j}$, g' is the corresponding element of $G_{i,j+1}$ under the appropriate weak contraction, and $i' > i$, then g' contains the closure of $g\cdot B(U_{i'})$; but no point of $B(U_i)$ is a limit point of $g'-g$;

(7) If g belongs to $G_{i,j}$, g' belongs to $G_{i',j'}$ ($i' > i$), and $\bar{g}\cdot\bar{g}'$ is not empty, then $T^{-1}(g)$ contains $\bar{g}'$, where T is the appropriate weak contraction; and

(8) Subject to all the above conditions, given $U_1, U_2, \cdots, U_n$

---

[14] To be exact, if $U_2 : U_1$ and g belongs to G, then Ind (U_2, U_1, g) is the number of components of $g\cdot(D-\bar{U}_2)$ that have a limit point in $B(U_1)$, and Ind (U_2, U_1, G) is $\sum_{g\in G}$ Ind (U_2, U_1, g). As in several of the definitions and lemmas given above, we have a sort of duality relation in which certain sets U, V are replaced by the complements (in D) of their closures.

and $G_{i,j}$ $(i \leqq n,\ j \leqq n)$, the immediately following terms U_{n+1} and $G_{i,n+1}$ $(i \leqq n+1)$ are chosen so as to minimize the numbers Ind $(U_{n+1}, U_{n-1}, G_{i,n+1}) = f(i)$ in the order of increasing i; that is to say, of all possible succeeding terms which satisfy (1)–(7) and *which can be continued to give a complete sequence satisfying* (1)–(7), we restrict ourselves to those for which $f(1)$ is minimal; among the latter possibilities, we limit ourselves to those for which $f(2)$ is minimal, and so on.

Of course we must start the sequences so that $R: U_1: U_2: R'$. Let G_i, for each i, be the collection obtained by adding together the ascending sequences of corresponding elements of the collections $G_{i,j}$. Let R'' be the sum of the sets U_{2i-1}. Then $B(R'')$ is the common part of the sets G_i^*.

One reason for using the triangular sequence $G_{i,j}$ rather than a simple sequence is that in order to obtain finite indices satisfying (5), we may need to enlarge our coverings slightly. On the other hand, in passing from U_i to U_{i+1}, we do not need to enlarge the coverings in the neighborhood of $B(U_i)$, so that (6) can be satisfied. Note that (6) implies that (9) Ind $(U_{i+2k}, U_i, G_{i,i+k'})$ is a nonincreasing function of k and k' simultaneously. Note also that (4) and (7) imply that (10) for k, $k'>0$, U_{i+2k} is a jumpless extension of U_{i-2} with respect to U_{i-1} and $G_{i,i+k}$.

It is clear that if there are sequences satisfying conditions (1)–(7), then there are sequences which also satisfy (8). To show that such sequences exist, we shall make use of an auxiliary triangular sequence $H_{i,j}$ which is to satisfy conditions (1)–(6) and also:

(7′) If h belongs to $H_{i,j}$, Z is the sum of two intersecting closures of elements of $H_{i',j'}$ $(i'>i)$ and $\bar{h}$ intersects $\bar{Z}$, then $T^{-1}(h)$ contains $\bar{Z}$, where T is the weak contraction throwing $H_{i,j}$ into $H_{i,j+1}$.

We shall assume that U_i and $H_{i,j}$ have been defined for $i \leqq j < n$ and that $G_{i,j}$ has been defined for $i \leqq j \leqq n-2$, and show that they may be extended a step further. In the transitivity step we shall assume that n is odd, the corresponding argument when n is even being entirely analogous.

We first define all the immediately succeeding terms of the H- and G-sequences except for $H_{n,n}$ and $G_{n-1,n-1}$, so as to satisfy (1)–(7) and (7′). Given these, let ϵ be a positive number such that any set $H_{n,n}$ with mesh less than ϵ will satisfy (7′). By Lemma 2, let U'_{n-1} be a set and let $H'_{n-1,n-1}$ be a collection satisfying all the hypotheses for $H_{n-1,n-1}$ such that U'_{n-1} is a jumpless extension of U_{n-1} with respect to U_{n-2}, $H'_{n-1,n-1}$, and $H'_{n-1,n-1}$, and such that U'_{n-1} is ϵ-approachable from U_{n-2} through $H'_{n-1,n-1}$. Let $H_{n,n}$ be an admissible covering of $B(U'_{n-1})$, with the required finite indices, and let U_n be an open set such that the mesh of $H_{n,n}$ is less than ϵ, and such that $H_{n,n}$ is contractible with respect to U'_{n-1} and U_n. Let $G_{n-1,n-1}$ be the set obtained by replacing each element h of $H_{n-1,n-1}$ by the sum g of h and all elements of $H_{n,n}$ that intersect h. By (10), U'_{n-1} is a jumpless extension of U_{n-3} with respect to U_{n-2}, G_{n-2}, and $H_{n,n}$. We are now in the same position as before, with n replaced by $n+1$.

LEMMA 4. *Let p be a point of $g\cdot B(R'')$, where g belongs to some G_i. Then there does not exist a sequence C_1, C_2, $\cdots$ of components of $R''\cdot g$ and a sequence p_1, p_2, $\cdots$ of points such that for each j, p_j belongs to C_j, and such that p is the sequential limit point of the sequence p_1, p_2, $\cdots$.*

PROOF. Since $R''\cdot g$ is locally connected, it contains no point of the limiting set of the C-sequence. Let n be a natural number such that if g' and g'' belong to G_n and have a point in common and $p\in g'$, then $g''\subset g$. Let P be the sum of all such elements g' of G_n. For each $2i+1$ greater than n, $B(U_{2i+1})$ intersects P, and P is the sum of two mutually separated sets H and K, H containing $P\cdot B(U_{2i+1})$ and K intersecting a term of the C-sequence. This contradicts condition (10) for the U-sequence.

LEMMA 5. *Let p and g be as in Lemma 4. Then there do not exist two components C and C' of $R''\cdot g$, each of which has p as a limit point.*

PROOF OF LEMMA. Assume that the lemma is false, let n be an even integer, and let g_n be an element of $G_{n,n}$ containing p. Let k be the least odd integer for which g_n contains a point of $C\cdot B(U_k)$ and a point of $C'\cdot B(U_k)$. Then C and C' contribute a count of at least two to Ind $(U_{k+2}, U_k, G_{n,k+2})$. We shall show that if the lemma is false, then U_{k+2} did not satisfy condition (8).

Let $G'_{n,k+1}$ be $G_{n,k+2}$. Let $G'_{k+1,k+1}$ be a certain $G_{m,m}$, m being an odd number, sufficiently large so that condition (7) will be satisfied. Let P be an open set containing p and lying in an element of $G_{m,m}$ such that $P\cdot D$ is connected, and such that $\bar{P}$ lies in U_{m+3}. Replace U_{k+2} by some odd-numbered element of the U-sequence which intersects P. By this process Ind $(U_{k+2}, U_k, G_{n,k+2})$ has been reduced. But given any ϵ, we may choose P and U'_{k+2} so as to make U'_{k+2} ϵ-approachable from U_{k+1} through $G_{m,m}$, so that the sequence can be continued satisfying conditions (1)–(7). Ind $(U_{k+2}, U_k, G_{n,k+2})$ was therefore not minimal, which contradicts condition (8), and the lemma is proved. The theorem now follows immediately.

DEFINITION 1. Let S be a Peano space and let G be a finite collection of mutually exclusive connected open sets such that $\overline{G^*}$ is S. Then G is a *connected grating decomposition of* S. If the elements of G are uniformly locally connected, then G is a *grille decomposition of* S. Two elements of G are *adjacent* if they have a boundary point in common. If $G_1, G_2, \cdots$ is a sequence of connected grating (or grille) decompositions of S such that each term of the sequence after the first is a refinement of its predecessors, and such that for each positive ϵ, only a finite number of terms of the sequence have mesh greater than ϵ, then the sequence $G_1, G_2, \cdots$ is a *complete sequence of connected grating* (*or grille*) *decompositions of* S.

THEOREM 2. *Every Peano space has a complete sequence of grille decompositions.*

PROOF. Let H and H' be finite coverings of S by open sets such that $H' \gg H$. Let h and h' be elements of H and H' respectively such that $\bar{h}' \subset h$. We now have the hypothesis of Theorem 1 satisfied with $D = S$. Let g_1 be the set R'' of the conclusion of Theorem 1 such that $h' : g_1 : h$. Now let the D of Theorem 1 be $S - \bar{g}_1$; we choose another element of H', and an element of H containing its closure; and we proceed as before to obtain a set g_2 which contains $\bar{h}'' \cdot D$. At each stage, we let D be the complement of the closure of the sum of the sets g_i previously obtained. Finally we let G_1 be the set of all components of the sets g_i. To obtain G_2, we work with the elements g of G_1 one at a time, starting the process for each g by letting $D = g$.

DEFINITION 2. Let $G_1, G_2, \cdots$ be a complete sequence of connected grating decompositions of the Peano space S; and let $g \in G_i$. Then the border Bord (g) of g is the set of all elements g' of G_{i+1} that lie in g and have a boundary point in common with $\beta(g)$. The core Core (g) of g is the set of all elements of G_{i+1} that lie in g but do not belong to Bord (g).

DEFINITION 3. A *chain* is a finite collection C of mutually exclusive open sets $c_1, c_2, \cdots, c_k$, such that the sets c_i and c_j have a boundary point in common if and only if i and j are identical or consecutive integers. The *end-links* of the chain are c_1 and c_k. If x and y are points of c_1 and c_k respectively (or of $\bar{c}_1$ and $\bar{c}_k$ respectively), then C is a *chain from* x *to* y (or a *chain spanning* x *and* y). If H and K are sets intersecting $\bar{c}_1$ and $\bar{c}_k$ respectively, then C *spans* H *and* K.

DEFINITION 4. A complete sequence of connected grating decompositions is *core-wise connected* if for each g of each G_i and for each pair g', g'' of elements of G_{i+1} lying in g, there is a chain of elements of G_{i+1}, having g' and g'' as its end-links, such that $C - g' - g''$ lies in Core (g).

THEOREM 3. *Every Peano space has a core-wise connected complete sequence of grating decompositions.*

PROOF. Let $G_1, G_2, \cdots$ be a complete sequence of connected grating decompositions of the space S. Let g be an element of G_1. By deleting a finite number of terms of the G-sequence, if necessary, we may arrange for Core (g) not to be empty. Let x be a point belonging to an element of Core (g). For each h of G_2 lying in g, let x_h be a point of h. Let A be a collection of arcs from x to the points x_h, not intersecting $\beta(g)$. Let n be an integer such that if g_n belongs to G_n, then $\bar{g}_n$ does not intersect both A^* and $\beta(g)$. Let Z be a collection of chains of elements of G_n, no links of which have boundary points in common with $\beta(g)$, and which contain x and the points x_h in the closures of their end-links. (Note that x and the points x_h do not necessarily belong to G_n^*.) For each h of Bord (g), let G_h be the set of all components of $h-\bar{Z}^{**}$ that have a boundary point in $\beta(g)$. Since each h is connected, each element of G_h will have a boundary point in $\bar{Z}^{**}$. Let G_h' be the set of all components of $h-\bar{G}_h^*$. Now let G be the sum of Core (g), the sets G_h, and the sets G_h'.

Let G_1' be G_1 and let G_2' be G. G_1' is clearly corewise connected and the same process may be repeated to give a complete sequence.

THEOREM 4. *Every Peano space can be given a convex metric.*

PROOF. Let $G_1, G_2, \cdots$ be a complete sequence of core-wise connected grating decompositions of the space S. We shall define a certain subsequence $G_1', G_2', \cdots$ of the G-sequence. For each g of each G_i, we shall define a real number $\delta(g)$. For each two points x, y of S, we shall let $d_i'(x, y)$ be the least real number d for which there is a chain C: $c_1, c_2, \cdots, c_k$ of elements of G_i' such that x and y belong to $\bar{c}_1$ and $\bar{c}_k$ respectively, and such that the length $l(c) = \sum \delta(c_j)$ of the chain C is equal to d.

For each g of $G_1' = G_1$, let $\delta(g) = 1$.

Now suppose that $G_1', \cdots, G_i'$ have been defined as a subsequence of the G-sequence and the δ function has been defined for each of their elements. Let G_{i+1}' be a term of the G-sequence following G_i' such that no element of G_i' has an empty core and satisfying a second condition which we shall state presently. Let p be the number of elements of G_{i+1}' and let η be $2^{-i-2}/p$. For each g of G_i' and each b of Bord (g), let $\delta(b)$ be $(1/2)(\delta(g)-\eta)$. Let q be the number of elements of Core (g), and for each c of Core (g), let $\delta(c)$ be η/q. The second condition imposed on G_{i+1}' is that with the above definition of δ, any chain of elements of G_{i+1}' which consists entirely of elements of the borders of elements of G_i' and spans two non-adjacent elements of G_i' shall have length greater than that of any chain of G_j'. Since δ is an increasing function of p on the borders of elements of G_j', this condition may be satisfied merely by going out far enough in the G-sequence.

Let m be the least of η and all numbers $\delta(b)$, for each c of each Core (g), let $\delta(c)=m/p$.

LEMMA 1. *Let x and y be points of S and let i be an integer such that no chain of elements of G_i' of less than five links spans x and y. For each j greater than i, let $d_j''(x, y)$ be the length of the shortest chain of elements of G_j' which spans the sum of* (1) *the set of all points x for which there is a two-link chain of elements of G_j' spanning x and x' and* (2) *the set of all points y' for which there is such a chain spanning y and y'. Then $D_{j+1}''(x, y)\geqq(1-2^{-j})d_j''(x, y)$.*

PROOF OF LEMMA. Let C' be a chain of elements of G_{j+1}' spanning (1) and (2) and let C be the set of all elements of G_j' that contain one or more elements of C', with the ordering naturally induced by that of C'. If we can show that each c of C contains two links of C' that belong to Bord (c), the lemma will be proved. Suppose that there is a c of C which does not have this property and let C_e be a subcollection of C which is maximal with respect to being a segment of consecutive elements of C none of which have the property. Then every element of C_e is adjacent to every other element of C_e and has a boundary point in common with (1) or (2).[15] Suppose, without loss of generality, that each element of C_e has a boundary point in common with (1). It follows that the first element c_1 of C that follows all elements of C_e also has a boundary point in common with (1); and this c_1 contains two links of C' that lie in Bord (c_1). But there is an element of Bord (c_1) that has a boundary point in common with (1); C' may therefore be modified in an obvious manner so as to reduce its length, which contradicts our hypothesis.

LEMMA 2. *If x and y are different points of S, then $d(x, y)=\lim_{i\to\infty} d_i'(x, y)$ exists and is positive.*

PROOF OF LEMMA. Since, for fixed x, y, d_i' is a diminishing function of i, we need only show that its limit is not zero. It is obvious that d_j' is not less than any d_j'' of the type defined in Lemma 1; our result therefore follows directly from the convergence of the infinite product $\prod(1-2^{-j})$.

LEMMA 3. *If x is the sequential limit of the sequence x_1, x_2, $\cdots$*

---

[15] This can be shown by an argument like the one immediately following.

of points of S, *then* $\lim_{i\to\infty} d(x, x_i) = 0$, *and conversely.*

LEMMA 4. *If* x, y, *and* z *belong to* S, *then* $d(x, y) + d(y, z) \geqq d(x, z)$.

LEMMA 5. *If* x *and* z *belong to* S, *then there is a point* y *such that* $d(x, y) = d(y, z) = (1/2)d(x, z)$.

PROOF OF LEMMA. By the construction of the distance function, it is clear that for each i there is a point y_i for which the required equalities hold with an error of less than $1/i$. Any point y which appears infinitely often in the sequence, or which is a limit point of $\sum y_i$, satisfies the conclusion of the lemma.

It follows that d is a convex metric.

UNIVERSITY OF MICHIGAN

ON THE NUMBER OF POSITIVE INTEGERS LESS THAN x AND FREE OF PRIME DIVISORS GREATER THAN x^c

V. RAMASWAMI

Dr. Chowla recently raised the following question regarding the number of positive integers less than x and free of prime divisors greater than x^c, which number is here denoted by $f(x, c)$: For every fixed positive c, is $\liminf_{x\to\infty} f(x, c)/x > 0$?

This paper, while incidentally answering this question in the affirmative, proves more, in fact the best[1] possible result in this direction, namely:

THEOREM A. *A function* $\phi(c)$ *defined for all* $c > 0$ *exists such that*

(1) $\phi(c) > 0$ *and is continuous for* $c > 0$;

(2) *for any fixed* c

$$f(x, c) = x\phi(c) + O(x/\log x)$$

where the "O" is uniform for c greater than or equal to any given positive number.

Notation. The following symbols are used for the entities mentioned against them:

p, p_r: any prime.

$S(x, p)$: the set of integers less than x each divisible by p and free of prime divisors greater than p.

$T(x, p)$: the set of integers less than x each free of prime divisors greater than p.

$N[K]$: the number of members of K, where K denotes any finite set of integers.

$F(t)$: $\sum_{p\leq t} (1/p)$ where p runs through primes.

Preliminary lemmas.

LEMMA I. *For* $c \geq 1$, *the theorem is true, and*

$$f(x, c) = x\phi(c) + O(1).$$

PROOF. This is obvious. In fact, for these values of c, $\phi(c) = 1$, and

$$|f(x, c) - x\phi(c)| \leq 1.$$

LEMMA II. *If* $p_1 \neq p_2$, *the sets* $S(x, p_1)$ *and* $S(x, p_2)$ *are distinct, and*

Received by the editors January 12, 1948, and, in revised form, July 22, 1948.

[1] The "O" of the theorem cannot be improved upon, as will be seen in the sequel.

$N[S(x, p)]=N[T(x/p, p)]$.

Proof. This is obvious, from the unique factorization theorem.

Lemma III. *The number of primes less than or equal to x is $O(x/\log x)$ and $F(x)=\log\log x+b+O(1/\log x)$ where b is constant.*

This is well known and is an "elementary theorem" in the theory of primes.

Lemma IV. *If $0<c_1\leqq 1$, and Theorem* A *is true for $c\geqq c_1$, then it is true for $c\geqq c_1/(1+c_1)$.*

Proof. By hypothesis $\phi(c)$ is defined for $c\geqq c_1$, and

$$\phi(c) > 0 \text{ and is continuous for } c \geqq c_1,$$

$$f(x, c) = x\phi(c) + O(x/\log x) \text{ uniformly for } c_1 \leqq c \leqq 1, \tag{3}$$

$$f(x, c) = x\phi(c) + O(1) \text{ for } c \geqq 1, \qquad \text{by Lemma 1.}$$

Also, obviously,

(4) $\phi(c)$ is bounded and monotonic increasing, though possibly not strictly so, for $c\geqq c_1$.

Let now

$$c_2=c/(1+c_1) \text{ and } c_2 \leqq d \leqq c_1 \text{ (obviously } c_2 < c_1). \tag{5}$$

Now

$$\begin{aligned} f(x, c_1) - f(x, d) &= \sum_{x^d<p\leqq x^{c_1}} N[S(x, p)] && \text{(by Lemma II)} \\ &= \sum_{x^d<p\leqq x^{c_1}} N\left[T\left(\frac{x}{p}, p\right)\right] && \text{(by Lemma II)} \\ &= \sum_{x^d<p\leqq x^{c_1}} f\left(\frac{x}{p}, \frac{\log p}{\log x - \log p}\right) \\ &= \sum_{x^d<p\leqq x^{c_1}} \left\{\frac{x}{p}\, \phi\left(\frac{\log p}{\log x - \log p}\right)\right\} \\ &\quad + \sum_{p\geqq x^{1/2}} O(1) + \sum_{p<x^{1/2}} O\left(\frac{x}{p\log(x/p)}\right) \end{aligned}$$

by (3) and (5), since in this range

$$\frac{\log p}{\log x - \log p} > \frac{d}{1-d} \geqq \frac{c_2}{1-c_2} = c_1. \tag{6}$$

Hence

$$\text{(6a)}\quad f(x, c_1) - f(x, d) = x \int_{x^d}^{x^{c_1}} \phi\left(\frac{\log t}{\log x - \log t}\right) dF(t) + O\left(\frac{x}{\log x}\right)$$

by Lemma III, where the "O" is uniform with respect to d in virtue of (3) and (6), and the Riemann-Stieltjes integral on the right exists in virtue of (4) and the continuity of $\phi(c)$ for $c \geqq c_1$. Using integration by parts for the integral and using (3) and (4) and Lemma III, we obtain (see Note 1a)

$$\text{(7)}\quad \begin{aligned} f(x, c_1) - f(x, d) &= x \int_{x^d}^{x^{c_1}} \phi\left(\frac{\log t}{\log x - \log t}\right) \frac{dt}{t \log t} + O\left(\frac{x}{\log x}\right) \\ &\qquad \text{(uniformly for } c_2 \leqq d \leqq c_1) \\ &= x \int_d^{c_1} \phi\left(\frac{u}{1-u}\right) \frac{du}{u} + O\left(\frac{x}{\log x}\right) \\ &\qquad \text{(uniformly for } c_2 \leqq d \leqq c_1). \end{aligned}$$

This, together with (3), proves the existence of $\phi(c)$ for $c \geqq c_1/(1+c_1)$, though it is not yet clear whether $\phi(c) > 0$ in $c_1/(1+c_1) \leqq c \leqq c_1$, and is continuous therein.

Now, let

$$\text{(8)}\quad \frac{c_1}{1 + c_1} \leqq d_1 < d_2 \leqq c_1.$$

Then (7) gives

$$\begin{aligned} \frac{f(x, d_2) - f(x, d_1)}{x} &= \int_{d_1}^{d_2} \phi\left(\frac{u}{1-u}\right) \frac{du}{u} + O\left(\frac{1}{\log x}\right) \\ &\to \int_{d_1}^{d_2} \phi\left(\frac{u}{1-u}\right) du \qquad \text{as } x \to \infty. \end{aligned}$$

Hence

$$\text{(9)}\quad \phi(d_2) - \phi(d_1) = \int_{d_1}^{d_2} \phi\left(\frac{u}{1-u}\right) \frac{du}{u} > 0,$$

by (3), since in the range $d_1 < u < d_2$, we have $u/(1-u) > c_1$.

This shows that $\phi(d_2) \neq 0$ for any d_2 of the kind specified in (8); for, if $\phi(d_2)$ were zero, then obviously $\phi(d_1)$ would be zero and their difference also would be so, contrary to (9). Also, obviously $\phi(d_2) \geqq 0$. Hence

$$\text{(10)}\quad \phi(c) > 0 \qquad \text{for } c > c_1/(1 + c_1).$$

Also by (9) and the hypothesis

(11) $\phi(c)$ is continuous for $c \geqq c_1/(1+c_1)$.

Using now the results (7), (10), and (11) and repeating the above argument with $c_1/(1+c_1)$ in place of c_1, and noting that the positiveness of $\phi(c_1)$ is not needed in the above argument, it follows that $\phi(c_1/(1+c_1))>0$.

This completes the proof of the lemma.

PROOF OF THE THEOREM. Lemma I holds for $c\geqq 1$ and Lemma IV for $0<c<1$, if we note that the hypotheses of the lemma are satisfied for $c_1=1$, and hence by the result of the lemma for $c_1=1/2$ and hence by induction for $c_1=1/n$ (n positive integral), and that $1/n\rightarrow 0$ as $n\rightarrow\infty$.

COROLLARY. *For* $0<c_1<c_2\leqq 1$,

$$\phi(c_2)-\phi(c_1)=\int_{c_1}^{c_2}\phi\left(\frac{u}{1-u}\right)\frac{du}{u}$$

(*see also Note* 1b).

PROOF. A finite set of numbers, $d_0, d_1, d_2, \cdots, d_n$, obviously exists such that

$$c_1=d_0<d_1<d_2<\cdots<d_n=c_2;$$

and

$$d_r\geqq\frac{d_{(r+1)}}{1+d_{(r+1)}}\qquad (r=0, 1, 2, \cdots, n-1).$$

To each of these intervals (d_r, d_{r+1}) apply the result (9) and add; the corollary follows at once.

REMARKS. The "O" of the theorem cannot be improved upon. This can be seen as follows:

Obviously $f(x, 1)=x+O(1)$ and

$$f(x,1)-f\left(x,\frac{1}{2}\right)=\sum_{x^{1/2}<p\leqq x}\left[\frac{x}{p}\right]=x\sum_{x^{1/2}<p\leqq x}\left(\frac{1}{p}\right)-G(x)$$

where $G(x)=\sum_{x^{1/2}<p\leqq x}\{x/p\}$ and $\{y\}$ denotes fractional part of y.

Hence $f(x, 1/2)=x(1-\log 2)+o(x/\log x)+G(x)$ by the prime number theory (see Note 1c). But $G(x)>kx/\log x$ where k is a fixed positive number, as can be easily seen from the prime number theory.

NOTE 1. (a) The deduction of (7) from (6a) is based on the following:

We observe that the Riemann-Stieltjes integral

$$\int_{x^d}^{x^{c_1}} \phi\left(\frac{\log t}{\log x - \log t}\right) dF(t) = \left[F(t)\phi\left(\frac{\log t}{\log x - \log t}\right)\right]_{x^d}^{x^{c_1}}$$

$$- \int_{x^d}^{x^{c_1}} F(t) d\phi$$

$$= \left[(\log \log t + b)\phi\left(\frac{\log t}{\log x - \log t}\right)\right]_{x^d}^{x^{c_1}}$$

$$- \int_{x^d}^{x^{c_1}} (\log \log t + b) d\phi + R$$

which, by the substitution $t = x^u$ performed after integration by parts of the integral on the right, equals

$$\int_d^{c_1} \phi\left(\frac{u}{1-u}\right)\frac{du}{u} + R$$

where, by Lemma III,

$$R = O\left(\frac{1}{d \log x}\right) - O\left(\frac{1}{d \log x}\right) - \int_{x^d}^{x^{c_1}} O\left(\frac{1}{\log t}\right) d\phi$$

where, on account of (4), the integral also is $O(d^{-1} \log^{-1} x)$.

(b) The integral equation for $\phi(c)$ may be used for successive computation of the function. Starting with $\phi(c) = 1$ for $c \geqq 1$, one observes that $c/(1-c) \geqq 1$ for $1/2 \leqq c < 1$ so that

$$\phi(c) = 1 - \int_c^1 \frac{du}{u} = 1 + \log c \quad \text{for } 1/2 \leqq c \leqq 1.$$

This result can be used to compute $\phi(c)$ on the interval $(1/3, 1/2)$ and so on; and an easy induction shows that

$$\phi(c) = 1 + \sum_{r=1}^{\infty} (-1)^r \psi_r(c), \qquad c > 0,$$

where $\psi_1(c) = \int_c^1 du/u$ for $c \leqq 1$, and $\psi_1(c) = 0$ for $c > 1$, and

$$\psi_r(c) = \int_c^1 \psi_{r-1}\left(\frac{u}{1-u}\right)\frac{du}{u}, \qquad \text{for } c > 0.$$

One notes that $\psi_r(c) = 0$ for $c \geqq 1/r$ (so that the infinite series is actually a finite sum) and also that the functions $\psi_r(c)$, $r \geqq 2$, are not elementary functions.

(c) From the known result $\pi(x)=x/\log x+x/\log^2 x+o(x/\log^2 x)$ follows

$$\sum_{x^{1/2}<p\leqq x}\frac{1}{p}=\int_{x^{1/2}}^{x}\frac{1}{t}\,d\pi(t)=\frac{\pi(x)}{x}-\frac{\pi(x^{1/2})}{x^{1/2}}+\int_{x^{1/2}}^{x}\frac{\pi(t)}{t^2}\,dt$$

$$=-\frac{1}{\log x}+o\left(\frac{1}{\log x}\right)$$

$$+\int_{x^{1/2}}^{x}\left[\frac{t}{\log t}+\frac{t}{\log^2 t}+o\left(\frac{t}{\log^2 t}\right)\right]\frac{dt}{t^2}$$

$$=\log 2+o\left(\frac{1}{\log x}\right).$$

NOTE 2. From the corollary to the theorem follows, for $0<c_1<c_2\leqq c_1/(1-c_1)$ and $c_2\leqq 1$,

$$\phi(c_2)-\phi(c_1)\geqq\phi(c_2)\log c_2/c_1>\phi(c_2)(1-c_1/c_2)$$

from which follows $\phi(c_2)/c_2>\phi(c_1)/c_1$, whence, arguing as in the proof of the corollary, one sees that $\phi(c)/c$ is a strictly monotonic increasing function in $0<c\leqq 1$.

$\phi(c)$ has other interesting properties, which will be published shortly. One such is that for every fixed n, $\phi(c)/c^n\to 0$ as $c\to+0$. This result, together with the now obvious result

$$\lim_{x\to\infty}\frac{f(x,c)}{x}=\phi(c)>0,\qquad c\text{ fixed and positive,}$$

was communicated to Dr. Chowla in August of 1947. I understand from his reply that Vijayaraghavan already was in possession of a proof of his (Chowla's) conjecture that

$$\liminf_{x\to\infty}\frac{f(x,c)}{x}>0,\qquad\text{for } c \text{ positive and fixed.}$$

In conclusion, I wish to thank the referees for their suggestions which have led to the clarification and additions of content contained in Note 1.

ANDHRA UNIVERSITY

ON STRICTLY MINIMAL TOPOLOGICAL DIVISION RINGS

LEOPOLDO NACHBIN

1. **Introduction.** It is well known that, for every real or complex topological vector space, the following statements hold good:

(1) A nonvanishing linear functional on the space is continuous if and only if its kernel is closed in the space;

(2) Every finite-dimensional space possesses only one admissible topology.

In this note we shall determine the widest classes of topological division rings that can be used as scalar domains of topological vector spaces so as to preserve these important propositions. The results thus obtained generalize part of the recent work of J. Braconnier [2] and I. Kaplansky [5].[1] In a subsequent paper we shall apply these results to other related questions.

A *topological ring* is a ring endowed with an *admissible* topology, that is, a Hausdorff topology on the ring with respect to which the ring operations $x+y$, $-x$, xy are continuous. A *topological vector space* is a vector space over a topological division ring endowed with an *admissible* topology, that is, a Hausdorff topology on the vector space with respect to which the vector space operations $x+y$, λx are continuous. Throughout this note we shall understand the notion of *completeness* and *completion* for these topological systems in the sense formulated by A. Weil [8] (see also Bourbaki [1]). $\mathfrak{T}_1$ and $\mathfrak{T}_2$ being two topologies on the same point set, we write $\mathfrak{T}_1 \leqq \mathfrak{T}_2$ to denote that every set open according to $\mathfrak{T}_1$ must be open according to $\mathfrak{T}_2$; and $\mathfrak{T}_1 < \mathfrak{T}_2$ if, in addition to this, $\mathfrak{T}_1 \neq \mathfrak{T}_2$.

2. **Strictly minimal rings.** Let K be a topological division ring and $\mathfrak{T}_K$ be its admissible topology. A topology $\mathfrak{T}$ on K is said to be *admissible with respect to* $\mathfrak{T}_K$ if K endowed with $\mathfrak{T}$ is a topological vector space over K endowed with $\mathfrak{T}_K$: this means that the mapping $(x, y) \to x+y$ is continuous from $\mathfrak{T} \times \mathfrak{T}$ to $\mathfrak{T}$, and the mapping $(x, y) \to xy$ is continuous from $\mathfrak{T}_K \times \mathfrak{T}$ to $\mathfrak{T}$. Putting $y=1$ in the last condition, we see that the identity mapping $x \to x$ is continuous from $\mathfrak{T}_K$ to $\mathfrak{T}$, that is, $\mathfrak{T} \leqq \mathfrak{T}_K$. An obvious partial converse to this fact is the following: if $\mathfrak{T}$ is an admissible topology on K and $\mathfrak{T} \leqq \mathfrak{T}_K$, then $\mathfrak{T}$ is admissible with respect to $\mathfrak{T}_K$.

Presented to the Society, October 30, 1948; received by the editors August 13, 1948.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

A topological division ring K is said to be ***minimal*** if its topology $\mathfrak{T}_K$ is a minimal element in the ordered set of all admissible topologies on K; that is, if there exists no admissible topology $\mathfrak{T}$ on K such that $\mathfrak{T} < \mathfrak{T}_K$.[2] The topological division ring is said to be *strictly minimal* if there exists no topology $\mathfrak{T}$ on K admissible with respect to $\mathfrak{T}_K$ such that $\mathfrak{T} < \mathfrak{T}_K$; it amounts to the same to say that the only topology on K admissible with respect to $\mathfrak{T}_K$ is $\mathfrak{T}_K$ itself.

THEOREM 1. *Every strictly minimal topological division ring is minimal.*

PROOF. Let $\mathfrak{T}_K$ be the admissible topology given on the division ring K, and let $\mathfrak{T}$ be another admissible topology on K. If $\mathfrak{T} \leqq \mathfrak{T}_K$ it follows that $\mathfrak{T}$ is admissible with respect to $\mathfrak{T}_K$; but $\mathfrak{T}_K$ is strictly minimal: therefore $\mathfrak{T} = \mathfrak{T}_K$, that is, $\mathfrak{T}_K$ is minimal.

Let K be a topological division ring. A set $A \subset K$ is said to be *l-bounded* if for any neighborhood W of 0 there exists a neighborhood V of 0 such that $VA \subset W$. If K is discrete, every set A is obviously bounded. Let us assume that K is not discrete. If the set A is bounded, for any neighborhood W of 0 there exists some $\lambda \in K$, $\lambda \neq 0$, such that $\lambda A \subset W$: it suffices to pick some $\lambda \neq 0$ in that neighborhood V of 0 for which $VA \subset W$. Conversely, assume that corresponding to every neighborhood W of 0 there exists $\lambda \in K$ with $\lambda \neq 0$ such that $\lambda A \subset W$. Given the neighborhood W of 0, take some neighborhood W_1 of 0 such that $W_1 W_1 \subset W$ and determine $\lambda \in K$ with $\lambda \neq 0$ such that $\lambda A \subset W_1$. Putting $V = W_1 \lambda$, we have a neighborhood of 0 and $VA \subset W$: therefore A is l-bounded. The set A is said to be *restricted* if $0 \notin \overline{B}$, where $B = (A-0)^{-1}$, where the bar denotes closure in K and $A-0$ is the set of all $x \in A$, $x \neq 0$. It amounts to the same to say that there exists a neighborhood V of 0 satisfying the following equivalent conditions: $1 \notin VA$, $1 \notin AV$.

Every topological division ring has at least one restricted neighborhood of 0. In fact, consider a neighborhood W of 0 such that $1 \notin W$ and take a neighborhood V of 0 such that $VV \subset W$: then $1 \notin VV$ and therefore V is restricted.

Every l-bounded set is restricted. Let W be a neighborhood of 0 such that $1 \notin W$. If the set $A \subset K$ is bounded, there exists some neighborhood V of 0 such that $VA \subset W$ and therefore $1 \notin VA$.

---

[2] A minimal admissible topology on K, even when K is complete under it, need not be the first element of the ordered set of all admissible topologies on K. In fact, let R be the field of real numbers and $\mathfrak{T}$ be the natural topology on R. It is known (see Dieudonné [3]) that there exists an admissible topology $\mathfrak{T}^*$ on R such that the completion of R under $\mathfrak{T}^*$ is isomorphic to the field of complex numbers endowed with its usual topology. By the remark after the proof of Theorem 2, we see that both $\mathfrak{T}$ and $\mathfrak{T}^*$ are minimal on R. But $\mathfrak{T} \neq \mathfrak{T}^*$: therefore $\mathfrak{T} \leqq \mathfrak{T}^*$ is false.

THEOREM 2. *Every nondiscrete topological division ring in which all restricted sets are l-bounded is strictly minimal.*

PROOF. Let K be the given division ring and $\mathfrak{T}_K$ be its admissible topology. Consider a topology $\mathfrak{T}$ on K admissible with respect to $\mathfrak{T}_K$. We have already seen that $\mathfrak{T} \leqq \mathfrak{T}_K$. Let V_0 be a neighborhood of 0 according to $\mathfrak{T}$ such that $1 \notin V_0$. The continuity of the mapping $(x, y) \rightarrow xy$ from $\mathfrak{T}_K \times \mathfrak{T}$ to $\mathfrak{T}$ implies the existence of a neighborhood V_1 of 0 according to $\mathfrak{T}_K$ and a neighborhood V_2 of 0 according to $\mathfrak{T}$ such that $V_1 V_2 \subset V_0$ and a fortiori $1 \notin V_1 V_2$. This shows that V_2 is restricted according to $\mathfrak{T}_K$. By the assumption, V_2 is l-bounded according to $\mathfrak{T}_K$ and therefore corresponding to every neighborhood V of 0 according to $\mathfrak{T}_K$ there exists some $\lambda \in K$, $\lambda \neq 0$, such that $\lambda V_2 \subset V$. Since λV_2 is a neighborhood of 0 according to $\mathfrak{T}$, we may infer that the same is true for V. This shows that $\mathfrak{T}_K \leqq \mathfrak{T}$ and finally $\mathfrak{T} = \mathfrak{T}_K$.

From the result just established, we may conclude that every nondiscrete topological division ring which admits a valuation preserving the topology (see Kaplansky [5]) is strictly minimal. More particularly, every nondiscrete locally compact division ring is strictly minimal.

3. **One-dimensional spaces.** At this moment, let us recall the usual notion of direct image of a topology in the vector space case. Let E be a topological vector space over a topological division ring K, let F be a vector space over K and consider a linear transformation $\phi: E \rightarrow F$ from E onto F, that is $\phi(E) = F$. Let $\mathfrak{T}_E$ and $\mathfrak{T}_K$ be the topologies given on E and K, respectively, and define a topology $\mathfrak{T}_F$ on F in the following way: a set $Y \subset F$ is said to be open according to $\mathfrak{T}_F$ if its inverse image $\phi^{-1}(Y)$ is open according to $\mathfrak{T}_E$. It is easy to see that: (1) $\mathfrak{T}_F$ is really a topology on F, (2) $\phi(X)$ is open according to $\mathfrak{T}_F$ for any X open according to $\mathfrak{T}_E$, (3) the vector space operations on F are continuous with respect to $\mathfrak{T}_F$ and $\mathfrak{T}_K$, (4) $\mathfrak{T}_F$ is a Hausdorff topology if and only if the kernel $\phi^{-1}(0)$ is closed according to $\mathfrak{T}_E$, and finally (5) ϕ is continuous from $\mathfrak{T}_E$ to $\mathfrak{T}_F$. We call $\mathfrak{T}_F$ the *direct image of* $\mathfrak{T}_E$ under ϕ.

THEOREM 3. *Let K be a given topological division ring. Consider two topological vector spaces E and F over K, where F is one-dimensional, and a nonvanishing linear transformation $\phi: E \rightarrow F$. If K is strictly minimal, then ϕ is continuous if and only if its kernel $\phi^{-1}(0)$ is closed in E. If K is not strictly minimal, this result need not be always true.*

PROOF. If ϕ is continuous, it is obvious that its kernel must be closed in E: this half of the proof does not depend on K. Conversely, let us assume that K is strictly minimal and let $\mathfrak{T}_K$ be its topology. Assume also that $\phi^{-1}(0)$ is closed in E. Noticing that $\phi(E)=F$, we may introduce on F the direct image $\mathfrak{T}_F$ of the topology $\mathfrak{T}_E$ given on E, and $\mathfrak{T}_F$ will be admissible on F. Since every one-dimensional vector space F over K is algebraically isomorphic to K considered as a vector space over K, the assumption that K is strictly minimal is equivalent to saying that every one-dimensional vector space over K possesses only one admissible topology. Therefore $\mathfrak{T}_F$ coincides with the admissible topology previously given on F and the proof is achieved by remarking that ϕ is continuous from $\mathfrak{T}_E$ to $\mathfrak{T}_F$. Now, let us assume that K is not strictly minimal and let $\mathfrak{T}$ be a topology on K admissible with respect to $\mathfrak{T}_K$ such that $\mathfrak{T}<\mathfrak{T}_K$. Consider the set $E=K$ endowed with $\mathfrak{T}$ as a topological vector space over K endowed with $\mathfrak{T}_K$ and the set $F=K$ endowed with $\mathfrak{T}_K$ as a topological vector space over K endowed with $\mathfrak{T}_K$. The nonvanishing linear transformation $\phi:E\rightarrow F$ defined by $\phi(x)=x$ for any $x\in E$ has a kernel $\phi^{-1}(0)=0$ closed in E, but ϕ is not continuous.

THEOREM 4. *Let $E=E_1\oplus E_2$ be a topological vector space over a strictly minimal topological division ring K, where E_1 and E_2 are vector subspaces of E, the first being one-dimensional. Let $\mathfrak{T}$ be the admissible topology given on E and $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be the admissible topologies induced by $\mathfrak{T}$ on E_1 and E_2, respectively. Then we have $\mathfrak{T}=\mathfrak{T}_1\oplus\mathfrak{T}_2$ if and only if E_2 is closed in E.*

PROOF. It is clear that the relation $\mathfrak{T}=\mathfrak{T}_1\oplus\mathfrak{T}_2$ implies that E_2 (and also E_1) is closed in E. Conversely, let us assume that E_2 is closed in E. Putting $\mathfrak{T}^*=\mathfrak{T}_1\oplus\mathfrak{T}_2$ we obtain an admissible topology on E. Consider any neighborhood W of 0 in E according to $\mathfrak{T}$ and determine some neighborhood V of 0 in E according to $\mathfrak{T}$ such that $V+V\subset W$. Then $W_1=V\cap E_1$ is a neighborhood of 0 in E_1 according to $\mathfrak{T}_1$ and $W_2=V\cap E_2$ is a neighborhood of 0 in E_2 according to $\mathfrak{T}_2$ and we have $W_1+W_2\subset W$. Since W_1+W_2 is a neighborhood of 0 in E according to $\mathfrak{T}^*$, the same is true for W and we have proved that $\mathfrak{T}\leqq\mathfrak{T}^*$. This half of the proof does not depend on the nature of K. On the other hand, every $x\in E$ may be expressed uniquely as $x=x_1+x_2$, where $x_1\in E_1$ and $x_2\in E_2$. Define $\pi_1:E\rightarrow E_1$ and $\pi_2:E\rightarrow E_2$ by $\pi_1(x)=x_1$ and $\pi_2(x)=x_2$. Since $E_2=\pi_1^{-1}(0)$ is closed in E, we may make use of Theorem 3 and say that $\pi_1:E\rightarrow E_1$ is continuous from $\mathfrak{T}$ to $\mathfrak{T}_1$. It amounts to the same to say that $\pi_1:E\rightarrow E$ is continuous from $\mathfrak{T}$ to $\mathfrak{T}$. Since $\pi_2(x)=x-\pi_1(x)$, we may also say that $\pi_2:E\rightarrow E$ is continuous from $\mathfrak{T}$ to $\mathfrak{T}$ or, equivalently, that $\pi_2 : E \rightarrow E_2$ is continuous from $\mathfrak{T}$ to $\mathfrak{T}_2$. Having proved this, let there be given any neighborhood W of 0 in E according to $\mathfrak{T}^*$; by definition there exists a neighborhood W_1 of 0 in E_1 according to $\mathfrak{T}_1$ and also a neighborhood W_2 of 0 in E_2 according to $\mathfrak{T}_2$ such that $W_1 + W_2 \subset W$. By the continuity of π_1, we may find a neighborhood V_1 of 0 in E according to $\mathfrak{T}$ such that $\pi_1(V_1) \subset W_1$, and similarly there exists a neighborhood V_2 of 0 in E according to $\mathfrak{T}$ for which $\pi_2(V_2) \subset W_2$. Putting $V = V_1 \cap V_2$, we see that $V \subset W_1 + W_2 \subset W$; therefore W is a neighborhood of 0 in E according to $\mathfrak{T}$ and we have proved that $\mathfrak{T}^* \leqq \mathfrak{T}$. In conclusion we have $\mathfrak{T} = \mathfrak{T}^*$.

E and F being two sets and $\phi : E \rightarrow F$ being a function from E to F, the set $G(\phi)$ of all points $\{x, \phi(x)\}$ of the product space $E \times F$ will be called the *graph* of the function.

THEOREM 5. *Let K be a given topological division ring. Consider two topological vector spaces E and F over K, where F is one-dimensional, and a linear transformation $\phi : E \rightarrow F$. If K is strictly minimal, then ϕ is continuous if and only if its graph $G(\phi)$ is closed in $E \times F$. If K is not strictly minimal, this result need not be always true.*

PROOF. It is clear that the graph of a continuous function into a Hausdorff space is closed. Therefore the continuity of ϕ implies that $G(\phi)$ is closed. Conversely assume that K is strictly minimal and $G(\phi)$ is closed. Define $\varphi(x, y) = \phi(x) - y$ for $x \in E$, $y \in F$. Then $\varphi : E \times F \rightarrow F$ is a linear transformation with a closed kernel $\varphi^{-1}(0) = G(\phi)$. By Theorem 3, φ is continuous and since $\varphi(x, 0) = \phi(x)$ we conclude that ϕ is continuous. The final part of the statement is proved exactly as in the case of Theorem 3.

4. **Finite-dimensional spaces.** A partial extension of Tychonoff's theorem about the uniqueness of admissible topologies is the following:

THEOREM 6. *Let K be a given topological division ring. If K is strictly minimal, every n-dimensional vector space E over K possesses only one admissible topology with respect to which all $n-1$ dimensional vector subspaces are closed. If K is not strictly minimal this result need not be true.*

PROOF. Let $\mathfrak{T}$ be the admissible topology given on E. Since every vector subspace of E is an intersection of $n-1$ dimensional vector subspaces, it follows from the assumptions that all vector subspaces of E are closed. Let $E = E_1 \oplus \cdots \oplus E_n$, each E_i being a one-dimensional vector subspace of E. Put $E^i = E_i \oplus \cdots \oplus E_n$. Let $\mathfrak{T}_i$ and $\mathfrak{T}^i$

be the topologies induced by $\mathfrak{T}$ on E_i and on E^i respectively. Since E^2 is closed in E, we have $\mathfrak{T}=\mathfrak{T}_1\oplus\mathfrak{T}^2$ by Theorem 4. Since E^3 is closed in E and $E^3\subset E^2$, we see that E^3 is closed in E^2. Therefore $\mathfrak{T}^2=\mathfrak{T}_2\oplus\mathfrak{T}^3$ again by Theorem 4, and so on. By induction we get $\mathfrak{T}=\mathfrak{T}_1\oplus\cdots\oplus\mathfrak{T}_n$. Since K is strictly minimal, each $\mathfrak{T}_i$ is the unique admissible topology on E_i; this implies the uniqueness of $\mathfrak{T}$. The final part of the statement is proved exactly as in the case of Theorem 3.

In the preceding theorem, it would be sufficient to assume the existence of n closed vector subspaces $S_1, \cdots, S_n$ of dimension $n-1$ such that $S_1\cap\cdots\cap S_n=0$. In fact, let S be another vector subspace of dimension $n-1$. Consider $n+1$ linear functionals $\phi_i:E\to K$ and $\phi:E\to K$ with kernels S_i and S $(i=1, \cdots, n)$ respectively. Then there are $c_1, \cdots, c_n\in K$ such that $\phi(x)=c_1\phi_1(x)+\cdots+c_n\phi_n(x)$ for any $x\in E$. Since S_i is closed, we see by Theorem 3 that each ϕ_i is continuous; therefore ϕ is continuous and S is closed.

The complete extension of Tychonoff's theorem is as follows:

THEOREM 7. *Let K be a given topological division ring. In order that every finite-dimensional vector space over K should have only one admissible topology it is necessary and sufficient that K be strictly minimal and complete.*

PROOF. We have already remarked that every one-dimensional vector space over K possesses only one admissible topology if and only if K is strictly minimal. Let us assume that K is not complete and consider its completion $\tilde{K}$, that is, the essentially unique complete topological ring containing K as a dense topological subring (see Bourbaki [1]). By the assumption, we may pick some $\zeta\in\tilde{K}-K$. Let $E\subset\tilde{K}$ be the set of all points $x\zeta+y$, where $x, y\in K$. Since $K\subset\tilde{K}$, we may say that $\tilde{K}$ is a topological vector space over K. But E is a vector subspace of $\tilde{K}$. Therefore, the topology $\mathfrak{T}_1$ induced on the vector space E by the topology of $\tilde{K}$ is admissible. Moreover $K\subset E$, $\overline{K}=E$ (the bar denoting closure in E according to $\mathfrak{T}_1$) because K is dense in $\tilde{K}$ and a fortiori in E. On the other hand, the mapping $(x, y)\to x\zeta+y$ is a vector space isomorphism between $K\times K$ and E. Since $K\times K$ is a topological vector space over K, we may transfer its topology to an admissible topology $\mathfrak{T}_2$ on E and it is clear that $\overline{K}=K$ (where the bar now denotes closure in E according to $\mathfrak{T}_2$). By this procedure, we have been able to set up two distinct admissible topologies $\mathfrak{T}_1$ and $\mathfrak{T}_2$ on the two-dimensional vector space E over K. Therefore, if every two-dimensional vector space over K must have a unique admissible topology, K is to be complete. Conversely, let us assume that K is strictly minimal and complete. This implies that every one-dimensional vector space over K possesses only one admissible topology and, in addition, is complete under this topology. Assume that every $n-1$ dimensional vector space over K possesses only one admissible topology and, in addition, is complete under this topology. Let us consider a vector space E of dimension n over K and an admissible topology $\mathfrak{T}$ on E. For any $n-1$ dimensional vector subspace of E, the induced topology is its unique admissible topology and the subspace is complete under it; a fortiori the subspace is closed in E. By Theorem 6 it follows that $\mathfrak{T}$ is uniquely determined; in addition to this, we may infer from $\mathfrak{T}=\mathfrak{T}_1\oplus \cdots \oplus\mathfrak{T}_n$ (in the notation of the proof of Theorem 6) that E is complete under $\mathfrak{T}$. The induction is thus completed.

THEOREM 8. *Let K be a given topological division ring. In order that every automorphism of any finite-dimensional topological vector space over K should be continuous it is necessary and sufficient that K be strictly minimal and complete.*

PROOF. Assume first that every automorphism is continuous. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be two admissible topologies on the vector space $K^n=K\times \cdots \times K$ (n times). Consider the vector space $K^n\times K^n$ endowed with the admissible topology $\mathfrak{T}_1\times\mathfrak{T}_2$. Since $K^n\times K^n$ is a square of a set, the symmetry $\{x, y\}\rightarrow\{y, x\}$ has a meaning (where $x, y\in K^n$). This transformation is an automorphism of $K^n\times K^n$. By the hypothesis it is continuous and this amounts to saying that $\mathfrak{T}_1=\mathfrak{T}_2$. This shows that every finite-dimensional vector space over K has a unique admissible topology. Applying the preceding theorem we see that K is strictly minimal and complete. Conversely, if K satisfies these conditions, we see by the same theorem that we have uniqueness of admissible topology in the finite-dimensional case: since the transform of an admissible topology under an algebraic isomorphism between two topological vector spaces is clearly admissible, we may infer that this algebraic isomorphism is also homeomorphic. More generally, if $\phi:E\rightarrow F$ is a linear transformation, with kernel $S=\phi^{-1}(0)$, between two finite-dimensional topological vector spaces, we may consider it as the product of the natural homomorphism $E\rightarrow E/S$ and the natural isomorphism $E/S\rightarrow\phi(E)$: therefore ϕ is continuous.

THEOREM 9. *Let K be a given topological division ring. Consider two topological vector spaces E and F over K, where F is finite-dimensional, and a linear transformation $\phi:E\rightarrow F$ from E onto F. If K is strictly minimal and complete, then ϕ is continuous if and only if its kernel $\phi^{-1}(0)$ is closed in E. If K is not strictly minimal, or is not complete, this result need not be always true.*

PROOF. The sufficiency of the assumptions is proved exactly as in the case of Theorem 3, but now with the help of the preceding theorem. If the assumptions concerning K are not fulfilled, we are able to find a vector space E over K of dimension one or two (see the first half of the proof of Theorem 7) endowed with two admissible topologies $\mathfrak{T}_1$ and $\mathfrak{T}_2$ such that $\mathfrak{T}_1<\mathfrak{T}_2$: then the identity transformation of E is not continuous from $\mathfrak{T}_1$ to $\mathfrak{T}_2$ but has a closed kernel.

THEOREM 10. *Let* $E=E_1\oplus E_2$ *be a topological vector space over a strictly minimal complete topological division ring* K, *where* E_1 *and* E_2 *are vector subspaces of* E, *the first being finite-dimensional. Let* $\mathfrak{T}$ *be the admissible topology given on* E *and* $\mathfrak{T}_1$ *and* $\mathfrak{T}_2$ *be the admissible topologies induced by* $\mathfrak{T}$ *on* E_1 *and* E_2, *respectively. Then we have* $\mathfrak{T}=\mathfrak{T}_1\oplus\mathfrak{T}_2$ *if and only if* E_2 *is closed in* E.

PROOF. The proof is exactly the same as that given for Theorem 4, but now with the help of the preceding theorem.

The preceding Theorem 10 can be reformulated in the customary fashion as a result about the continuity of a projection with closed kernel of a topological vector space onto a finite-dimensional vector subspace, or a projection with finite-dimensional kernel of a topological vector space onto a closed vector subspace; similarly for Theorem 4 and projections with one-dimensional ranges or kernels.

THEOREM 11. *Let* K *be a given topological division ring. Consider two topological vector spaces* E *and* F *over* K, *where* F *is finite-dimensional, and a linear transformation* $\phi:E\rightarrow F$. *If* K *is strictly minimal and complete, then* ϕ *is continuous if and only if its graph is closed in* $E\times F$. *If* K *is not strictly minimal, or is not complete, this result need not be always true.*

PROOF. The first part is proved as in the case of Theorem 5 by using Theorem 9. The final part is proved as in Theorem 9.

A topological vector space E is said to be *absolutely closed* if E is closed in every topological vector space containing it as a topological vector subspace. It is known that this is equivalent to saying that E is complete. From Theorem 7 if follows that *every finite-dimensional topological vector space over a strictly minimal complete topological division ring is complete.* In different words, *every finite-dimensional vector subspace of a topological vector space over a strictly minimal complete topological division ring is closed.* A stronger result is the following:

THEOREM 12. *Let* E *be a topological vector space over a strictly minimal complete topological division ring* K. *If* E_1 *and* E_2 *are vector subspaces* *of E, the first being finite-dimensional and the second closed in E, then E_1+E_2 is a closed vector subspace of E.*

PROOF. Without loss of generality, we may assume that E_1 and E_2 are disjoint vector subspaces and therefore $E_1+E_2=E_1\oplus E_2$. Consider the completion $\hat{E}$ of E, that is, the essentially unique complete topological vector space over K containing E as a dense topological vector subspace. Since E_1 is disjoint from E_2, we have $E_1-0\subset E-E_2$. But $E-E_2$ is open in E and this enables us to find a set $A\subset\hat{E}$ open in $\hat{E}$ such that $A\cap E=E-E_2$. Then $E_1-0\subset A$, $E_2\subset\hat{E}-A$ and therefore $\overline{E}_2\subset\hat{E}-A$ (here and in the sequel the bar denotes closure in $\hat{E}$). Thus $E_1-0\subset\hat{E}-\overline{E}_2$ and E_1 and $\overline{E}_2$ are disjoint vector subspaces of $\hat{E}$. Let us prove the relation $\overline{F}=E_1\oplus\overline{E}_2$, where $F=E_1\oplus E_2$. The inclusion $\supset$ is obvious. By Theorem 10, the topology on $E_1\oplus\overline{E}_2$ is the direct sum of the topologies on E_1 and on $\overline{E}_2$. But E_1, being finite-dimensional, is complete and $\overline{E}_2$, being closed in $\hat{E}$, is also complete. This shows that $E_1\oplus\overline{E}_2$ is complete and thus closed in $\hat{E}$. From this we conclude that the inclusion $\subset$ is true. Finally observing that $E_1\subset E$ and making use of the modularity law we have $E\cap\overline{F}=E\cap(E_1\oplus\overline{E}_2)=E_1\oplus(E\cap\overline{E}_2)=E_1\oplus E_2$, that is, $E_1\oplus E_2$ is closed in E.

Theorem 12 was proved by Mackey (see [6, Lemma 2.1]) in the case of normed spaces. It is known (see Stone [7, p. 21]) that this theorem need not be true even in separable Hilbert space if we assume merely that E_1 and E_2 are closed in E.

BIBLIOGRAPHY

1. N. Bourbaki, *Éléments de mathématiques*: *Topologie générale*, Actualités Scientifiques et Industrielles, nos. 858, 916, and 1045, Paris, 1940–1948.

2. J. Braconnier, *Sur les espaces vectoriels localement compacts*, C. R. Acad. Sci. Paris vol. 222 (1946) pp. 777–778.

3. J. Dieudonné, *Sur les corps topologiques connexes*, C. R. Acad. Sci. Paris vol. 221 (1945) pp. 396–398.

4. I. Kaplansky, *Topological rings*, Amer. J. Math. vol. 69 (1947) pp. 153–183.

5. ———, *Topological methods in valuation theory*, Duke Math. J. vol. 14 (1947) pp. 527–541.

6. G. W. Mackey, *Isomorphisms of normed linear spaces*, Ann. of Math. vol. 43 (1942) pp. 244–260.

7. M. H. Stone, *Linear transformations in Hilbert spaces and their applications to analysis*, Amer. Math. Soc. Colloquium Publications, vol. 15, New York, 1932.

8. A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, Actualités Scientifiques et Industrielles, no. 551, Paris, 1938.

UNIVERSITY OF BRAZIL

A NOTE ON THE UNIQUE FACTORIZATION OF ABSTRACT ALGEBRAS

FREDERICK B. THOMPSON

In their monograph[1] *Direct decompositions of finite algebraic systems*, Jónsson and Tarski have established the unique factorization theorem and several related results for a comprehensive class of finite abstract algebras. Every algebra $\mathfrak{A}$ of this class is constituted by a set A of arbitrary elements, by a binary operation $+$, and possibly by other operations $0_0, 0_1, \cdots$; the only conditions imposed on $\mathfrak{A}$ are that A be closed under all the operations involved and that it contain an element z which is a (both-hand) zero element for the operation $+$ and is idempotent under the remaining operations. At the end of their paper, Jónsson and Tarski raised the problem of whether their results can be extended to an even more comprehensive class of algebras, in fact to algebras $\mathfrak{A}$ differing from those mentioned above in that the element z is only to be idempotent under all operations involved (not necessarily being a zero element for $+$). The purpose of this note is to show that the solution of the problem just mentioned is a negative one.

We consider algebras $\mathfrak{A}_{p,t}$ defined in the following way: p and t are positive integers; $\mathfrak{A}_{p,t}$ is constituted by the set A_t of all non-negative integers less than t, and by the operation $\oplus_p$ defined by the following formula:

$$(hp + m)\oplus_p(kp + n) = mp + n$$

where h, k, m, and n are non-negative integers, $m<p$ and $n<p$. In particular, $u\oplus_1 v=0$ for all non-negative integers u and v. In what follows we shall be interested in only those algebras $\mathfrak{A}_{p,t}$ in which $p|t$ and $p^2\leqq t$; we denote by K the class of all such algebras. As is easily seen, in every algebra $\mathfrak{A}_{p,t}$ of K the set of all elements A_t is closed under the operation $\oplus_p$ and contains an element, in fact 0, which is idempotent under this operation. The following two theorems express the fundamental properties of the algebras of the class K with respect to cardinal multiplication.

Presented to the Society, November 27, 1948; received by the editors September 27, 1948.

[1] B. Jónsson and A. Tarski, *Direct decompositions of finite algebraic systems*, Notre Dame Mathematical Lectures, No. 5, Notre Dame, Indiana, 1947. The notations of this monograph are applied in the present note, except that the relation of isomorphism between two algebras is denoted by the symbol $\cong$.

THEOREM 1. *If* $\mathfrak{A}_{m,r}$ *and* $\mathfrak{A}_{n,s}$ *are any two algebras in* K, *then* $\mathfrak{A}_{mn,rs}$ *is also an algebra in* K *and we have* $\mathfrak{A}_{mn,rs} \cong \mathfrak{A}_{m,r} \times \mathfrak{A}_{n,s}$.

PROOF. The first part of the conclusion is obvious. To obtain the second part, remember that $m|r$, and hence $r = um$ for some positive integer u. Consider any two elements: h of the algebra $\mathfrak{A}_{m,r}$ and k of the algebra $\mathfrak{A}_{n,s}$. Clearly we can write h and k in the forms $h = h_0 m + h_1$, $k = k_0 n^2 + k_1 n + k_2$, where $h_1 < m$, $k_1 < n$, $k_2 < n$. We now correlate with the ordered couple h, k the number:

$$f(h, k) = (k_0 un + h_0 n + k_1)mn + (h_1 n + k_2).$$

It is now a routine matter to check that this correlation establishes the desired isomorphism.

THEOREM 2. *Let* $\mathfrak{A}_{p,t}$ *be an algebra in the class* K, *and let* $\mathfrak{B}$ *and* $\mathfrak{C}$ *be any two algebras each constituted by a set of elements and a binary operation. If*

$$\mathfrak{A}_{p,t} \cong \mathfrak{B} \times \mathfrak{C},$$

then there are algebras $\mathfrak{A}_{m,r}$ and $\mathfrak{A}_{n,s}$ *in* K *such that*

$$\mathfrak{B} \cong \mathfrak{A}_{m,r}, \qquad \mathfrak{C} \cong \mathfrak{A}_{n,s}.$$

PROOF. Let $\mathfrak{B}$ consist of the set B of r elements and the binary operation $\bigcirc$; let $\mathfrak{C}$ consist of the set C of s elements and the binary operation $\square$. Clearly $rs = t$. Let f map $\mathfrak{B} \times \mathfrak{C}$ isomorphically onto $\mathfrak{A}_{p,t}$. For $b \in B$, let B_b denote the set of non-negative integers $a < p$ for which $f(b, c) \equiv a \pmod{p}$ for some $c \in C$. Similarly C_c will denote the set of non-negative integers $a < p$ for which $f(b, c) \equiv a \pmod{p}$ for some $b \in B$.

(1) For $b_1, b_2 \in B$, the sets B_{b_1} and B_{b_2} are either disjoint or identical.

For suppose $a_1 \in B_{b_1} \cap B_{b_2}$ (that is, a_1 is in the common part of B_{b_1} and B_{b_2}), $a_2 \in B_{b_1}$, $a_2 \notin B_{b_2}$. Then there are integers a_1', a_1'', a_2', a_3, a_3' such that

$$f(b_1, c_1) = a_1'p + a_1, \qquad f(b_2, c_1') = a_1''p + a_1,$$
$$f(b_1, c_2) = a_2'p + a_2, \qquad f(b_2, c_2) = a_3'p + a_3,$$

for some c_1, c_1', $c_2 \in C$. We see that $a_3 \in B_{b_2}$, thus $a_3 \neq a_2$.

$$\begin{aligned} f(b_1 \bigcirc b_1, c_1 \square c_1) &= f(b_1, c_1) \oplus f(b_1, c_1) = (a_1'p + a_1) \oplus (a_1'p + a_1) \\ &= a_1 p + a_1 = (a_1''p + a_1) \oplus (a_1''p + a_1) \\ &= f(b_2, c_1') \oplus f(b_2, c_1') = f(b_2 \bigcirc b_2, c_1' \square c_1'). \end{aligned}$$

Therefore since f is biunique, $b_1 \bigcirc b_1 = b_2 \bigcirc b_2$. Thus, by a similar manipulation:

$$a_2 p + a_2 = f(b_1 \bigcirc b_1, c_2 \Box c_2) = f(b_2 \bigcirc b_2, c_2 \Box c_2) = a_3 p + a_3.$$

Consequently $a_2 = a_3$, which contradicts the above assertion that $a_2 \neq a_3$.

(2) For $c \in C$ and $b \in B$, the set $C_c \cap B_b$ contains exactly one element.

Indeed, let $f(b, c) = a'p + a$; thus $a \in C_c \cap B_b$. Suppose $a_1 \neq a$ were also in $C_c \cap B_b$. Thus for some b_1 and c_1, $f(b_1, c) = a_1'p + a_1$ and $f(b, c_1) = a_1''p + a_1$. Thus

$$f(b_1 \bigcirc b_1, c \Box c) = a_1 p + a_1 = f(b \bigcirc b, c_1 \Box c_1);$$

and since f is biunique, $b \bigcirc b = b_1 \bigcirc b_1$, $c \Box c = c_1 \Box c_1$. Therefore

$$f(b \bigcirc b, c \Box c) = ap + a = a_1 p + a_1$$

and $a = a_1$.

(3) For $b_1, b_2 \in B$, the sets B_{b_1} and B_{b_2} have the same number of elements.

For suppose $B_{b_1} = \{a_1, a_2, \cdots, a_n\}$, $B_{b_2} = \{a_1', \cdots, a_m'\}$, with $n > m$. Let $c_i \in C$ be such that $C_{c_i} \cap B_{b_1} = \{a_i\}$ for $i = 1, \cdots, n$. Clearly $C_{c_i} \cap B_{b_2} = C_{c_j} \cap B_{b_2}$ for some i, j, $i \neq j$. Therefore C_{c_i} and C_{c_j} are neither disjoint nor, by (2), are they identical, thus contradicting (1).

(4) For $b \in B$, let b/C be the set of all $f(b, c)$ for $c \in C$. Either B_b and b/C are disjoint or $B_b \subset b/C$.

Suppose $a_1 \in B_b \cap b/C$ and $a_2 \in B_b$. Then, for some $c_1, c_2 \in C$ and integer a_2', $f(b, c_1) = a_1$, $f(b, c_2) = a_2'p + a_2$. Let b_0, c_0 be such that $f(b_0, c_0) = 0$. Thus $f(b_0 \bigcirc b, c_0 \Box c_1) = 0 \oplus a_1 = a_1$; therefore $b_0 \bigcirc b = b$. Consequently

$$f(b, c_0 \Box c_2) = f(b_0 \bigcirc b, c_0 \Box c_2) = 0 \oplus (a_2'p + a_2) = a_2;$$

and $a_2 \in b/C$.

Let n be the number of elements in each B_b; let m be the number of elements in each C_c.

$$mn = p. \tag{5}$$

This follows immediately from (1) and (2).

$$m^2 \leqq r; \qquad n^2 \leqq s. \tag{6}$$

Suppose $a_1, a_2 \in B_b$. Thus $f(b, c_1) = a_1'p + a_1$, $f(b, c_2) = a_2'p + a_2$. $f(b \bigcirc b, c_1 \Box c_2) = a_1 p + a_2$. Therefore the set $b \bigcirc b/C$ of all $f(b \bigcirc b, c)$ for

$c \in C$ contains at least n^2 elements. But $b \bigcirc b/C$ contains s elements. Thus $n^2 \leqq s$. Similarly $m^2 \leqq r$.

$$(7) \qquad m \mid r; \qquad n \mid s.$$

For some $b_0 \in B$, consider B_{b_0}. There are nt/p elements of A_t whose residues mod p are in B_{b_0}. If k is the number of sets b/C such that $B_{b_0} = B_b$, then $nt/p = ks$, since b/C contains s elements. Thus $k = nt/ps = nrs/nms = r/m$; therefore $m \mid r$. Similarly $n \mid s$.

$$(8) \qquad \mathfrak{B} \cong \mathfrak{A}_{m,r}; \qquad \mathfrak{C} \cong \mathfrak{A}_{n,s}.$$

By (1), (3), (4), and (5), we see that there are just m elements $b \in B$ such that $B_b \subset b/C$. Let these elements be $b_0, b_1, \cdots, b_{m-1}$, where the numbering is arbitrary except that $0 \in B_{b_0}$. Let $b_i \bigcirc b_j = b_{im+j}$, $i, j < m$. It is easily checked that this definition is permissible. For each i, $0 \leqq i < m$, we assign (in arbitrary order) indices m^2+i, m^2+m+i, $m^2+2m+i, \cdots, r-m+i$ to all those elements $b \in B$ which have not been previously numbered and for which $B_b = B_{b_i}$. Let $\phi(b_i) = i$. Clearly ϕ is biunique on B to A_r, the set of elements of $\mathfrak{A}_{m,r}$. Consider $b_{hm+k}, b_{um+v} \in B$, $k, v < m$;

$$\begin{aligned} f(b_{hm+k} \bigcirc b_{um+v}, c \,\square\, c') &= (k'p + k) \oplus (v'p + v) = kp + v \\ &= f(b_k \bigcirc b_v, c'') = f(b_{km+v}, c'') \end{aligned}$$

for some $c'' \in C$. Thus $b_{hm+k} \bigcirc b_{um+v} = b_{km+v}$. Therefore

$$\phi(b_{hm+k} \bigcirc b_{um+v}) = \phi(b_{hm+k}) \oplus_m \phi(b_{um+v}).$$

Consequently $\mathfrak{B} \cong \mathfrak{A}_{m,r}$. Similarly $\mathfrak{C} \cong \mathfrak{A}_{n,s}$.

The theorem is now a consequence of (5), (6), (7), and (8).

If, instead of algebras $\mathfrak{A}_{p,t}$ of the class K, we speak of their isomorphism types $\alpha_{p,t}$, and denote the class of all such isomorphism types by K', then an essential part of the contents of Theorems 1 and 2 can be expressed in the following way.

COROLLARY 3. *The cardinal product $\beta \times \gamma$ of two isomorphism types is in K' if and only if both β and γ are in K'.*

Furthermore, in each particular case we can easily determine whether an isomorphism type $\alpha_{p,t}$ is decomposable and can describe all of its decompositions into indecomposable factors. We apply here Theorems 1 and 2 and make use of the obvious facts that $\alpha_{1,1}$ is the unit type (the isomorphism type of a one element algebra) and that two isomorphism types $\alpha_{m,r}$ and $\alpha_{n,s}$ are identical if and only if $m = n$ and $r = s$. In this way we see, for instance, that the isomorphism type $\alpha_{2,4}$ is indecomposable; $\alpha_{2,8}$ has, apart from order, the unique decomposition into indecomposable factors:

$$\alpha_{2,8} = \alpha_{1,2} \times \alpha_{2,4};$$

and finally $\alpha_{2,12}$ has two different decompositions into indecomposable factors:

$$\alpha_{2,12} = \alpha_{2,6} \times \alpha_{1,2} = \alpha_{2,4} \times \alpha_{1,3}.$$

Thus the last example shows that the refinement theorems 4.7 and 4.8, as well as the unique factorization theorem 4.9, of Jónsson and Tarski cannot be extended to algebras which have an idempotent element but not a zero element. The problem whether the cancellation theorem 4.10 can be extended to such algebras still remains open.

UNIVERSITY OF CALIFORNIA

NOTE ON A PAPER BY C. E. RICKART

R. P. DILWORTH AND MORGAN WARD

In a recent issue of this Bulletin,[1] C. E. Rickart proves the following two theorems:

THEOREM 1. *Any one-to-one multiplicative mapping of a Boolean ring onto an arbitrary ring is necessarily additive.*

THEOREM 3. *Any one-to-one meet preserving mapping of a distributive lattice onto a distributive lattice is also join preserving.*

We should like to point out that both of these theorems are simple consequences of the following well known principle of lattice theory: *Any one-to-one mapping of one lattice onto another lattice which preserves order both ways is a lattice isomorphism.*

Now a one-to-one meet preserving mapping of one lattice onto another preserves order both ways; for if x and x' denote corresponding elements,

$$a \geqq b \rightleftarrows a \cap b = b \rightleftarrows a' \cap b' = b' \rightleftarrows a' \geqq b'.$$

CALIFORNIA INSTITUTE OF TECHNOLOGY

Received by the editors September 20, 1948.

[1] Vol. 54 (1948) pp. 758–764.

THE EUCLIDEAN ALGORITHM

TH. MOTZKIN

In this note a constructive criterion for the existence of a Euclidean algorithm within a given integral domain is derived, and from among the different possible Euclidean algorithms in an integral domain one is singled out. The same is done for "transfinite" Euclidean algorithms. The criterion obtained is applied to some special rings, in particular rings of quadratic integers. By an example it is shown that there exist principal ideal rings with no Euclidean algorithm. Finally, different sets of axioms for the Euclidean algorithm and related notions are compared, and the possible implications for the classification of principal ideal rings, and other integral domains, indicated.

The question of the relationship between different Euclidean algorithms in the same integral domain was raised (orally) by O. Zariski.

1. The derived sets. Let Q be an integral domain. A subset P of $Q-0$ (Q except zero) shall be called a *product ideal* if $P(Q-0)\subseteq P$.

For any subset S of Q, the set B of all b in Q for which there exists an a in Q such that $a+bQ\subseteq S$ is called the *total derived set* of S, and the intersection $B\cap S$ is called the *derived set* S'. With S also S' is a product ideal. If $S_1\subseteq S$, then $S_1'\subseteq S'$.

A *Euclidean algorithm* (or process) is given by a norm $|a|$ defined in $Q-0$, with positive integral (or zero) values and such that $|a| \geqq |b|$ for b dividing a and that for any b in $Q-0$ and any a not divisible by b there exist q and r in Q satisfying $a=qb+r$, $|r|<|b|$.

Let P_i, $i=0, 1, 2, \cdots$, be the set of all b in Q with $|b|\geqq i$. Obviously P_i is a product ideal. For any b in P_i', let a be an element with $a+bQ\subseteq P_i$, whence $a-bq\neq 0$ and (for any $r=a-bq$ with $|r|<|b|$) $|r|\geqq i$, $|b|\geqq i+1$; we see that $P_i'\subseteq P_{i+1}$. Conversely, given a sequence $Q-0=P_0\supseteq P_1\supseteq \cdots$ of product ideals with empty intersection $\bigcap P_i$ such that $P_i'\subseteq P_{i+1}$, the norm defined by $|b|=i$ for every b in P_i-P_{i+1} will fulfil the conditions for a Euclidean algorithm. Hence *there is a one-one correspondence between sequences of this kind and Euclidean algorithms.*

If for another Euclidean algorithm, with the sequence $\bar{P}_i$, always $P_i\subseteq \bar{P}_i$, we say that the first algorithm is the *faster* one (under cer-

---

Presented to the Society, October 30, 1948; received by the editors September 28, 1948.

tain additional conditions, indeed less algorithm steps are needed).

If there exists at all a Euclidean algorithm in Q, then *there exists a fastest Euclidean algorithm* defined by the sequence $P_0, P_0', P_0'', \cdots$. Hence *the emptiness of the intersection* $\bigcap P_0^{(i)}$ *is a criterion* (n.a.s.c.) *for the existence of a Euclidean algorithm in* Q.

Any sequence (P_i) as above may be changed to a new sequence $P_1, \cdots, P_{i-1}, \overline{P}_i, \overline{P}_i', \overline{P}_i'', \cdots$, where $\overline{P}_i$ is a product ideal with $P_{i-1} \supseteq \overline{P}_i \supseteq (P_{i-1})'$, so (besides the trivial repetition of identical product ideals) there always exist, if any, different Euclidean algorithms except if there are no product ideals between any $P_0^{(i)}$ and $P_0^{(i+1)}$.

2. **Generalization.** These considerations may be generalized as follows. Let a *tea* (*transfinite Euclidean algorithm*) be an algorithm as before but where (1) we allow $|b|$ to take any ordinal numbers as values; (2) we do not require $|a| \geqq |b|$ for b dividing a. Then it is seen in the usual way that the existence of a *tea* implies that Q is a principal ideal ring.

Further, such a *tea* determines, and is determined by, a transfinite sequence S_λ, $0 \leqq \lambda \leqq \mu$, of subsets of $Q-0$ with (1) $S_\lambda' \subseteq S_{\lambda+1}$, (2) $S_\lambda \subseteq S_{\lambda-1}$, but $S_\lambda = \bigcap S_i$, $i < \lambda$, if $\lambda - 1$ does not exist, (3) empty S_μ.

Defining "faster" as before there is again a *fastest tea* given by $P_0^{(\lambda)}$, where $S^{(\lambda)}$ is defined as $(S^{(\lambda-1)})'$ or $\bigcap S^{(i)}$, $i < \lambda$.

Hence *the criterion for the existence of a tea is the emptiness of some* $P_0^{(\mu)}$.

For the fastest *tea* the sequence consists of product ideals, so that the monotonity condition $|a| \geqq |b|$ for b dividing a is automatically fulfilled.

If no $P_0^{(\lambda)}$ vanishes then there is no *tea*. If Q is not a principal ideal ring this is certainly so, but even for a principal ideal ring the constant $P_0^{(\mu)}$ (which is the largest subset S of Q with $S = S'$, and therefore never a principal ideal) may not be empty, as shown by some of the following examples.

3. **Examples.** The derived set S' of a given set S may also be defined as the set obtained from S by exemption of all b such that for every a, b divides some $a+c$ with c not in S. In particular P_0' is the set of all non-units except 0. Now call a non-unit $b \neq 0$ a *side divisor* of a if b divides some $a+e$, where e is a unit or 0. Then P_0'' is obtained from P_0' by exemption of the *universal side divisors*, that is, of those elements b which are side divisors of every a in Q, or equivalently for which there is a unit, or 0, in every residue class mod b. Such an element is obviously prime; the principal ideal (b) must even be

maximal. If no universal side divisors exist, then $P_0''=P_0'$, and there is, except for the trivial case of a field, no *tea* in Q.

For the ring of rational integers, $Q-P_0'$ has three elements 0, ± 1. Hence the universal side divisors are ± 2 and ± 3, and $Q-P_0''$ contains all c with $|c|<2^i$. By induction, b is in $P_0^{(i)}$ if and only if $|b|\geqq 2^i$. The fastest Euclidean algorithm is given by $a=bq+r$ with minimal $|r|$ (in a fixed algorithm q and r need not be unique).

$P_0''=P_0'$ holds for the algebraic integers of every quadratic number field with negative discriminant d except -1, -2, -3, -7, -11. This can be seen as follows. It is well known that the above integers are the numbers $f+gd^{1/2}$, where f and g denote arbitrary rational integers, and in addition, if 4 divides $d-1$, the numbers $f+1/2+(g+1/2)d^{1/2}$. It follows easily that, except for the five stated values of d, 2 and 3 are irreducible and ± 1 the only units, so that the only side divisors of 2 are ± 2, ± 3; but these are not side divisors of $(1+d^{1/2})/2$, and if this is not an integer, of $d^{1/2}$. Hence there are no universal side divisors. For the excepted values of d there are respectively 12, 4, 24, 4, 4 universal side divisors, among which all the 22 non-real quadratic integers b with $1<b\bar{b}\leqq 3$ occur. (Dedekind [2, Supplement XI, §159 (4th ed., 1894, p. 451)][1] stated that the usual norm gives no Euclidean algorithm for the principal ideal ring belonging to $d=-19$. Hasse [4, p. 11] asked whether a Euclidean algorithm might be obtained by another norm, retaining the multiplicativity condition $|ab|=|a|\,|b|$. We see that this is not the case, even without this condition and allowing ordinal numbers as norm values. Hence we have an example of a principal ideal ring with no Euclidean algorithm. The given result for arbitrary negative discriminant generalizes, and contains a new proof of, the similar result of Dickson [3, pp. 150–151] for the usual norm. For this norm and positive discriminant the question is not entirely solved, see Chatland [1], with further references.)

We have also $P_0''=P_0'$ for the ring of all polynomials, or power series, of one variable over an integral domain that is not a field. Likewise, for a valuation ring with no smallest positive value, $P_0''=P_0'$. If a smallest positive value v exists, then $P_0^{(i)}$ is the set of all elements whose value is at least i, and $P_0^{(\omega)}=P_0^{(\omega+1)}$. In particular for the power series of one variable over a field, $P_0^{(\omega)}=0$. Similarly for the polynomials of one variable over a field, $P_0^{(i)}$ is the set of all polynomials of degree not less than i. This is a special case of the next example.

---

[1] Numbers in brackets refer to the references cited at the end of the paper.

4. Quotient rings. If, within the affine space over an algebraically closed field K of arbitrary characteristic, C is a rational curve with no singular point at finite distance, then the ring of all rational functions on C with no poles at finite distance is a principal ideal ring (for example, since we shall see that it has a Euclidean algorithm). The ring consists of all those rational functions with coefficients in K, of a parameter t of the curve, whose poles belong to a given finite set $(t_1, \cdots, t_k)$. Subjecting t, if necessary, to a broken linear transformation, we may suppose $t_1=\infty$. In this case $P^{(i)}$ is the set of all the functions in the ring that have at least i zeros (with due counting of multiplicities) outside $(t_1, \cdots, t_k)$. Indeed, define $|a|$ accordingly as the number of zeros of a outside $(t_1, \cdots, t_k)$; to prove that $|a-bq|<|b|$ can be solved we may (multiplying both by a polynomial c with $|c|=0$) suppose that a and b are polynomials and (shifting other factors of b to q) that b has no zero in $(t_1, \cdots, t_k)$, in which case the usual polynomial q will do. And obviously, since for $|a_1|<|b|$, $|a_2|<|b|$, $a_1\neq a_2$, never $a_1-a_2=bq$, no faster Euclidean algorithm exists.

Similarly a *tea* may be defined in any quotient ring of an integral domain with a given *tea* by letting $|a|$ be the smallest value of $|am/n|$ in the original ring, where m and n are elements of a fixed multiplicatively closed set of denominators that yields the quotient ring considered.

5. Related notions. If we modify the definition of the derived set by demanding the existence of an a (not divisible by b) such that $aM+bN\subseteq(S, 0)$, where M and N are given subsets of Q (for instance, the set $0, \pm 1, \pm 2, \cdots$), we obtain sequences of subsets quite similar to those obtained before, which contract the faster the larger the sets M and N are. Here too P_0 is the set of non-units except 0. For $M=N=Q$ and if Q is a principal ideal ring, $P_0^{(i)}$ is the set of all elements that are products of at least i primes, so $P_0^{(\omega)}=0$; the corresponding (multiplicative) norm being γ^i with fixed $\gamma>1$.

Comparing the strength of different notions similar to the usual Euclidean algorithm, we may consider an algorithm (j, k, l), where $j=1$ means that the norm shall be a positive integer, $j=2$ that it be within a set of real positive numbers with no limit point except ∞, $j=3$ that it be an ordinal number; $k=1$ that $|ab|=|a|\,|b|$, $k=2$ that $|a|\geqq|b|$ for b dividing a, $k=3$ no such condition; $l=1$ that, for any $b\neq 0$ and a not divisible by b, there exists some q with $|a-bq|<|b|$, $l=2$ that, in the only relevant case $|a|\geqq|b|$, there only need exist m and q with $|am-bq|<|b|$ and m prime to b (that is, b shall divide

mn only if it divides n), $l=3$ the same, demanding only $|am-bq| < |a|$, $l=4$ again demanding that $|am-bq| < |b|$, but with no restriction for m. So l determines the stepping down condition characteristic for the "descente infinie" application of Euclidean algorithms. We exclude the case $j=3$, $k=1$.

Then the existence of any algorithm (j, k, l) clearly implies that the integral domain Q is a principal ideal ring. It is easily seen that in every principal ideal ring the before mentioned norm γ' fulfils (1, 1, 2) (see, for example, [4, pp. 7–8]), so that every combination with $l>1$ gives a n.a.s.c. for principal ideal rings. On the other hand, even the weakest condition with $l=1$, which is (3, 3, 1), is not always fulfilled in principal ideal rings, as we have shown; and (3, 3, 1) is equivalent to (3, 2, 1), (2, 3, 1) is equivalent to (2, 2, 1), (1, 3, 1), and (1, 2, 1), and finally (2, 1, 1) to (1, 1, 1), while it remains open whether these three sets of conditions are really of different strength.

A further classification of integral domains may be made according to the number μ of §2, or by (M, N)-stepping down conditions. Thus the fact that (1, 1, 3) is fulfilled for $M=(1)$, $N=(\pm 1)$ in the ring of rational integers is the essence of the simple Kronecker-Zermelo proof [4, p. 3] of unique decomposition into primes. For a similar, still weaker condition than $l=4$ characterizing integral domains with unique decomposition into primes, see Krull [5, pp. 107–108]; also with respect to that condition derived sets may be defined and integral domains grouped according to whether the constant $P^{(\mu)}$ is or is not empty, and according to μ.

REFERENCES

1. H. Chatland, *On the Euclidean Algorithm in quadratic number fields*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 948–953.

2. L. Dickson, *Algebren und ihre Zahlentheorie*, Zürich and Leipzig, 1927.

3. P. G. Lejeune Dirichlet, *Vorlesungen über Zahlentheorie*, ed. by R. Dedekind.

4. H. Hasse, *Über eindeutige Zerlegung in Primelemente oder in Primhauptideale in Integritätsbereichen*, J. Reine Angew. Math. vol. 159 (1928) pp. 3–12.

5. W. Krull, *Idealtheorie*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vols. 4, no. 3, 1935.

HEBREW UNIVERSITY AND
HARVARD UNIVERSITY

ON MINKOWSKI BODIES OF CONSTANT WIDTH

PAUL J. KELLY

A metric set is entire if the addition of any point to the set increases the diameter. A convex body has constant width if all pairs of parallel supporting planes are the same distance apart. These concepts are known to be equivalent in euclidean space.[1] The present paper shows that they are also equivalent in a minkowski space.

A proof for this equivalence for the minkowski plane was given by Meissner.[2] He showed also that two curves of the same constant width have the same circumference, and that a three-dimensional body of constant width has plane sections of constant width, in terms of the corresponding section of the minkowski sphere as plane indicatrix. However, Meissner's three-space proof for the equivalence of entireness to constant width is incomplete.[3] He assumed, moreover, that the indicatrix had no singular points. The equivalence here is shown for the n-dimensional case with the assumption merely that the indicatrix is convex.

In a euclidean E^n space let C be a closed, convex hypersurface with O as center. In terms of C as indicatrix let a minkowski distance be defined in the usual way to make the space an M^n.[4]

Let r and s be half-rays emanating from O at an angle θ and let λ_r, λ_s be the euclidean lengths of the minkowski radii of C in these directions. Then $\operatorname{sm}(r, s)$, a positional sine with respect to C, is defined to mean $\lambda_r\lambda_s \sin\theta$.[5]

LEMMA. *If r, s, t are half-rays through O, which lie in a plane, with s between r and t (in terms of an angle not greater than π) then* $\operatorname{sm}(r, s) + \operatorname{sm}(s, t) \geqq \operatorname{sm}(r, t)$.

If X_1, X_2, X_3 are the end points of radii λ_r, λ_s, λ_t, then from the convexity of C it follows that the euclidean area of ΔOX_1X_2 plus the area of ΔOx_2x_3 is not less than the area of ΔOx_1x_3. Since the area of

---

Received by the editors August 16, 1948.

[1] Bonnesen and Fenschel, *Theorie der konvexen Körper*, Ergebnisse der Mathematik, 1934, p. 128. Jessen, *Über konvexe Punktmengen konstanter Breite*, Math. Zeit. vol. 29 (1928) pp. 378–380.

[2] E. Meissner, *Über Punktmengen konstanter Breite*, Vierteljahrschrift der Naturforschenden Gesellschaft, 1911.

[3] This was already noticed in the references under footnote 1.

[4] Bonnesen and Fenschel, cf. footnote 1, p. 23.

[5] This was taken from a more general minkowski sine function defined by H. Busemann who has in preparation a paper on the subject.

ΔOX_1X_2 is $(1/2)\,\mathrm{sm}(r, s)$, and similarly for the other terms, the lemma follows.

THEOREM. *If K is a body in M^n space which is entire, in terms of the minkowski metric, it is of constant width, in terms of this metric, and conversely.*

PROOF. (1) Let K be entire. Assume it is not of constant width.

(2) It is easily shown that K is convex and that its diameter is its maximum minkowski width.[6] Take O interior to K and let H and E be the euclidean supporting functions of K and C (C plus its interior). If ξ is a unit euclidean vector, the minkowski distance between the supporting planes of K in the direction ξ is given by $W(\xi) = [H(\xi) + H(-\xi)]/E(\xi)$.[7] From (1), $W(\xi)$ will assume a maximum and a minimum value.

(3) Let K^* be the euclidean polar of K with respect to the unit euclidean sphere ("polar," throughout, will refer to the unit euclidean sphere). To a point U of K^* there corresponds a unit euclidean vector ξ in the direction from O to U, and a width, as in (2) $W(\xi)$. Points of K^* will be called maximal or minimal whenever the corresponding width is maximal or minimal. Since $W(\xi) = W(-\xi)$, to each U of K^* there is a counterpoint U' on K^* in the direction $-\xi$ and to which the same width corresponds.

(4) If K^* contains a non-maximal, extreme point, then the assumption in (1) is false. For suppose U_0 such a point. From continuity, U_0 has a neighborhood of non-maximal points. Let a plane π, at a distance δ from U_0, cut K^* into two parts and let K_1^* be that part separated from U_0 by π. Since K_1^* is properly contained in K^*, then K_1, the polar of K_1^*, properly contains K. Since the width of K_1 differs from that of K only in the neighborhood of a non-maximal direction, it follows from continuity that if this neighborhood, that is, δ, is chosen sufficiently small, then the maximal widths of K and K_1 will be the same. Hence the diameters will be the same. Since K_1 properly contains K, this contradicts the entireness of K and the assumption in (1) must be false.

(5) If U_1, U_2, U_3 are points of K^* which are collinear, in that order, and which have collinear counterpoints, then $W(\xi_2) \geq \min [W(\xi_1), W(\xi_3)]$. We prove this as follows.

(5.1) Let C^* be the polar of C. This is again a central, convex body and can be used as indicatrix for a second minkowski metrization of E^n. Let $|u_i|^*$ represent the minkowski distance, with respect to C^*, from O to U_i, $|u_i|$ the corresponding euclidean distance, λ_i the euclidean length of the radius of C^* in the direction ξ_i, and θ_{ij} the angle from ξ_i to ξ_j. By definition, then, $|u_i| = \lambda_i |u_i|^*$. In this notation, the fact that the euclidean area of $\Delta O u_1 u_3$ equals that of $\Delta O U_1 U_2$ plus that of $\Delta O U_2 U_3$ can be expressed as $\lambda_1\lambda_3 |u_1|^* |u_3|^* \sin \theta_{13} = \lambda_1\lambda_2 |u_1|^* |u_2|^* \sin \theta_{12} + \lambda_2\lambda_3 |u_2|^* |u_3|^* \sin \theta_{23}$. Using the positional sine of the lemma (now with respect to C^*), the above relation can be put in the form

---

[6] Cf. H. Busemann, *Intrinsic area*, Ann. of Math. vol. 48 (1947) pp. 234–267.

[7] Ibid.

$$\frac{1}{|u_2|^*} = \frac{1}{|u_3|^*} \frac{\text{sm}\ (\xi_1, \xi_2)}{\text{sm}\ (\xi_1, \xi_3)} + \frac{1}{|u_1|^*} \frac{\text{sm}\ (\xi_2, \xi_3)}{\text{sm}\ (\xi_1, \xi_3)} .$$

(5.2) Directly, from the definitions of minkowski distance and polar body, we have $|u_i|^* = [H(\xi_i)]^{-1}/[E(\xi_i)]^{-1} = E(\xi_i)/H(\xi_i)$, $i = 1, 2, 3$. Hence the last relation in (5.1) can be written

$$\frac{H(\xi_2)}{E(\xi_2)} = \frac{H(\xi_3)\ \text{sm}\ (\xi_1, \xi_2)}{E(\xi_3)\ \text{sm}\ (\xi_1, \xi_3)} + \frac{H(\xi_1)\ \text{sm}\ (\xi_2, \xi_3)}{E(\xi_1)\ \text{sm}\ (\xi_1, \xi_3)} .$$

(5.3) From the linearity of the counterpoints, U_i', the same area argument can be applied to the three triangles $OU_i'U_j'$, $i \neq j$. The fact that C^* is central, and the equality of vertical angles, allows the conclusion to be put in the form

$$\frac{H(-\ \xi_2)}{E(\xi_2)} = \frac{H(-\ \xi_3)\ \text{sm}\ (\xi_1, \xi_2)}{E(\xi_3)\ \text{sm}\ (\xi_1, \xi_3)} + \frac{H(-\ \xi_1)\ \text{sm}\ (\xi_2, \xi_3)}{E(\xi_1)\ \text{sm}\ (\xi_1, \xi_3)}$$

and this, added to (5.2), gives

$$W(\xi_2) = \frac{W(\xi_3)\ \text{sm}\ (\xi_1, \xi_2)}{\text{sm}\ (\xi_1, \xi_3)} + \frac{W(\xi_1)\ \text{sm}\ (\xi_2, \xi_3)}{\text{sm}\ (\xi_1, \xi_3)}$$

or

$$W(\xi_2) \geqq \min \{ [W(\xi_1), W(\xi_3)] \} \cdot \left[\frac{\text{sm}\ (\xi_1, \xi_2) + \text{sm}\ (\xi_2, \xi_3)}{\text{sm}\ (\xi_1, \xi_3)} \right].$$

From the lemma, the last factor above is equal to or greater than one, which establishes (5).

(6) If K^* contains no non-maximal, extreme points then the assumption in (1) is false. For suppose every extreme point is also a maximal point. Being convex, K^* is the union of all convex simplices whose vertices are extreme points.[8] Then every nonextreme boundary

---

[8] Bonnesen and Fenschel, cf. footnote 1, p. 9.

point lies on a simplex, of dimension equal to or less than $n-1$, which has maximal vertices. From (5), the edge points of this simplex must be maximal and this, again from (5), implies the face points must be maximal. Hence all the boundary points must be maximal, which is to say that only the maximal width is assumed, or that K is of constant width.

(7) Since (4) or (6) must hold, the assumption in (1) is false, and K is of constant width if it is entire.

(8) The counter equivalence, that if K is of constant width it is entire, is easy to establish and can be argued exactly as in the euclidean case.

UNIVERSITY OF SOUTHERN CALIFORNIA

HYPERGEODESIC CURVATURE AND TORSION

DAVID B. DEKKER

1. Introduction. A *hypergeodesic curve*[1] on a surface satisfies an ordinary second-order differential equation similar to that for the geodesic curves except that the coefficients, instead of being Christoffel symbols, are taken to be arbitrary functions of the surface coordinates. At a point of the surface the envelope of all the osculating planes of hypergeodesic curves of a family determined by such an equation which pass through the point will be a cone usually of order four and class three with its vertex at the point. If the cone at each point degenerates into a single line of a congruence of lines, then the family of hypergeodesics is the family of *union curves*[2] relative to this congruence. Recently, a union curvature[3] and a union torsion[4] have been defined relative to a family of union curves as generalizations of geodesic curvature and geodesic torsion.

The following investigation deals with the differential equation for an arbitrary family of hypergeodesic curves on a surface and the definition of a hypergeodesic curvature and a hypergeodesic torsion relative to a family of hypergeodesic curves in such a manner that the definitions will reduce to those for union curvature and union torsion when the family is taken to be a family of union curves. A geometric interpretation of hypergeodesic curvature is given which generalizes the geometric interpretation for union curvature and geodesic curvature. Finally, a geometric condition that a hypergeodesic be a plane curve is obtained as a generalization of a theorem known for union curves.

The notation of Eisenhart[5] is used throughout except that $\Gamma^{\alpha}_{\beta\gamma}$ is used for Christoffel symbols of the second kind with respect to the coefficients of the first fundamental form for the surface. The summation convention of tensor analysis with regard to repeated indices

---

Presented to the Society, April 30, 1949; received by the editors September 16, 1948.

[1] E. P. Lane, *Projective differential geometry*, University of Chicago Press, 1942, p. 192.

[2] P. Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, Amer. J. Math. vol. 40 (1918) p. 213.

[3] C. E. Springer, *Union curves and union curvature*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 686–691.

[4] C. E. Springer, *Union torsion of a curve on a surface*, Amer. Math. Monthly vol. 54 (1947) pp. 259–262.

[5] L. P. Eisenhart, *Differential geometry*, Princeton University Press, 1940.

will be observed with Greek letter indices taking on the values 1, 2 and the Latin letter indices taking on the values 1, 2, 3.

2. The differential equation of a family of hypergeodesics. Consider a surface S in ordinary space of three dimensions represented by the three equations $x^i = x^i(u^1, u^2)$ $(i=1, 2, 3)$ referred to a rectangular cartesian coordinate system. The functions x^i and their partial derivatives of the second order with respect to the parameters u^1 and u^2 are assumed continuous at any point of the surface. The solutions $u^2 = u^2(u^1)$ of an ordinary second order differential equation of the form

$$\frac{d}{du^1}\left(\frac{du^2}{du^1}\right) = A + B\frac{du^2}{du^1} + C\left(\frac{du^2}{du^1}\right)^2 + D\left(\frac{du^2}{du^1}\right)^3, \tag{2.1}$$

in which A, B, C, and D are analytic functions of u^1 and u^2, are the curves of a two-parameter family of hypergeodesics. If $u^\alpha = u^\alpha(s)$ $(\alpha = 1, 2)$, where the parameter s is arc length, is a hypergeodesic of the family, then equation (2.1) takes the form

$$e_{\alpha\beta}u'^\alpha u''^\beta = A_{\alpha\beta\gamma}u'^\alpha u'^\beta u'^\gamma, \tag{2.2}$$

in which $e_{12}=1$, $e_{21}=-1$, $e_{11}=e_{22}=0$, and

$$\begin{aligned} A_{111} &= A, & A_{112}+A_{121}+A_{211} &= B,\\ A_{122}+A_{212}+A_{221} &= C, & A_{222} &= D, \end{aligned} \tag{2.3}$$

and the primes indicate differentiation with respect to s.

Under a transformation $\bar{u}^\alpha = \bar{u}^\alpha(u^1, u^2)$ $(\alpha = 1, 2)$ of surface coordinates for which the Jacobian does not vanish, the differential equation

$$e_{\alpha\beta}\bar{u}'^\alpha\bar{u}''^\beta = \bar{A}_{\tau\sigma\rho}\bar{u}'^\tau\bar{u}'^\sigma\bar{u}'^\rho$$

is obtained in the new coordinates, where

$$\bar{A}_{\tau\sigma\rho} = \frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(u^1, u^2)}\left[A_{\alpha\beta\gamma}\frac{\partial u^\alpha}{\partial\bar{u}^\tau}\frac{\partial u^\beta}{\partial\bar{u}^\sigma}\frac{\partial u^\gamma}{\partial\bar{u}^\rho} - e_{\alpha\beta}\frac{\partial u^\alpha}{\partial\bar{u}^\tau}\frac{\partial^2 u^\beta}{\partial\bar{u}^\sigma\partial\bar{u}^\rho}\right]. \tag{2.4}$$

It is clear, however, that $A_{\alpha\beta\gamma}$ is not in general a tensor.

In order to express the equation (2.2) in tensor form, consider the geodesic curvature vector[6] whose contravariant components ρ^τ are

$$\rho^\tau = u''^\tau + \Gamma^\tau_{\beta\gamma}u'^\beta u'^\gamma. \tag{2.5}$$

Substituting the values of u''^τ from (2.5), equation (2.2) becomes

---

[6] Eisenhart, loc. cit. p. 187.

$$e_{\alpha\tau} u'^{\alpha} \rho^{\tau} = [A_{\alpha\beta\gamma} + e_{\alpha\tau}\Gamma^{\tau}_{\beta\gamma}] u'^{\alpha} u'^{\beta} u'^{\gamma}. \tag{2.6}$$

The left-hand member of this equation becomes a scalar if multiplied by $g^{1/2}$, where g is the determinant of the coefficients of the first fundamental form, namely,

$$g = e^{\alpha\beta} g_{1\alpha} g_{2\beta}$$

with

$$g_{\alpha\beta} = \overset{i}{x}_{,\alpha} \overset{i}{x}_{,\beta}$$

and

$$e^{12} = 1, \qquad e^{21} = -1, \qquad e^{11} = e^{22} = 0.$$

Hence, the differential equation may be written in the tensor form

$$\epsilon_{\alpha\tau} u'^{\alpha} \rho^{\tau} = \Omega_{\alpha\beta\gamma} u'^{\alpha} u'^{\beta} u'^{\gamma}, \tag{2.7}$$

where $\epsilon_{\alpha\tau} = g^{1/2} e_{\alpha\tau}$ and the functions $\Omega_{\alpha\beta\gamma}$ of u^1 and u^2 are defined by

$$\Omega_{\alpha\beta\gamma} \equiv \epsilon_{\alpha\tau}\Gamma^{\tau}_{\beta\gamma} + g^{1/2} A_{\alpha\beta\gamma}. \tag{2.8}$$

Since the left member of (2.7) is a scalar, so also is the right member. Consequently, the functions $\Omega_{\alpha\beta\gamma}$ are the triply covariant components of a tensor of the third order; that is, under a transformation of surface coordinates,

$$\bar{\Omega}_{\alpha\beta\gamma} = \Omega_{\tau\sigma\rho} \frac{\partial u^{\tau}}{\partial \bar{u}^{\alpha}} \frac{\partial u^{\sigma}}{\partial \bar{u}^{\beta}} \frac{\partial u^{\rho}}{\partial \bar{u}^{\gamma}},$$

as may be verified by direct computation. Now equation (2.7) may be written

$$\epsilon_{\alpha\tau} u'^{\alpha} \rho^{\tau} = \Omega, \tag{2.9}$$

in which the scalar Ω is defined by the right-hand member of (2.7).

To specialize the components $\Omega_{\alpha\beta\gamma}$ to be symmetric in each pair of indices actually constitutes no restriction on the family of hypergeodesics to be considered. Employing this symmetry in (2.8) gives

$$\begin{aligned} A_{211} - A_{112} &= \Gamma^{1}_{11} + \Gamma^{2}_{21}, \\ A_{112} - A_{121} &= 0, \\ A_{212} - A_{122} &= \Gamma^{2}_{22} + \Gamma^{1}_{12}, \\ A_{212} - A_{221} &= 0, \end{aligned} \tag{2.10}$$

which with equations (2.3) are found to have the unique solutions

$$
\begin{aligned}
A_{111} &= A,\\
A_{112} = A_{121} &= (B - \overset{1}{\Gamma}_{11} - \overset{2}{\Gamma}_{21})/3,\\
A_{211} &= (B + 2\overset{1}{\Gamma}_{11} + 2\overset{2}{\Gamma}_{21})/3,\\
A_{212} = A_{221} &= (C + \overset{2}{\Gamma}_{22} + \overset{1}{\Gamma}_{12})/3,\\
A_{122} &= (C - 2\overset{2}{\Gamma}_{22} - 2\overset{1}{\Gamma}_{12})/3,\\
A_{222} &= D.
\end{aligned}
\tag{2.11}
$$

Hence, *a necessary and sufficient condition that* $\Omega_{\alpha\beta\gamma}$ *be a symmetric tensor in any pair of indices is that the functions* $A_{\alpha\beta\gamma}$ *be given by the relations* (2.11).

3. **Hypergeodesic curvature.** The differential equation (2.9) is transformed into two differential equations by simply multiplying it through by the expressions $\epsilon^{\delta\mu}g_{\mu\gamma}u'^{\gamma}$ $(\delta=1,\ 2)$, where $\epsilon^{\delta\mu}=e^{\delta\mu}/g^{1/2}$. The left-hand members of the equations so obtained reduce to $-\rho^{\delta}$ $(\delta=1,\ 2)$ from the facts $g_{\beta\gamma}u'^{\beta}u'^{\gamma}=1$, $g_{\tau\gamma}\rho^{\tau}u'^{\gamma}=0$, and a subtraction of $g_{\tau\gamma}\rho^{\tau}u'^{\gamma}u'^{\delta}=0$ $(\delta=1,\ 2)$. Hence, a family of hypergeodesics can be represented by the two differential equations

$$
\lambda^{\delta} \equiv \rho^{\delta} + \epsilon^{\delta\mu}g_{\mu\gamma}u'^{\gamma}\Omega = 0 \qquad (\delta = 1,\ 2). \tag{3.1}
$$

The left members of equations (3.1) define the contravariant components λ^{δ} of a vector which will be called the *hypergeodesic curvature vector* of a curve C given by $u^{\alpha}=u^{\alpha}(s)$. If λ^{δ} is a null vector along a curve, then equations (3.1) are satisfied and the curve is a hypergeodesic of the family, and conversely.

It is easily shown[7] that μ^{δ} defined by

$$
\mu^{\delta} \equiv -\epsilon^{\delta\mu}g_{\mu\gamma}u'^{\gamma} \qquad (\delta = 1,\ 2) \tag{3.2}
$$

are the contravariant components of a unit vector which makes a right angle with the unit vector u'^{α}, so that

$$
\epsilon_{\alpha\delta}u'^{\alpha}\mu^{\delta} = 1. \tag{3.3}
$$

Now, geodesic curvature[8] K_g is defined by

$$
K_g \equiv \epsilon_{\alpha\delta}u'^{\alpha}\rho^{\delta}, \tag{3.4}
$$

which implies that

$$
\rho^{\delta} = K_g\mu^{\delta}, \tag{3.5}
$$

so that (3.1) assumes the form

---

[7] Eisenhart, loc. cit., p. 136.

[8] Eisenhart, loc. cit., p. 187.

$$\lambda^\delta \equiv (K_g - \Omega)\mu^\delta = 0. \tag{3.6}$$

Therefore the hypergeodesic curvature vector λ^δ makes a right angle with the unit vector u'^α.

Hypergeodesic curvature K_h will be defined by

$$K_h \equiv \epsilon_{\alpha\delta} u'^\alpha \lambda^\delta, \tag{3.7}$$

which by (3.6) and (3.3) gives

$$K_h \equiv K_g - \Omega; \tag{3.8}$$

that is, hypergeodesic curvature is geodesic curvature diminished by the scalar Ω. With the definition (3.4) equation (2.9) may be written

$$K_g = \Omega, \quad \text{or} \quad K_g - \Omega = 0, \tag{3.9}$$

so that a family of hypergeodesics is characterized by the property (3.9); namely, the geodesic curvature is equal to the scalar Ω which is a function of u^1, u^2, u'^1, and u'^2 only. From (3.8) and (3.9) arises the following theorem which is a generalization of a corresponding theorem for geodesics: *a necessary and sufficient condition that a curve* $u^\alpha = u^\alpha(s)$ *be a hypergeodesic of a family is that the hypergeodesic curvature along the curve be identically zero in* s.

4. **The elements of the cone related to the hypergeodesics through a point.** The elements of the cone enveloped by the osculating planes of a family of hypergeodesics through a point P of the surface S are to be determined in tensor form.

If the curve $u^\alpha = u^\alpha(s)$ is a hypergeodesic on the surface S given by $x^i = x^i(u^1, u^2)$, then the osculating plane of the hypergeodesic at the point $x^i(u^1(s), u^2(s))$ is given by the determinantal equation

$$\epsilon_{ijk}(\bar{x}^i - x^i)\frac{dx^j}{ds}\frac{d^2x^k}{ds^2} = 0, \tag{4.1}$$

where the $\bar{x}^i$ are the current coordinates. But from

$$\frac{dx^j}{ds} = x^j_{,\sigma} u'^\sigma, \qquad \frac{d^2x^k}{ds^2} = x^k_{,\sigma} u''^\sigma + \frac{\partial^2 x^k}{\partial u^\alpha \partial u^\beta} u'^\alpha u'^\beta, \tag{4.2}$$

the Gauss equations[9]

$$\frac{\partial^2 x^k}{\partial u^\alpha \partial u^\beta} = \Gamma^\gamma_{\alpha\beta} x^k_{,\gamma} + d_{\alpha\beta} X^k, \tag{4.3}$$

and equations (2.5), equation (4.1) becomes

---

[9] Eisenhart, loc. cit. pp. 215–216.

$$\epsilon_{ijk}(\bar{x}^i - x^i)x^j_{,\sigma}u'^\sigma(\rho^\tau x^k_{,\tau} + d_{\alpha\beta}u'^\alpha u'^\beta X^k) = 0, \tag{4.4}$$

where the X^k are the components of the normal unit vector to S at P, and the $d_{\alpha\beta}$ are the doubly covariant components of the second fundamental quadratic tensor of the surface. Since

$$\epsilon_{ijk}x^j_{,\sigma}x^k_{,\tau} = \epsilon_{ijk}x^j_{,1}x^k_{,2}e_{\sigma\tau}, \tag{4.5}$$

and by use of (2.9), equation (4.4) may be written

$$\epsilon_{ijk}(\bar{x}^i - x^i)\left[\frac{x^j_{,1}x^k_{,2}}{g^{1/2}}\Omega + x^j_{,\sigma}X^k K_n u'^\sigma\right] = 0, \tag{4.6}$$

where the normal curvature K_n is defined by

$$K_n \equiv d_{\alpha\beta}u'^\alpha u'^\beta. \tag{4.7}$$

Differentiation with respect to each of the parameters u'^1 and u'^2 gives the two equations

$$\epsilon_{ijk}(\bar{x}^i - x^i)\left[\frac{x^j_{,1}x^k_{,2}}{g^{1/2}}\frac{\partial\Omega}{\partial u'^\tau} + x^j_{,\sigma}X^k\left\{K_n\frac{\partial u'^\sigma}{\partial u'^\tau} + u'^\sigma\frac{\partial K_n}{\partial u'^\tau}\right\}\right] = 0 \qquad (\tau = 1, 2). \tag{4.8}$$

Since $K_n u'^\sigma$ and Ω are homogeneous of degree three in u'^1 and u'^2, Euler's relation for homogeneous functions shows that the three planes given by (4.6) and (4.8) are coaxial. Consequently, the characteristic lines of the envelope of the one-parameter family of osculating planes (4.6) are the intersections of the planes given by (4.8).

The direction numbers of the normals to the planes (4.8) are

$$a_m \equiv \frac{\partial\Omega}{\partial u'^1}X^m + \left(K_n + u'^1\frac{\partial K_n}{\partial u'^1}\right)\epsilon_{mjk}x^j_{,1}X^k + \left(u'^2\frac{\partial K_n}{\partial u'^1}\right)\epsilon_{mjk}x^j_{,2}X^k \qquad (m = 1, 2, 3),$$

$$b_n \equiv \frac{\partial\Omega}{\partial u'^2}X^n + \left(u'^1\frac{\partial K_n}{\partial u'^2}\right)\epsilon_{njk}x^j_{,1}X^k + \left(K_n + u'^2\frac{\partial K_n}{\partial u'^2}\right)\epsilon_{njk}x^j_{,2}X^k \qquad (n = 1, 2, 3),$$

since $\epsilon_{ijk}x^j_{,1}x^k_{,2}/g^{1/2} = X^i$. The direction numbers c^h of the intersection of the two planes (4.8) are $c^h \equiv \epsilon^{hmn}a_m b_n$, which by neglecting zero determinants, interchanging columns, and renaming dummy indices when desirable, become

$$\begin{aligned} c^h \equiv \epsilon^{hmn}\Bigg[& \left\{\frac{\partial\Omega}{\partial u'^1}\left(u'^1\frac{\partial K_n}{\partial u'^2}\right) - \frac{\partial\Omega}{\partial u'^2}\left(K_n + u'^1\frac{\partial K_n}{\partial u'^1}\right)\right\} X^m \epsilon_{njk} x^j_{,1} X^k \\ & + \left\{\frac{\partial\Omega}{\partial u'^1}\left(K_n + u'^2\frac{\partial K_n}{\partial u'^2}\right) - \frac{\partial\Omega}{\partial u'^2}\left(u'^2\frac{\partial K_n}{\partial u'^1}\right)\right\} X^m \epsilon_{njk} x^j_{,2} X^k \\ & + \left\{\left(K_n + u'^1\frac{\partial K_n}{\partial u'^1}\right)\left(K_n + u'^2\frac{\partial K_n}{\partial u'^2}\right)\right. \\ & \qquad \left. - \left(u'^1\frac{\partial K_n}{\partial u'^2}\right)\left(u'^2\frac{\partial K_n}{\partial u'^1}\right)\right\} \epsilon_{mjk} x^j_{,1} X^k \epsilon_{nrs} x^r_{,2} X^s \Bigg]. \end{aligned}$$

Using the facts $\epsilon_{ijk} x^i_{,1} x^j_{,2} X^k = g^{1/2}$, $X^i X^i = 1$, $x^i_{,\alpha} X^i = 0$, and a well known formula for the simplification of a triple cross product of vectors gives

$$\begin{aligned} \epsilon^{hmn} X^m \epsilon_{njk} x^j_{,1} X^k &= x^h_{,1}, \\ \epsilon^{hmn} X^m \epsilon_{njk} x^j_{,2} X^k &= x^h_{,2}, \\ \epsilon^{hmn} \epsilon_{mjk} x^j_{,1} X^k \epsilon_{nrs} x^r_{,2} X^s &= g^{1/2} X^h. \end{aligned} \tag{4.9}$$

Euler's formulas for the homogeneous functions K_n and Ω are

$$u'^1\frac{\partial K_n}{\partial u'^1} + u'^2\frac{\partial K_n}{\partial u'^2} = 2K_n, \qquad u'^1\frac{\partial\Omega}{\partial u'^1} + u'^2\frac{\partial\Omega}{\partial u'^2} = 3\Omega. \tag{4.10}$$

Use of (4.9) and (4.10) reduces the expression for c^h to the simple form

$$\begin{aligned} c^h \equiv {} & 3\left(\Omega\frac{\partial K_n}{\partial u'^2} - K_n\frac{\partial\Omega}{\partial u'^2}\right) x^h_{,1} + 3\left(K_n\frac{\partial\Omega}{\partial u'^1} - \Omega\frac{\partial K_n}{\partial u'^1}\right) x^h_{,2} \\ & + 3g^{1/2}(K_n)^2 X^h, \end{aligned} \tag{4.11}$$

giving c^h as a single-valued function of the point (u^1, u^2) and the direction u'^1, u'^2, except at points and directions on the surface for which both $K_n = 0$ and $\Omega = 0$. In this case there is no unique plane (4.6) which has contact of the second order with the hypergeodesic in the direction u'^α for which $K_n = 0$ and $\Omega = 0$; that is, all the planes containing the tangent line to this hypergeodesic will have contact of the second order with it. Consequently, the tangent line may be considered an element of the cone and in an asymptotic direction since $K_n = 0$.

At a point (u^1, u^2) and in a direction u'^1, u'^2 such that $K_n=0$ and $\Omega\neq 0$, (4.11) has the form

$$c^{\lambda} = 3\Omega g^{1/2} \epsilon^{\alpha\beta} \frac{\partial K_n}{\partial u'^{\beta}} x^{\lambda}_{,\alpha}. \tag{4.12}$$

Now the vectors u'^{α} and $\epsilon^{\alpha\beta}\partial K_n/\partial u'^{\beta}$ are in the same direction, as is evident from the fact that the sine of the angle between the two vectors is zero since the cross product

$$\epsilon_{\alpha\gamma}\epsilon^{\alpha\beta} \frac{\partial K_n}{\partial u'^{\beta}} u'^{\gamma} = \delta^{\beta}_{\gamma} \frac{\partial K_n}{\partial u'^{\beta}} u'^{\gamma} = \frac{\partial K_n}{\partial u'^{\gamma}} u'^{\gamma} = 2K_n = 0.$$

Therefore, also in this case the element of the cone is tangent to the surface in an asymptotic direction.

At a point (u^1, u^2) and in a direction u'^1, u'^2 such that $K_n\neq 0$, (4.11) may be written

$$c^{\lambda} = 3g^{1/2}(K_n)^2 \left[\epsilon^{\beta\alpha} \frac{\partial}{\partial u'^{\beta}} \left(\frac{\Omega}{K_n}\right) x^{\lambda}_{,\alpha} + X^{\lambda}\right]. \tag{4.13}$$

If we discard the nonzero multipliers, the expressions for c^{λ} in (4.12) and (4.13) are replaced by

$$c^{\lambda} = \epsilon^{\alpha\beta} \frac{\partial K_n}{\partial u'^{\beta}} x^{\lambda}_{,\alpha} \qquad (K_n = 0,\ \Omega \neq 0) \tag{4.14}$$

and

$$c^{\lambda} = r^{\alpha} x^{\lambda}_{,\alpha} + X^{\lambda} \qquad (K_n \neq 0), \tag{4.15}$$

where

$$r^{\alpha} \equiv \epsilon^{\beta\alpha} \frac{\partial}{\partial u'^{\beta}} \left(\frac{\Omega}{K_n}\right) \qquad (K_n \neq 0) \tag{4.16}$$

defines the contravariant components of a vector which is the projection of c^{λ} onto the tangent plane to the surface. The assumption that the surface be expressible with the asymptotic curves as a coordinate net insures that in (4.14) $\partial K_n/\partial u'^{\beta}=0$ for both $\beta=1$ and $\beta=2$ is impossible; so (4.14) will have meaning.

5. A geometric interpretation of hypergeodesic curvature. It will be shown that the hypergeodesic curvature of a curve C, $u^{\alpha}=u^{\alpha}(s)$, is the curvature of the curve C' which is the projection of C upon the tangent plane at the point P, the lines of projection being parallel to that element of the cone determined by the direction u'^{α} at P.

Let S' denote the cylindrical surface consisting of the parallel lines of projection. If $1/R$ is the normal curvature of S' in the direction of C at P, and α is the angle between the principal normal to C and the normal to S' at P, then by the theorem of Meusnier

$$e/R = \rho^{-1} \cos \alpha, \tag{5.1}$$

where $e = \pm 1$, and ρ^{-1} is the curvature of C at P. Also, if $1/r$ is the curvature of C' at P, and if β is the angle between the principal normal to C' and the normal to S' at P, then

$$e/R = r^{-1} \cos \beta. \tag{5.2}$$

After eliminating R from these two equations, it is found that

$$e/r = \frac{\cos \alpha}{\rho \cos \beta} \cdot \tag{5.3}$$

Computation of cos α and cos β yields the formulas

$$\cos \alpha = \frac{-\rho}{L \sin \phi} \left[K_g - K_n \epsilon_{\gamma\delta} u'^{\gamma} r^{\delta}\right] \qquad (K_n \neq 0) \tag{5.4}$$

and

$$\cos \beta = \frac{-1}{L \sin \phi} \qquad (K_n \neq 0), \tag{5.5}$$

where ϕ is the angle between the vector c^i, as given by (4.15), and the unit tangent vector $x^i_{,\sigma} u'^{\sigma}$ to the curves C and C', and L is the length of the vector c^i. Now, by (4.16) and the fact that Ω/K_n is homogeneous of degree one in the parameters u'^{α}, it can be shown that

$$\epsilon_{\gamma\delta} u'^{\gamma} r^{\delta} = \Omega/K_n \qquad (K_n \neq 0). \tag{5.6}$$

Substitution of the formulas for cos α and cos β into (5.3) with use of (5.6) gives

$$e/r = K_g - \Omega \equiv K_h \qquad (K_n \neq 0), \tag{5.7}$$

that is, the curvature of the curve C' is exactly the hypergeodesic curvature of the curve C whenever the curve C is not in an asymptotic direction. When the curve C is in an asymptotic direction at P, the element of the cone determining the direction of the required projection of C will lie in the tangent plane, and consequently will not give a proper projection of C onto the tangent plane.

6. **Hypergeodesic torsion of a curve.** The torsion at P of the hyper-

geodesic $u^\alpha = u^\alpha(s)$ having the same direction as the curve C at P will be called the hypergeodesic torsion τ_h of C at the point P.

Let α^i, β^i, γ^i, be the tangent, principal normal, and binormal unit vectors of the hypergeodesic in the direction of C at P. Since the principal normal β^i of this hypergeodesic lies in the plane of α^i and c^i, the element of the cone given by (4.15), it follows that

$$\beta^i = a\alpha^i + bc^i, \tag{6.1}$$

where a and b are to be determined. Multiplying (6.1) by α^i and summing gives

$$a = -bc^i\alpha^i = -bL\cos\phi, \tag{6.2}$$

and multiplying (6.1) by β^i and summing gives

$$1/b = \beta^i c^i = L\sin\phi. \tag{6.3}$$

Substitution of the values of a and b obtained from (6.2) and (6.3) into (6.1) yields

$$\beta^i = \csc\phi\left[\frac{c^i}{L} - \cos\phi\alpha^i\right]. \tag{6.4}$$

The restriction that $K_n \neq 0$ occurs in (6.4) since in general a hypergeodesic in an asymptotic direction has c^i in coincidence with α^i making $\phi = 0$ so that (6.4) would be meaningless.

From the Frenet-Serret formula $d\gamma^i/ds = \tau_h\beta^i$ multiplied by β^i and summed arises

$$\tau_h = \beta^i\frac{d\gamma^i}{ds}. \tag{6.5}$$

Now γ^i is given by $\epsilon_{ijk}\alpha^j\beta^k$, which by $\alpha^j = dx^j/ds$ and (6.4) reduces to

$$\gamma^i = \csc\phi\epsilon_{ijk}\frac{dx^j}{ds}\frac{c^k}{L}. \tag{6.6}$$

Differentiation of (6.6) with respect to s yields

$$\begin{aligned}\frac{d\gamma^i}{ds} = &-\csc\phi\cot\phi\frac{d\phi}{ds}\epsilon_{ijk}\frac{dx^j}{ds}\frac{c^k}{L} + \csc\phi\epsilon_{ijk}\frac{d^2x^j}{ds^2}\frac{c^k}{L}\\ &+ \csc\phi\epsilon_{ijk}\frac{dx^j}{ds}\frac{d}{ds}\left[\frac{c^k}{L}\right].\end{aligned} \tag{6.7}$$

By the substitution of the expressions for β^i and $d\gamma^i/ds$, omission of the zero determinants, and a change of dummy indices, (6.5) becomes

$$\tau_k = \csc^2\phi\,\epsilon_{ijk}\frac{c^i}{L}\,\frac{dx^j}{ds}\left[\frac{d}{ds}\left(\frac{c^k}{L}\right) - \cos\phi\,\frac{d^2x}{ds^2}\right]. \tag{6.8}$$

Since the components c^k/L are scalars relative to transformations of surface coordinates, the partial derivatives of c^k/L with respect to u^α are the same as the covariant derivatives of c^k/L with respect to u^α; therefore, the derivative of c^k/L with respect to s may be written

$$\frac{d}{ds}\left(\frac{c^k}{L}\right) = \left(\frac{c^k}{L}\right)_{,\alpha}u'^\alpha + \frac{\partial}{\partial u'^\gamma}\left(\frac{c^k}{L}\right)u''^\gamma. \tag{6.9}$$

From (4.15) the covariant derivative of c^k/L takes the form

$$\left(\frac{c^k}{L}\right)_{,\alpha} = \frac{r^\beta}{L}\,x^{k}_{,\beta\alpha} + \frac{1}{L}\,X^{k}_{,\alpha} + \left(\frac{r^\beta}{L}\right)_{,\alpha}x^{k}_{,\beta} + \left(\frac{1}{L}\right)_{,\alpha}X^{k},$$

which, by use of the Weingarten and Gauss equations[10]

$$X^{k}_{,\alpha} = -\,d_{\alpha\gamma}g^{\gamma\beta}x^{k}_{,\beta}, \qquad x^{k}_{,\alpha\beta} = d_{\alpha\beta}X^{k},$$

may be written

$$\left(\frac{c^k}{L}\right)_{,\alpha} = a^{\beta}_{\alpha}x^{k}_{,\beta} + b_\alpha X^{k}, \tag{6.10}$$

where a^β_α and b_α are defined by

$$\begin{aligned} a^{\beta}_{\alpha} &\equiv \left(\frac{r^\beta}{L}\right)_{,\alpha} - \frac{1}{L}\,d_{\alpha\gamma}g^{\gamma\beta}, \\ b_\alpha &\equiv \frac{r^\beta}{L}\,d_{\alpha\beta} + \left(\frac{1}{L}\right)_{,\alpha}. \end{aligned} \tag{6.11}$$

Also, the partial derivatives of c^k/L with respect to u'^γ are given by

$$\frac{\partial}{\partial u'^\gamma}\left(\frac{c^k}{L}\right) = \frac{\partial}{\partial u'^\gamma}\left(\frac{r^\beta}{L}\right)x^{k}_{,\beta} + \frac{\partial}{\partial u'^\gamma}\left(\frac{1}{L}\right)X^{k}. \tag{6.12}$$

From (2.5), (4.2), and (4.3) it is evident that

$$\frac{d^2x^k}{ds^2} = \rho^{\beta}x^{k}_{,\beta} + K_n X^{k}. \tag{6.13}$$

Hence, from the relations (6.9) through (6.13) it follows that

---

[10] Eisenhart, loc. cit. pp. 216–217.

$$
\begin{aligned}
\frac{d}{ds}\left(\frac{c^h}{L}\right) - \cos\phi\,\frac{d^2x^h}{ds^2}\\
\text{(6.14)}\qquad = \left\{\left[a^{\beta}_{\alpha} - \frac{r_\alpha}{L}\rho^\beta\right] x^h_{,\beta} + \left[b_\alpha - \frac{r_\alpha}{L}K_n\right] X^h\right\} u'^{\alpha}\\
+ \left\{\frac{\partial}{\partial u'^{\gamma}}\left(\frac{r^\beta}{L}\right) x^h_{,\beta} + \frac{\partial}{\partial u'^{\gamma}}\left(\frac{1}{L}\right) X^h\right\} u''^{\gamma},
\end{aligned}
$$

in which use was made of the formula $\cos\phi = g_{\delta\alpha}(r^\delta/L)u'^\alpha = (r_\alpha/L)u'^\alpha$.

Substitution of (6.14) in (6.8), if one uses (4.2) and (4.15) and neglects the zero determinants, gives

$$
\begin{aligned}
\tau_h = \csc^2\phi\,\epsilon_{ijh}\Bigg\{ & x^i_{,\beta}x^j_{,\delta}X^h\,\frac{r^\beta}{L}\,u'^\delta\left[b_\alpha - \frac{r_\alpha}{L}K_n\right]u'^\alpha\\
& + X^i x^j_{,\delta}x^h_{,\beta}\left(\frac{1}{L}\right)u'^\delta\left[a^\beta_\alpha - \frac{r_\alpha}{L}\rho^\beta\right]u'^\alpha\\
& + x^i_{,\beta}x^j_{,\delta}X^h\,\frac{r^\beta}{L}\,u'^\delta\,\frac{\partial}{\partial u'^\gamma}\left(\frac{1}{L}\right)u''^\gamma\\
& + X^i x^j_{,\delta}x^k_{,\beta}\left(\frac{1}{L}\right)u'^\delta\,\frac{\partial}{\partial u'^\gamma}\left(\frac{r^\beta}{L}\right)u''^\gamma\Bigg\},
\end{aligned}
$$

which by the fact that $\epsilon_{ijk}x^i_{,}x^j_{,\delta}X^k = \epsilon_{\beta\delta}$ and by a rearrangement and simplification of terms becomes

$$
\begin{aligned}
\tau_h = & \frac{\csc^2\phi}{L}\epsilon_{\beta\delta}\left\{r^\beta b_\alpha - a^\beta_\alpha + \frac{r_\alpha}{L}\left[\rho^\beta - r^\beta K_n\right]\right\}u'^\alpha u'^\delta\\
\text{(6.15)}\qquad & - \frac{\csc^2\phi}{L^2}\epsilon_{\beta\delta}\frac{\partial}{\partial u'^\gamma}(r^\beta)u''^\gamma u'^\delta.
\end{aligned}
$$

However, for a hypergeodesic of the family under consideration it follows from (3.4), (5.6), and (3.9) that

$$
\text{(6.16)}\qquad \epsilon_{\beta\delta}\left[\rho^\beta - r^\beta K_n\right]u'^\delta = -K_g + \Omega = 0;
$$

and from (4.16) it follows that

$$
\begin{aligned}
\epsilon_{\beta\delta}\frac{\partial}{\partial u'^\gamma}(r^\beta)u''^\gamma u'^\delta & = \epsilon_{\beta\delta}\epsilon^{\sigma\beta}\frac{\partial^2}{\partial u'^\gamma\partial u'^\sigma}\left(\frac{\Omega}{K_n}\right)u''^\gamma u'^\delta\\
& = -u'^\delta\frac{\partial}{\partial u'^\delta}\left[\frac{\partial}{\partial u'^\gamma}\left(\frac{\Omega}{K_n}\right)u''^\gamma\right],
\end{aligned}
$$

or, since $\partial(\Omega/K_n)u''^\gamma/\partial u'^\gamma$ is homogeneous of degree zero in the parameters u'^α,

$$\epsilon_{\beta\delta}\frac{\partial}{\partial u'^\gamma}(r^\beta)u''^\gamma u'^\delta = 0. \tag{6.17}$$

Therefore, (6.15) is reduced to

$$\tau_h = \frac{\csc^2\phi}{L}\epsilon_{\beta\delta}\{r^\beta b_\alpha - a_\alpha^\beta\}u'^\alpha u'^\delta, \tag{6.18}$$

which, by the definitions (6.11) of a_α^β and b_α, may be written in the final form

$$\begin{aligned}\tau_h = \frac{\csc^2\phi}{L}\epsilon_{\beta\delta}\Big\{r^\beta\Big[\Big(\frac{1}{L}\Big)_{,\alpha} + \frac{r^\gamma}{L}d_{\alpha\gamma}\Big]\\ - \Big[\Big(\frac{r^\beta}{L}\Big)_{,\alpha} - \frac{1}{L}d_{\alpha\gamma}g^{\gamma\beta}\Big]\Big\}u'^\alpha u'^\delta.\end{aligned} \tag{6.19}$$

The analytic expression (6.19) for the hypergeodesic torsion at P of the curve C with direction u'^α, not an asymptotic direction, is a function of the point P, the direction u'^α, and of course the particular family of hypergeodesics under consideration. Therefore, all curves C with the same direction u'^α at P will have the same hypergeodesic torsion at P relative to the family under consideration.

7. A geometric condition that a hypergeodesic be a plane curve. In order to obtain a geometric condition equivalent to the condition that the torsion of a hypergeodesic be zero, namely that the expression (6.19) vanish, consider the developables of the complex of cone elements and their intersections with the surface. If only a very special set of developables is admitted, then the differential equation of the net of curves of intersection of these developables with the surface will be precisely the expression (6.19) set equal to zero. Hence, a hypergeodesic not in an asymptotic direction will be a plane curve if and only if it is a curve of this special related *intersector net* of the complex of cone elements of the family of hypergeodesics under consideration. The word net is used here in a more general sense to mean a set of curves consisting of two or more one-parameter families of curves.

To arrive at the differential equation of this special intersector net, a point with coordinates $x^i + tc^i/L$ is taken on the line of a cone element through P, and the point P is allowed to move along a curve C, $u^\alpha = u^\alpha(s)$, for which the point $x^i + tc^i/L$ will describe a curve tangent to the line of the cone element through P. That is, x^i+tc^i/L will generate the edge of regression of a developable. This developable will belong to the desired special set if c^i/L at P is taken as only that cone element which corresponds to the values u'^α for the curve C.

The above mentioned tangency to the cone element requires that

$$(7.1)\qquad d\left(x^i+t\frac{c^i}{L}\right)=dx^i+td\left(\frac{c^i}{L}\right)+\frac{c^i}{L}dt=m\frac{c^i}{L},$$

where m is to be determined. Multiplication by c^i/L and use of $(c^i/L)(c^i/L)=1$ and $(c^i/L)d(c^i/L)=0$ gives $m=dt+(c^i/L)dx^i$ so that (7.1) after a division by ds becomes

$$(7.2)\qquad \frac{dx^i}{ds}-\frac{c^i}{L}\frac{c^j}{L}\frac{dx^j}{ds}+t\frac{d}{ds}\left(\frac{c^i}{L}\right)=0.$$

Replacing the derivatives with respect to s by use of the relations (4.2) and (6.9) and multiplying by $(c^i/L)_{,\beta}$ yields

$$(7.3)\qquad \left[\left(\frac{c^i}{L}\right)_{,\beta}x^{i}_{,\alpha}-\left(\frac{c^i}{L}\right)_{,\beta}\frac{c^i}{L}\frac{c^j}{L}x^{j}_{,\alpha}\right]u'^{\alpha}$$
$$+\left[\left(\frac{c^i}{L}\right)_{,\beta}\left(\frac{c^i}{L}\right)_{,\alpha}u'^{\alpha}+\left(\frac{c^i}{L}\right)_{,\beta}\frac{\partial}{\partial u'^{\alpha}}\left(\frac{c^i}{L}\right)u''^{\alpha}\right]t=0.$$

From (4.15) and (6.10) it follows that

$$(7.4)\qquad \left(\frac{c^i}{L}\right)_{,\beta}x^{i}_{,\alpha}=(a^{\delta}_{\beta}x^{i}_{,\delta}+b_\beta X^{i})x^{i}_{,\alpha}=a^{\delta}_{\beta}g_{\delta\alpha},$$

$$(7.5)\qquad \frac{c^i}{L}x^{i}_{,\beta}=\left(\frac{r^\alpha}{L}x^{i}_{,\alpha}+\frac{1}{L}X^{i}\right)x^{i}_{,\beta}=\frac{r^\alpha}{L}g_{\alpha\beta},$$

$$(7.6)\qquad \left(\frac{c^i}{L}\right)_{,\alpha}\left(\frac{c^i}{L}\right)_{,\beta}=a^{\sigma}_{\alpha}a^{\tau}_{\beta}g_{\sigma\tau}+b_\alpha b_\beta,$$

$$(7.7)\qquad \left(\frac{c^i}{L}\right)_{,\beta}\frac{\partial}{\partial u'^{\alpha}}\left(\frac{c^i}{L}\right)=a^{\sigma}_{\beta}g_{\sigma\delta}\frac{\partial}{\partial u'^{\alpha}}\left(\frac{r^\delta}{L}\right)+b_\beta\frac{\partial}{\partial u'^{\alpha}}\left(\frac{1}{L}\right),$$

and

$$(7.8)\qquad \frac{c^i}{L}\left(\frac{c^i}{L}\right)_{,\beta}=\frac{r^\alpha}{L}a^{\gamma}_{\beta}g_{\alpha\gamma}+\frac{1}{L}b_\beta=0,$$

where the zero in (7.8) occurs since covariant differentiation of $(c^i/L)(c^i/L)=1$ gives $(c^i/L)(c^i/L)_{,\beta}=0$. Therefore, (7.3) reduces to the two equations

$$\begin{aligned} (7.9)\qquad & a_{\beta}^{\delta} g_{\delta\alpha} u'^{\alpha} + \Big[\{a_{\alpha}^{\sigma} a_{\beta}^{\tau} g_{\sigma\tau} + b_{\alpha} b_{\beta}\} u'^{\alpha} \\ & + \left\{a_{\beta}^{\sigma} g_{\sigma\delta} \frac{\partial}{\partial u'^{\alpha}}\left(\frac{r^{\delta}}{L}\right) + b_{\beta} \frac{\partial}{\partial u'^{\alpha}}\left(\frac{1}{L}\right)\right\} u''^{\alpha}\Big] t = 0 \\ & \qquad (\beta = 1, 2). \end{aligned}$$

Eliminating t from these two equations gives the determinantal equation

$$\begin{aligned} (7.10)\qquad & e^{\beta\epsilon} a_{\beta}^{\delta} g_{\delta\alpha} u'^{\alpha} \Big[\{a_{\gamma}^{\sigma} a_{\epsilon}^{\tau} g_{\sigma\tau} + b_{\gamma} b_{\epsilon}\} u'^{\gamma} \\ & + \left\{a_{\epsilon}^{\sigma} g_{\eta\sigma} \frac{\partial}{\partial u'^{\gamma}}\left(\frac{r^{\eta}}{L}\right) + b_{\epsilon} \frac{\partial}{\partial u'^{\gamma}}\left(\frac{1}{L}\right)\right\} u''^{\gamma}\Big] = 0. \end{aligned}$$

The part of (7.10) involving u''^{γ} may be written, after replacing b_{ϵ} by $-r^{\eta} a_{\epsilon}^{\sigma} g_{\eta\sigma}$ as obtained from (7.8), in the form

$$(7.11)\qquad e^{\beta\epsilon} a_{\beta}^{\delta} g_{\delta\alpha} a_{\epsilon}^{\sigma} g_{\eta\sigma} \left[\frac{\partial}{\partial u'^{\gamma}}\left(\frac{r^{\eta}}{L}\right) - r^{\eta} \frac{\partial}{\partial u'^{\gamma}}\left(\frac{1}{L}\right)\right] u'^{\alpha} u''^{\gamma}.$$

However,

$$(7.12)\qquad e^{\beta\epsilon} a_{\beta}^{\delta} a_{\epsilon}^{\sigma} = a e^{\delta\sigma}, \qquad e^{\delta\sigma} g_{\delta\alpha} g_{\eta\sigma} = g e_{\alpha\eta},$$

where $a \equiv e_{\gamma\tau} a_{1}^{\gamma} a_{2}^{\tau}$ and $g \equiv e^{\gamma\tau} g_{\gamma 1} g_{\tau 2}$; so that after some simplification (7.11) may be written

$$\frac{ag}{L} e_{\alpha\eta} \frac{\partial r^{\eta}}{\partial u'^{\gamma}} u'^{\alpha} u''^{\gamma},$$

which vanishes by (6.17). Consequently, equation (7.10) is reduced to

$$e^{\beta\epsilon} a_{\beta}^{\delta} g_{\delta\alpha} [a_{\gamma}^{\sigma} a_{\epsilon}^{\tau} g_{\sigma\tau} + b_{\gamma} b_{\epsilon}] u'^{\alpha} u'^{\gamma} = 0,$$

which by replacing b_{ϵ} as obtained from (7.8) and by use of (7.12) becomes

$$a g e_{\alpha\sigma} [a_{\gamma}^{\sigma} - r^{\sigma} b_{\gamma}] u'^{\alpha} u'^{\gamma} = 0.$$

If we discard the nonzero factors a and $g^{1/2}$, and if we use the fact that $g^{1/2} e_{\alpha\sigma} = -g^{1/2} e_{\sigma\alpha} = -\epsilon_{\sigma\alpha}$, the differential equation of the special intersector net of the complex of cone elements takes the form

$$(7.13)\qquad \epsilon_{\sigma\alpha} [r^{\sigma} b_{\gamma} - a_{\gamma}^{\sigma}] u'^{\alpha} u'^{\gamma} = 0.$$

When equation (7.13) is compared with the formula (6.18) it is apparent that *a hypergeodesic not in an asymptotic direction is a plane* *curve if and only if it is a curve of the special intersector net of the complex of cone elements.*[11]

This result may be stated in a purely geometric manner as follows: *a hypergeodesic not in an asymptotic direction is a plane curve if and only if the one-parameter family of cone elements, which are the elements of contact of the osculating planes of the hypergeodesic with the cones, constitutes a developable.*

The differential equation (7.13), after use of (6.11) and (4.16) and clearing of fractions, will in general be homogeneous of degree eight in the parameters u'^α; consequently, the differential equation is of the first order and the eighth degree. Therefore, the special intersector net in general has the property that through any point on the surface pass eight curves of the net. However, there are many cases of families of hypergeodesics for which an equation of lower degree than eight is obtained. Such is the case when the family of hypergeodesics under consideration is a family of union curves.

8. **Union curves.** A family of union curves is a family of hypergeodesics for which the osculating planes of all curves of the family through a point on S form an axial pencil. In this case the cone degenerates into a line of a congruence, one through each point of the surface and not in the tangent plane. If the direction cosines ξ^i of the lines of congruence are

$$\xi^i = p^\sigma x^i_{,\sigma} + qX^i \qquad (q > 0), \tag{8.1}$$

with p^σ and q as functions of u^1 and u^2 alone, then from (4.15) it follows that for a family of union curves

$$r^\sigma = \frac{p^\sigma}{q}, \qquad \frac{r^\sigma}{L} = p^\sigma, \qquad \frac{1}{L} = q, \tag{8.2}$$

so that, by (5.6), Ω must have the form

$$\Omega = K_n e_{\alpha\sigma} u'^\alpha \frac{p^\sigma}{q}. \tag{8.3}$$

It is apparent that for a family of union curves the factor K_n may be discarded from equation (4.6) of the osculating planes, since Ω has K_n as a factor. Therefore, for a union curve in an asymptotic direction equation (4.6) can be considered to define a unique osculating plane which will contain the line ξ^i of the congruence. Hence, the difficulties and exceptions which occur in the asymptotic directions for a general family of hypergeodesics are eliminated in the case of a family of union curves by taking (8.3) to define Ω even for $K_n=0$.

---

[11] A generalization of a corresponding theorem first proved for union curves by Sperry, loc. cit. p. 220.

Now, (8.3) is stated as a theorem as follows: *a necessary and sufficient condition that* (3.9) *be a differential equation of the family of union curves relative to the congruence* ξ^i *is that the scalar* Ω *be given by* (8.3). Substitution of the expression (4.7) for K_n and the expression $\Omega_{\alpha\beta\gamma}u'^{\alpha}u'^{\beta}u'^{\gamma}$ for Ω in (8.3) yields a non-symmetric form for $\Omega_{\alpha\beta\gamma}$, namely,

$$\Omega_{\alpha\beta\gamma} = \epsilon_{\alpha\sigma}\frac{p^{\sigma}}{q}d_{\beta\gamma}. \tag{8.4}$$

When the value of Ω in (8.3) is substituted in the expression (3.8) for hypergeodesic curvature, the result is an expression for union curvature K_u as found by Springer; namely, $K_u=K_g-K_n\epsilon_{\alpha\sigma}u'^{\alpha}\,p^{\sigma}/q$. Also, by use of (8.2), the expression (6.19) for hypergeodesic torsion reduces to the expression for union torsion τ_u as found by Springer; namely,

$$\tau_u = \csc^2\phi\,\epsilon_{\beta\delta}\{p^{\beta}[q_{,\alpha}+p^{\gamma}d_{\alpha\gamma}] - q[p^{\beta}_{,\alpha} - qd_{\alpha\gamma}g^{\gamma\beta}]\}u'^{\alpha}u'^{\delta}.$$

Likewise, the differential equation of the intersector net of the congruence ξ^i obtained from (7.13) after a multiplication by q and use of (6.11) and (8.2) is

$$\epsilon_{\sigma\alpha}\{p^{\sigma}[q_{,\gamma}+p^{\beta}d_{\beta\gamma}] - q[p^{\sigma}_{,\gamma} - qd_{\gamma\beta}g^{\beta\sigma}]\}u'^{\alpha}u'^{\gamma} = 0,$$

which is homogeneous of the second degree in the parameters u'^1 and u'^2. Consequently, in the case of a family of union curves the intersector net has only two curves through each point of the surface instead of eight.[12]

The geometric interpretation of union curvature is the same as that for hypergeodesic curvature with the one simplification that the method of projection (parallel to ξ^i at P) is a function of the point P only and not the direction u'^{α}.

9. **Geodesics.** If the congruence ξ^i for a family of union curves is normal to the surface, then the curves are the geodesic curves of the surface. The congruence is normal if and only if

$$p^{\alpha} \equiv 0, \qquad q \equiv 1 \qquad (\alpha = 1, 2). \tag{9.1}$$

---

[12] The curves of this net were originally called the torsal curves relative to the congruence by Sperry, and later referred to as Γ-curves by Lane and more recently as curves of the intersector net by Springer.

From (8.3) it is apparent that the property (9.1) is equivalent to the condition that the scalar Ω be identically zero. As a result, the following theorem may be stated: *a necessary and sufficient condition that a family of hypergeodesics defined by equation* (3.9) *be the family of geodesics is that the scalar* Ω *be identically zero in* u^1, u^2, u'^1, u'^2, *or what is equivalent, that the symmetric components* $\Omega_{\alpha\beta\gamma}$ *be identically zero in* u^1 *and* u^2.

It is observed then from (3.8) that hypergeodesic curvature reduces to geodesic curvature when the family of hypergeodesics is the family of geodesics. Also, the geometric interpretation of hypergeodesic curvature reduces to the correct geometric interpretation for geodesic curvature, and the expression (6.19) for hypergeodesic torsion will reduce to the expression

$$\tau_g = \epsilon^{\sigma\mu} d_{\alpha\sigma} g_{\delta\mu} u'^{\alpha} u'^{\delta}$$

for geodesic torsion[13] τ_g.

From the above discussion it is seen that hypergeodesic curvature and hypergeodesic torsion are proper generalizations of geodesic curvature and geodesic torsion since they are identical when the family of hypergeodesics under consideration is taken to be the family of geodesics.

UNIVERSITY OF WASHINGTON

---

[13] Eisenhart, loc. cit. p. 247.

SOME REMARKS ON RULED SURFACES

J. ERNEST WILKINS, JR.

In a previous paper [1][1] the author showed that the projective differential geometry of a nondevelopable ruled surface S in three-dimensional space could be studied by means of the expansion

$$
\begin{aligned}
z = xy &- \frac{1}{3}\gamma y^3 - \frac{1}{3}\gamma_u x y^3 - \frac{1}{12}\gamma_v y^4 - \frac{1}{6}\gamma_{uu} x^2 y^3 - \frac{1}{12}\gamma_{uv} x y^4 \\
&+ \frac{1}{60}(10\gamma\gamma_u - \gamma_{vv}) y^5 - \frac{1}{24}\gamma_{uuv} x^2 y^4 \\
&+ \frac{1}{60}(10\gamma\gamma_{uu} + 10\gamma_u^2 - \gamma_{uvv}) x y^6 \\
&+ \frac{1}{360}(20\gamma_u\gamma_v + 15\gamma\gamma_{uv} - \gamma_{vvv}) y^6 + \cdots ,
\end{aligned} \tag{1}
$$

for one nonhomogeneous coordinate z as a power series in the other two nonhomogeneous coordinates x and y. Here γ is a function of the form $A(v)u^2 + B(v)u + C(v)$. It was also shown that there is a one-parameter family of cubic surfaces with fifth order contact with S, namely,

$$
\begin{aligned}
\frac{1}{3}\gamma y^3 + Dz^2 + (z - xy)\left(Px - \frac{1}{4}(\gamma_v/\gamma)y + Mz + 1\right)& \\
+ yz(Iy + Jz) &= 0,
\end{aligned} \tag{2}
$$

where D is the parameter and P, M, I, and J are defined as follows:

$$
\begin{aligned}
P &= (15\gamma_v^2 + 40\gamma^2\gamma_u - 12\gamma\gamma_{vv})/80\gamma^3, \\
M &= (40\gamma^2\gamma_u\gamma_v + 12\gamma\gamma_v\gamma_{vv} - 80\gamma^3\gamma_{uv} - 15\gamma_v^3)/320\gamma^4, \\
I &= (5\gamma_v^2 + 40\gamma^2\gamma_u - 4\gamma\gamma_{vv})/80\gamma^2, \\
J &= (15\gamma_u\gamma_v^2 + 40\gamma^2\gamma_u^2 - 12\gamma\gamma_u\gamma_{vv} + 40\gamma^3\gamma_{uu})/240\gamma^3.
\end{aligned}
$$

In this paper we shall report some further results on nondevelopable ruled surfaces which can be obtained with the help of the above formulas.

It was shown in [1] that S is a cubic surface if and only if $\mathcal{A} = \mathcal{B} = 0$, where

---

Received by the editors October 7, 1947, and in revised form September 30, 1948.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$\mathcal{A} = [(15\gamma_v^2 - 12\gamma\gamma_{vv} - 40\gamma^2\gamma_u)(\gamma_u\gamma_v - \gamma\gamma_{uv}) \\ + 40\gamma^3(\gamma_u\gamma_v - \gamma\gamma_{uv})_u]/960\gamma^4,$$

$$\mathcal{B} = [1600\gamma^4(2\gamma\gamma_{uu} - \gamma_u^2) + 960\gamma^3(\gamma_u\gamma_v - \gamma\gamma_{uv})_v \\ - 2400\gamma^2\gamma_v(\gamma_u\gamma_v - \gamma\gamma_{uv}) + (12\gamma\gamma_{vv} - 15\gamma_v^2)^2]/57600\gamma^4.$$

Since this criterion requires derivatives of γ of the third order, it is of interest to notice that the following theorem is true.

THEOREM. *The nondevelopable ruled surface* (1) *is cubic if and only if* $P=M=J=0$. *It is Cayley's cubic scroll if and only if* $I=0$ *also.*

If S is cubic we have seen [1] that we can assume that either $\gamma=3$ or $\gamma=3u/10v^2$. By direct substitution it is found in the first case, which corresponds to Cayley's cubic scroll, that $P=M=J=I=0$, and in the second case that $P=M=J=0$, $I=1/10v^2$. The converse follows from the identities

$$960\gamma^3\mathcal{A} = (15\gamma_v^2 - 12\gamma\gamma_{vv})(4\gamma M + \gamma_v P) + 40\gamma^3(4\gamma M + \gamma_v P)_u,$$

$$360\gamma\mathcal{B} = 40\gamma^2(3J - 2\gamma_u P + \gamma P^2) + 6\gamma(4\gamma M + \gamma_v P)_v \\ - 9\gamma_v(4\gamma M + \gamma_v P),$$

which imply that $\mathcal{A}=\mathcal{B}=0$ if $M=P=J=0$.

If S is cubic, the surface (2) will actually have sixth order contact with S if and only if $D=P\gamma_{uu}/6$, and it will then coincide with S.

Let us now seek to find the double points of the cubic surface (2). The homogeneous coordinates of such a point must satisfy the equations $F_i=0$, where

$$F_1 = x_4\left(Px_2 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Mx_4 + x_1\right) + (x_1x_4 - x_2x_3),$$

$$F_2 = -x_3\left(Px_2 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Mx_4 + x_1\right) + P(x_1x_4 - x_2x_3),$$

$$F_3 = \gamma x_3^2 - x_2\left(Px_2 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Mx_4 + x_1\right) \\ - \frac{1}{4}(\gamma_v/\gamma)(x_1x_4 - x_2x_3) + 2Ix_3x_4 + Jx_4^2,$$

$$F_4 = 3Dx_4^2 + x_1\left(Px_2 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Mx_4 + x_1\right) \\ + M(x_1x_4 - x_2x_3) + Ix_3^2 + 2Jx_3x_4.$$

Setting $x_4=0$, we readily find that $(-P, 1, 0, 0)$ is the only double point in the tangent plane, and that this point is a double point for all the cubic surfaces (2).

To find double points not in the tangent plane, suppose that $x_4 \neq 0$. Then $F_1=F_2=0$ if and only if either

$$(3) \qquad Px_2 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Mx_4 + x_1 = 0, \quad x_1x_4 - x_2x_3 = 0,$$

or

$$(4) \qquad F_1 = 0, \qquad Px_4 + x_3 = 0.$$

When (3) holds we may set $x_4=1$ and see that a double point must satisfy (3) and

$$(5) \qquad \gamma x_3^2 + 2Ix_3 + J = 0, \qquad Ix_3^2 + 2Jx_3 + 3D = 0.$$

Let us suppose for the moment that $I^2 \neq \gamma J$. Then two values of x_3 are determined by the first of equations (5). Each value of x_3 determines a value of D by the second of equations (5). These values of D are different since $I^2 \neq \gamma J$. For each value of x_3 we determine x_1 and x_2 by solving the equations

$$(6) \qquad x_1 + Px_2 = \frac{1}{4}(\gamma_v/\gamma)x_3 - M, \qquad x_1 - x_2x_3 = 0.$$

Hence a unique x_1 and x_2 can be found unless $x_3 = -P$. We conclude that if the surface S is such that $I^2 \neq \gamma J$, $\gamma P^2 - 2IP + J \neq 0$, then there are two cubic surfaces (2) each of which has a (unique) double point which is not in the tangent plane and satisfies equations (3).

On the other hand, if $\gamma P^2 - 2IP + J = 0$ and $I^2 \neq \gamma J$, then one root x_3 of the first of equations (5) is equal to $-P$ and the other is not. Corresponding to the root which is different from $-P$ there exists one cubic surface (2) which has a unique double point not in the tangent plane which satisfies equations (3). When $x_3 = -P$, equations (3.4) have a solution if and only if

$$(7) \qquad 0 = M + \frac{1}{4}(\gamma_v/\gamma)P = (\gamma_u\gamma_v - \gamma\gamma_{uv})/4\gamma^2.$$

Since it is easy to see that

$$(8) \qquad \begin{aligned} 19200\gamma^5(\gamma P^2 - 2IP + J) = {} & (15\gamma_v^2 - 12\gamma\gamma_{vv})^2 \\ & - 1600\gamma^4(\gamma_u^2 - 2\gamma\gamma_{uu}), \end{aligned}$$

it is clear that when $\gamma P^2-2IP+J=0$ and $x_3=-P$, equations (3.4) have a solution if and only if $\mathcal{A}=\mathcal{B}=0$. In this case S is a cubic surface itself and the entire line $x_1+Px_2=x_3+Px_4=0$ consists of double points for the cubic surface (2) determined by the parameter $D=(-IP^2+2JP)/3$. Since S is cubic, $P=0$. Therefore, $D=0$ and hence the cubic surface (2) is the surface S itself. Since $I^2\neq\gamma J=0$, S is not Cayley's cubic scroll. We conclude that if $\gamma P^2-2IP+J=0$, $I^2\neq\gamma J$, then there is one cubic surface (2) which has a (unique) double point which is not in the tangent plane and satisfies equations (3). There is another cubic surface (2) which has a double point not in the tangent plane which satisfies equations (3) if and only if S is itself a cubic surface, not Cayley's cubic scroll, in which case the cubic surface (2) coincides with S which has the whole line $x_1=x_3=0$ as a line of double points.

When $I^2=\gamma J$, then only one value of x_3 is determined by the first of equations (5). If also $\gamma P^2-2IP+J\neq 0$, then this value of x_3 is different from $-P$ and so there exists one cubic surface (2) with a (unique) double point which is not in the tangent plane and satisfies equations (3).

When $I^2=\gamma J$ and $\gamma P^2-2IP+J=0$, reasoning like that used two paragraphs earlier shows that there is a cubic surface (2) with a double point not in the tangent plane which satisfies equations (3) if and only if S is Cayley's cubic scroll, in which case the cubic surface (2) is S itself and has the whole line $x_1=x_3=0$ as a line of double points.

Now let us consider double points which satisfy equations (4). Setting $x_3=-P$, $x_4=1$, we see that the equations $F_1=F_2=F_4=0$ become

$$(9)\quad 2(x_1+Px_2)=-M-\frac{1}{4}(\gamma_v/\gamma)P=-(\gamma_u\gamma_v-\gamma\gamma_{uv})/4\gamma^2,$$

$$(10)\quad (4\gamma x_2-\gamma_v)(\gamma_u\gamma_v-\gamma\gamma_{uv})=32\gamma^3(\gamma P^2-2IP+J),$$

$$(11)\quad D=-(x_1-M)(\gamma_u\gamma_v-\gamma\gamma_{uv})/24\gamma^2-(IP^2-2JP)/3.$$

Unique values for x_1, x_2, and D can thus be determined provided that $\gamma_u\gamma_v-\gamma\gamma_{uv}\neq 0$. The cubic surface (2) so determined may, however, coincide with a cubic surface previously determined which has a double point satisfying equations (3). To find the condition that this occurs we explicitly solve equations (9), (10), and (11). Let $\eta=\gamma_u\gamma_v-\gamma\gamma_{uv}$, $\theta=5\gamma_v^2-4\gamma\gamma_{vv}$, $\Delta=\gamma_u^2-2\gamma\gamma_{uu}$. It follows from equations (10) and (8) that

$$x_2=\gamma_v/4\gamma+3\theta^2/800\gamma^2\eta-2\gamma\Delta/3\eta$$

if $\eta \neq 0$. From (9) and the definitions of P and M we then find that

$$M - x_1 = 9\theta^3/64000\gamma^6\eta + 3\theta^2\gamma_u/1600\gamma^4\eta - \Delta\theta/40\gamma^2\eta \\ + 3\eta/8\gamma^2 - \gamma_u\Delta/3\eta,$$

and finally we conclude from equation (11) and the definitions of I, P, and J that

$$3D = (6\theta\gamma\gamma_{uu} + 15\eta^2 + 80\gamma^2\gamma_u\gamma_{uu})/320\gamma^4. \tag{12}$$

The cubic surface (2) with this value of D coincides with a cubic surface (2) with a double point which is not in the tangent plane and satisfies equations (3) if and only if there is a value of x_3 which satisfies equations (5) simultaneously when D satisfies equation (12) and which belongs to a double point obtained from equations (3). Eliminating x_3 between equations (5) we see that this coincidence will occur if and only if

$$(3D\gamma - IJ)^2 - 4(I^2 - \gamma J)(J^2 - 3ID) = 0,$$

an equation which can be reduced in a straightforward manner to

$$\begin{aligned} - 81\Delta\theta^4 + 405\eta^2\theta^3 + 28800\gamma^4\Delta^2\theta^2 - 648000\gamma^4\eta^2\Delta\theta \\ + 2430000\gamma^4\eta^4 - 2560000\gamma^8\Delta^3 = 0. \end{aligned} \tag{13}$$

We conclude that if $\eta \neq 0$ and equation (13) fails to hold, then there is another cubic surface (2) in addition to the ones already found which has a double point not in the tangent plane.

If $\eta \neq 0$ and equation (13) holds, then there is a value of x_3 which satisfies equations (5) when D has the value given by equation (12). This value of x_3 is equal to $-P$ if and only if $\gamma P^2 - 2IP + J = 0$ and $3D = -IP^2 + 2JP$, and hence if and only if $M - x_1 = 0$ and $9\theta^2 = 1600\gamma^4\Delta$. When $9\theta^2 = 1600\gamma^4\Delta$ it is easy to see that $M - x_1 = 3\eta/8\gamma^2 \neq 0$. Hence the value of x_3 thus obtained can never be equal to $-P$. It follows that if either $I^2 - \gamma J$ or $\gamma P^2 - 2IP + J$ is different from zero no new cubic surfaces (2) having a double point not in the tangent plane can be found by considering equations (4) if $\eta \neq 0$ and equation (13) holds. When $I^2 - \gamma J = \gamma P^2 - 2IP + J = 0$, it is seen that $\Delta = \theta = 0$ and hence that the left-hand side of equation (13) reduces to $2430000\gamma^4\eta^4$ which does not vanish.

If $\eta = 0$, equations (9), (10), and (11) have a solution if and only if $\gamma P^2 - 2IP + J = 0$ also, and then $3D = -IP^2 + 2JP$. We have already encountered this case among the solutions of equations (3) and so no new cubic surface can be obtained from equations (4) when $\eta = 0$.

We may conveniently summarize our results in the following table, in which $\mathfrak{D}$ is an abbreviation for the left-hand side of equation (13).

TABLE I

$I^2-\gamma J$	$\gamma P^2-2IP+J$	$\gamma_u\gamma_v-\gamma\gamma_{uv}$	$\mathfrak{D}$	(3)	(4)
*	*	*	*	2	1
*	*	*	0	2	0
*	*	0	*	2	0
*	*	0	0	Impossible	
*	0	*	*	1	1
*	0	*	0	1	0
*	0	0	*	Impossible	
*	0	0	0	2	0
0	*	*	*	1	1
0	*	*	0	1	0
0	*	0	*	1	0
0	*	0	0	Impossible	
0	0	*	*	0	1
0	0	*	0	Impossible	
0	0	0	*	Impossible	
0	0	0	0	1	0

In the first four columns a * is entered if the quantity at the head of the column is different from zero and a 0 is entered if it vanishes. In the fifth column is entered the number of cubic surface (2) with a double point not in the tangent plane which can be obtained by considering equations (3), and in the sixth column is entered the number of such additional cubic surfaces which can be obtained by considering equations (4). We have already seen that when $I^2-\gamma J=\gamma P^2-2IP+J=0$, then $\mathfrak{D}$ and η either both vanish or neither vanishes. Hence the cases in the fourteenth and fifteenth lines do not occur. Moreover, when $\eta=0$ it is easy to see that $\mathfrak{D}= -\Delta(9\theta^2-1600\gamma^4\Delta)^2$, so that $\mathfrak{D}$ and $\gamma P^2-2IP+J$ either both vanish or neither vanishes. Hence the cases in the fourth, seventh, twelfth, and fifteenth lines do not occur either. In line eight one of the two surfaces is the surface S itself which is cubic and not Cayley's cubic scroll. In line sixteen the surface is S itself, which is Cayley's cubic scroll. In both cases there is an entire line of double points not in the tangent plane.

It is worth noticing that if none of the four quantities $I^2-\gamma J$, $\gamma P^2-2IP+J$, $\gamma_u\gamma_v-\gamma\gamma_{uv}$, and $\mathfrak{D}$ vanishes, then there are three cubic surfaces (2) which have a double point not in the tangent plane. If exactly one of these quantities vanishes there are two such cubic surfaces. If exactly two vanish there is one such cubic surface. If exactly three vanish, there are two such cubic surfaces, one of which is S itself and is not Cayley's cubic scroll. If all four vanish, there is one such cubic surface, namely S itself, which is Cayley's cubic scroll.

Let us now study the geometry in the tangent plane of S. It has been shown [1] that the tangent plane intersects S in a curve with two branches, one of which is the generator $y=z=0$, and the other is

$$\begin{aligned} x = {} & \frac{1}{3}\gamma y^2 + \frac{1}{12}\gamma_v y^3 + \frac{1}{180}(3\gamma_{vv} - 10\gamma\gamma_u)y^4 \\ & - \frac{1}{360}(10\gamma_u\gamma_v + 5\gamma\gamma_{uv} - \gamma_{vvv})y^5 + \cdots, \quad z = 0, \end{aligned} \tag{14}$$

and that the equation of the osculating conic of this curve is

$$z = Px^2 + x - \frac{1}{3}\gamma y^2 - \frac{1}{4}(\gamma_v/\gamma)xy = 0. \tag{15}$$

There exists a one-parameter family of noncomposite cubic curves with a node at the origin and asymptotic tangents for nodal tangents each of which has fifth order contact with the curve (14). The equation of such a cubic curve is

$$z = kx^3 - \frac{1}{3}\gamma y^3 + Px^2y - \frac{1}{4}(\gamma_v/\gamma)xy^2 + xy = 0, \tag{16}$$

where k is the parameter. The points of inflexion of this nodal cubic curve lie on the line

$$z = Px - \frac{1}{4}(\gamma_v/\gamma)y + 1 = 0, \tag{17}$$

some one of the inflexions lying on each of the lines $3kx^3=\gamma y^3$. The line (17) is tangent to the osculating conic (15) at the point whose homogeneous coordinates are $(-P, 1, 0, 0)$. This is the point characterized above as the only double point of a cubic surface (2) which lies in the tangent plane. It may also be characterized as the intersection different from the origin of the generator $y=z=0$ and the osculating conic (15).

Since the point $(-P, 1, 0, 0)$ is a double point of the cubic surface (2), the polar quadric of $(-P, 1, 0, 0)$ with respect to (2) must decompose into two planes. These planes are

$$Px_4 + x_3 = 0, \qquad Mx_4 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Px_2 + x_1 = 0.$$

The first of these planes intersects the tangent plane in the generator, and the second intersects the tangent plane in the line (17).

The nodal cubic curve (16) has sixth order contact with the curve (14) if and only if $k=27\mathcal{C}/\gamma^3$, where $\mathcal{C}$ was defined in [1] as

$$\mathcal{C} = [18\gamma\gamma_v\gamma_{vv} - 15\gamma_v^3 - 4\gamma^2\gamma_{vvv} - 20\gamma^2(\gamma_u\gamma_v - \gamma\gamma_{uv})]/1440\gamma^2.$$

Therefore, there exists a noncomposite nodal cubic curve, nodal tangents being the asymptotic tangents, with sixth order contact with the curve (14) if and only if $\mathcal{C}\neq 0$. If $\mathcal{C}=0$, the only such cubic decomposes into the generator $y=z=0$ and the conic (15) which then hyperosculates the curve (14).

If we do not insist that the nodal tangents be the asymptotic tangents, we can prove that there exists a one-parameter family of cubic curves with a node at the origin and sixth order contact with the curve (14). The equation of such a cubic curve is

$$\begin{aligned} z = (EP+k)x^3 - \frac{1}{3}\gamma y^3 + \left[P - \frac{1}{4}E(\gamma_v/\gamma)\right]x^2y \\ - \left[\frac{1}{3}E\gamma + \frac{1}{4}(\gamma_v/\gamma)\right]xy^2 + Ex^2 + xy = 0, \end{aligned} \tag{18}$$

where $k=27\mathcal{C}/\gamma^3$ and E is the parameter. The nodal tangents are

$$x = 0, \qquad y = -Ex, \tag{19}$$

and the line of inflexions is

$$(4E^2\gamma^2 + 12P\gamma)x + (8E\gamma^2 - 3\gamma_v)y + 12\gamma = 0. \tag{20}$$

As E varies this line of inflexion envelops a conic which turns out to be precisely the osculating conic (15), which thus receives another geometrical characterization. Moreover, the nodal tangent $y+Ex=0$ intersects the line of inflexion (20) in the point whose homogeneous coordinates are $(4E^2\gamma^2-3E\gamma_v-12P\gamma,\ 12\gamma,\ -12E\gamma,\ 0)$, the locus of which as E varies is also the osculating conic (15).

BIBLIOGRAPHY

1. Wilkins, J. Ernest, Jr., *The contact of a cubic surface with a ruled surface*, Amer. J. Math. vol. 67 (1945) pp. 71–82.

AMERICAN OPTICAL COMPANY

ON INTERPOLATION TO AN ANALYTIC FUNCTION IN EQUIDISTANT POINTS: PROBLEM β

J. L. WALSH AND W. E. SEWELL

The present note is an addendum to a paper by the same authors,[1] to which (especially §8) the reader may refer for detailed notation and definitions.

Two phases of the direct problem of the study of degree of convergence of a sequence of functions approximating to a given function are (i) proof of the existence of a sequence approximating with a certain degree of convergence and (ii) study of the degree of convergence of a sequence defined by a specific method. We are here concerned with the second phase of the problem:

THEOREM 1. *Let the function* $f(z)$ *be analytic and of class* $L(p, \alpha)$ *in the annulus* $\gamma_\rho : \rho > |z| > 1/\rho < 1$, *where* ρ *is given. Let*

$$p_n(z) = a_{nn}z^n + a_{n,n-1}z^{n-1} + \cdots + a_{n0} + \cdots + a_{n,-n}z^{-n}$$

be the unique polynomial in z *and* $1/z$ *of degree* n *which coincides with* $f(z)$ *in the* $(2n+1)$*st roots of unity. Then for* z *on* $\gamma : |z| = 1$ *we have*

$$(1) \qquad | f(z) - p_n(z) | \leqq M/\rho^n \cdot n^{p+\alpha},$$

where M *is independent of* n *and* z.

The polynomial $p_n(z)$ may be defined by the equation

$$(2) \qquad f(z) - p_n(z) = \frac{1}{2\pi i}\int_{|z|=r} + \int_{|z|=1/r} \frac{t^n(z^{2n+1} - 1)f(t)dt}{z^n(t^{2n+1} - 1)(t - z)},$$

for z in the annulus $1 > r > |z| > 1/r$; the function $f(z)$ is assumed analytic for $r > |z| > 1/r$, continuous for $r \geqq |z| \geqq 1/r$. In particular, if $f(z)$ itself is here chosen as a polynomial $P_n(z)$ in z and $1/z$ of degree n, the interpolating polynomial $p_n(z)$ must coincide with $P_n(z)$, so we have for $r > |z| > 1/r$

$$(3) \qquad 0 = \frac{1}{2\pi i}\int_{|t|=r} + \int_{|t|=1/r} \frac{t^n(z^{2n+1} - 1)P_n(t)dt}{z^n(t^{2n+1} - 1)(t - z)}.$$

Return to the original function $f(z)$ in (2) with subtraction of (3) from (2) thus yields for $r > |z| > 1/r$

Received by the editors September 24, 1948.

[1] Trans. Amer. Math. Soc. vol. 49 (1941) pp. 229–257.

$$(4)\quad f(z) - p_n(z) = \frac{1}{2\pi i}\int_{|t|=r} + \int_{|t|=1/r} \frac{t^n(z^{2n+1} - 1)[f(t) - P_n(t)]dt}{z^n(t^{2n+1} - 1)(t - z)}.$$

Suppose now $f(z)$ is of class $L(p, \alpha)$ in the annulus $\rho > |z| > 1/\rho < 1$, where p is non-negative, $0 < \alpha \leqq 1$. Then by Cauchy's integral formula for the annulus, $f(z)$ can be written $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ is analytic in the region $|z| < \rho$ and is of class $L(p, \alpha)$ on $|z| = \rho$, and where $f_2(z)$ is analytic in the region $|z| > 1/\rho$ even at infinity, and $f_2(1/z)$ is of class $L(p, \alpha)$ on $|z| = \rho$. Consequently (the result is due to Curtiss) there exist polynomials $P_n'(z)$ and $P_n''(z)$ in z and $1/z$ respectively of degree n such that we have

$$|f_1(z) - P_n'(z)| \leqq M_1/n^{p+\alpha}, \qquad \text{for } |z| = \rho,$$
$$|f_2(z) - P_n''(z)| \leqq M_2/n^{p+\alpha}, \qquad \text{for } |z| = 1/\rho.$$

By the principle of maximum modulus we may write

$$|f_1(z) - P_n'(z)| \leqq M_1/n^{p+\alpha}, \qquad \text{for } |z| = 1/\rho,$$
$$|f_2(z) - P_n''(z)| \leqq M_2/n^{p+\alpha}, \qquad \text{for } |z| = \rho.$$

Then if we set $P_n(z) = P_n'(z) + P_n''(z)$, the function $P_n(z)$ is a polynomial in z and $1/z$ of degree n, and on both $|z| = \rho$ and $|z| = 1/\rho$ we have

$$(5)\qquad |f(z) - P_n(z)| \leqq (M_1 + M_2)/n^{p+\alpha}.$$

When $|t| = \rho > 1$, we have $|t^n/(t^{2n+1} - 1)| < m/\rho^n$, where m is independent of t and n, and we have a similar relation for $|t| = 1/\rho$, so inequality (5) and equation (4) with $r = \rho$ yield the conclusion of the theorem, $p \geqq 0$.

Suppose now that $f(z)$ is of class $L(p, \alpha)$ in the annulus $\rho > |z| > 1/\rho$, where p is negative; we have by definition $(r < \rho)$

$$|f(z)| \leqq N \cdot (\rho - r)^{p+\alpha}$$

for $|z| = r > 1$ and for $|z| = 1/r$, where N is independent of z and r. If we set $r = r_n = (1 - 1/n)\rho$ directly in (2), we also obtain the conclusion of the theorem; compare §3 of the paper already mentioned. The proof is now complete.

The degree of approximation asserted in Theorem 1 is also the degree of approximation obtained (loc. cit.) for the partial sums of the Laurent series for $f(z)$.

If $f(z)$ is an arbitrary function continuous on γ, and if for the polynomials $t_n(z)$ in z and $1/z$ of degree n of best approximation to $f(z)$ on γ in the sense of Tchebycheff we have

$$|f(z) - t_n(z)| \leqq \epsilon_n, \qquad z \text{ on } \gamma,$$

then for the polynomials $p_n(z)$ in z and $1/z$ of degree n which interpolate to $f(z)$ in the $(2n+1)$st roots of unity, we have[3]

$$|f(z) - p_n(z)| \leqq \epsilon_n \log n, \qquad z \text{ on } \gamma.$$

Theorem 1 represents an improvement on this result for the special functions $f(z)$ of class $L(p, \alpha)$ in $\rho \geqq |z| \geqq 1/\rho$, provided $0<\alpha<1$, because for such functions we have in the above notation

$$|f(z) - t_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha}, \qquad z \text{ on } \gamma,$$

an inequality which for the case $0<\alpha<1$ and for the entire class of functions cannot be improved.

We formulate explicitly two consequences of our preceding discussion:

COROLLARY 1. *Let $f(z)$ be analytic in the annulus $\rho>|z|>1/\rho<1$' continuous in the corresponding closed region, and let us suppose polynomials $P_n(z)$ in z and $1/z$ of degree n to exist such that we have*

$$(6) \qquad |f(z) - P_n(z)| \leqq \epsilon_n, \qquad \textit{for } |z| = \rho \textit{ and } |z| = 1/\rho.$$

Then for the polynomials $p_n(z)$ defined in Theorem 1 we have

$$|f(z) - p_n(z)| \leqq M\epsilon_n/\rho^n, \qquad \textit{for } |z| = 1,$$

where M is independent of n and z.

A sufficient condition for (6) is that $f(z)$ can be written $f(z) \equiv f_1(z) + f_2(z)$ in the closed region $\rho \geqq |z| \geqq 1/\rho$, where $f_1(z)$ is analytic for $|z|<\rho$, and $f_2(z)$ is analytic for $|z|>1/\rho$ including the point at infinity, and that polynomials $P_n'(z)$ and $P_n''(z)$ in z and $1/z$ respectively of degree n exist so that we have

$$|f_1(z) - P_n'(z)| \leqq \epsilon_n, \qquad \text{for } |z| = \rho,$$
$$|f_2(z) - P_n''(z)| \leqq \epsilon_n, \qquad \text{for } |z| = 1/\rho.$$

COROLLARY 2. *Let $f(z)$ be analytic in the annulus $\rho>|z|>1/\rho<1$, let the function $\phi(r)$ be defined in the interval $1<r<\rho$, and suppose we have $|f(z)| \leqq \phi(\rho-r)$ on the two circles $|z|=r$ and $z=1/r$. Then for the polynomials $p_n(z)$ defined in Theorem 1 we have*

$$|f(z) - p_n(z)| \leqq M \cdot \phi(\rho/n)/\rho^n, \qquad \textit{for } |z| = 1,$$

---

[3] D. Jackson, *The theory of approximation*, Amer. Math. Soc. Colloquium Publications, vol. 11, p. 121.

where M is independent of n and z.

There can be developed extensions of the above results to approximation on an arbitrary analytic Jordan curve or on more general point sets bounded by analytic Jordan curves, by rational functions with poles in prescribed points or uniformly distributed on given curves. The present results are intended primarily as illustrations of this general theory.

Harvard University and
Fort Riley, Kansas

NOTE ON THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL

P. T. BATEMAN

Erdős[1] has proved that if A_n denotes the largest coefficient (in absolute value) of the nth cyclotomic polynomial, then for infinitely many n

$$A_n > \exp\left\{c_1(\log n)^{4/3}\right\}.$$

He also conjectured that a much stronger statement may be true, namely that[2]

$$\text{(A)} \qquad A_n > \exp\left\{n^{(c_{13}/\log\log n)}\right\}$$

holds for some c_{13} and infinitely many n, but pointed out that this would be a best result, since

$$\text{(B)} \qquad A_n < \exp\left\{n^{(c_{14}/\log\log n)}\right\}$$

for some c_{14} and all n. Erdős suppressed the proof of (B), because his proof was complicated. It is the purpose of this note to give the following short proof of (B).

The cyclotomic polynomial $F_n(x) = \prod_{d|n} (1-x^d)^{\mu(n/d)}$ is majorized by the power series

$$\prod_{d|n} (1 + x^d + x^{2d} + \cdots).$$

Received by the editors September 20, 1948.

[1] Paul Erdős, *On the coefficients of the cyclotomic polynomial*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 179–184.

[2] Formulas (A) and (B) were printed incorrectly in Erdős' paper (on the bottom of p. 182).

Since $F_n(x)$ is of degree less than n, it is also in fact majorized by the polynomial

$$\prod_{d|n} (1 + x^d + x^{2d} + \cdots + x^{(n/d-1)d}).$$

Hence A_n is less than the sum of the coefficients of this polynomial. Thus, if $d(n)$ denotes the number of divisors of n, we have for sufficiently large n

$$\begin{aligned} A_n < \prod_{d|n} (n/d) &= n^{d(n)/2} = \exp\left\{\frac{1}{2} d(n) \log n\right\} \\ &< \exp\left\{\frac{1}{2} 2^{(1+\epsilon/2)\log n/\log\log n} \log n\right\} \\ &< \exp\left\{2^{(1+\epsilon)\log n/\log\log n}\right\} \\ &= \exp\left\{n^{(1+\epsilon)\log 2/\log\log n}\right\}, \end{aligned}$$

where we have used Wigert's estimation[3] of $d(n)$. Thus (B) is proved.

Added in proof: In a paper to be published in Portugaliae Mathematica, Erdös has given a proof of (A) (for some positive c_{13} and infinitely many n).

INSTITUTE FOR ADVANCED STUDY

---

[3] Cf. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, pp. 219–222, or S. Ramanujan, *Collected papers*, pp. 44–46.

ON THE EQUICONVERGENCE OF FOURIER SERIES AND FOURIER INTEGRALS

FÉLIX EDUARDO HERRERA

Given any function $f(x)$ defined for $-\infty < x < \infty$, let

$$\int_0^\infty \{a(u)\cos ux + b(u)\sin ux\}\,du \tag{1}$$

be the Fourier integral (F.i.) of $f(x)$. Here

$$a(u) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos ut\,dt; \qquad b(u) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\sin ut\,dt \tag{2}$$

are the cosine and sine transforms of $f(x)$. They exist as absolutely convergent integrals if $f\in L(-\infty, \infty)$, or in a certain generalized sense, if $f\in L^p(-\infty, \infty)$ with $1<p\leqq 2$.

In all these cases, the partial integrals of (1) are given by the formula

$$\begin{aligned} S_\omega(x, f) &= \int_0^\infty \{a(u)\cos ux + b(u)\sin ux\}\,du \\ &= \frac{1}{\pi}\int_{-\infty}^\infty f(x+t)\frac{\sin \omega t}{t}\,dt, \end{aligned}$$

but the last integral has meaning if

$$\int_{-\infty}^\infty \frac{|f(t)|}{1+|t|}\,dt < \infty \tag{3}$$

even if the transforms (2) do not exist.

Now, it is well known that the problem of the representation of $f(x)$ as the limit of

$$S_\omega(x, f) = \frac{1}{\pi}\int_{-\infty}^\infty f(x+t)\frac{\sin \omega t}{t}\,dt \tag{3a}$$

for $\omega\rightarrow\infty$ can be reduced to the representation of $f(x)$ by a Fourier series (F.s.). More precisely, we have the following theorem:

(A) *Given any function satisfying* (3), *and any interval* $I_a = (a, a+2\pi)$ *of length* 2π, *let* $f_a(x)$ *be the function of period* 2π *and coinciding with* $f(x)$ *in* I_a. *Let*

Received by the editors September 29, 1948.

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{4}$$

be the F.s. of $f(x)$ *and*

$$s_\omega(x, f_a) = \frac{1}{2}a_0 + \sum_{n \leqq \omega}(a_n \cos nx + b_n \sin nx), \qquad \omega \geqq 0,$$

the partial sums of (4). *Then the difference*

$$\Delta_\omega(x) = \Delta_\omega(x, f, a) = S_\omega(x, f) - s_\omega(x, f_a) \tag{5}$$

tends to zero at every point inside I_a, *and the convergence is uniform in every closed interval interior to* I_a.[1]

Thus, tests of convergence which are valid for F.s. are immediately extensible to F.i. Moreover, since the convergence of (5) to zero implies its summability (C, α) for any $\alpha > 0$, we immediately obtain for the summability of F.i. results analogous to those for F.s.

Now, let

$$\int_0^\infty \{a(u) \sin ux - b(u) \cos ux\} du \tag{6}$$

be the integral conjugate to (1). The partial integrals of (6) can be written

$$\tilde{S}_\omega(x, f) = -\frac{1}{\pi}\int_{-\infty}^{\infty} f(x + t)\,\frac{1 - \cos \omega t}{t}\,dt \tag{6a}$$

if $f \in L^p(-\infty, \infty)$ with $1 \leqq p \leqq 2$, but again the right-hand side here has meaning only if (3) is satisfied. The following result completes (A):

(B) *Let*

$$\tilde{s}_\omega(x, f_a) = \sum_{n \leqq \omega}(a_n \sin nx - b_n \cos nx), \qquad \omega \geqq 1.$$

On the assumptions of (A), *the difference*

$$\tilde{\Delta}_\omega(x) = \tilde{\Delta}_\omega(x, f, a) = \tilde{S}_\omega(x, f) - \tilde{s}_\omega(x, f_a)$$

tends to a limit (*not necessarily zero*) *at every point* x *interior to* I_a, *and the convergence is uniform over any closed interval interior to* I_a.[2]

---

[1] See Zygmund, *Trigonometrical series*, p. 306. The book will henceforth be quoted T.S.

[2] Pi Calleja, P., *Über die Konvergenzbedingungen der komplexen Form des Fourierschen Integralen*, Math. Zeit. vol. 40 (1935–1936).

In a number of problems of the theory of F.s., we have to consider the behaviour of a F.s. differentiated term by term k times. A typical result in this order of ideas is the following:

(C) *Let* $f_{(k)}(x)$ *be the kth generalized derivative of* $f(x)$. *If* $f_{(k)}(x_0)$ *exists, the F.s. of* $f(x)$, *differentiated term by term k times, is, at the point* x_0, *summable* (C, α), $\alpha > k$, *to the value* $f_{(k)}(x_0)$.[3]

Similar problems exist for F.i.

In the present paper we prove a theorem that enables us to reduce a F.i. problem of the aforementioned kind to the corresponding problem on F.s., namely:

THEOREM. *Let* $f(x)$ *satisfy condition* (3). *Let* $I_a = (a, a+2\pi)$ *and let* $f_a(x)$ *be the function of period* 2π *coinciding with* $f(x)$ *in* I_a. *Let* $S_\omega(x, f)$ *be the partial integral* (3a) *and let* $s_\omega(x, f_a)$ *be the partial sums of the F.s. of* $f_a(x)$. *Let* $k = 1, 2, \cdots$. *Then the difference*

$$\frac{d^k}{dx^k} S_\omega(x, f) - \frac{d^k}{dx^k} s_\omega(x, f_a)$$

is uniformly summable (C, k) *to zero in every closed interval interior to* I_a.

Similarly, the difference

$$\frac{d^k}{dx^k} \tilde{S}_\omega(x, f) - \frac{d^k}{dx^k} \tilde{s}_\omega(x, f_a)$$

is uniformly summable (C, k) *in every closed interval interior to* I_a.

This theorem complements the results (A) and (B) stated above and is reduced to these results when $k=0$. An immediate application of it leads us, when (C) is taken into account, to:

(C') *If* $f_{(k)}(x_0)$ *exists, the F.i. of* $f(x)$, *differentiated k times under the integral sign, is, at the point* x_0, *summable* (C, α), $\alpha > k$, *to the value* $f_{(k)}(x_0)$.

In this connection we may recall the definition of summability of a series by M. Riesz's method of order α or summability (R, α) and the fact that if a series is summable to sum s by the method (R, α), it is also summable to the same sum by the method (C, α) and conversely.[4]

We shall base the proof of the theorem on the following lemmas:

(i) *Let* $g(t)$ *be an integrable function in* $(-\pi, \pi)$ *and of period* 2π. *Let* $r > 1$ *and* $0 < \delta < \pi$. *Then the integral*

---

[3] T.S. p. 257.

[4] T.S. p. 312.

$$\int_{|t|\geqq\pi}\frac{g(t)}{(x-t)^r}\,dt$$

converges absolutely and uniformly for $-\pi+\delta\leqq x\leqq\pi-\delta$.

The proof of this lemma is immediate.

(ii) *Let* $0<\eta<1/2$. *Under the conditions of Lemma* (i), *the integral*

$$\left\{\int_{\pi}^{\infty}+\int_{-\infty}^{-\pi}\right\}\frac{g(t)}{x-t}\,e^{i\omega(x-t)}dt = \lim_{T\to\infty}\left\{\int_{\pi}^{T}+\int_{-T}^{-\pi}\right\}\frac{g(t)}{x-t}\,e^{i\omega(x-t)}dt$$

converges:

(a) *uniformly for* $-\pi+\delta\leqq x\leqq\pi-\delta$ *and any fixed* ω,

(b) *uniformly for* $-\pi+\delta\leqq x\leqq\pi-\delta$, $\eta\leqq\omega\leqq 1-\eta$,

(c) *boundedly for* $-\pi+\delta\leqq x\leqq\pi-\delta$ *and any range of variation of* ω.

Clearly it is sufficient to suppose that $T=(2n+1)\pi$. On such an assumption we have

$$\left\{\int_{\pi}^{T}+\int_{-T}^{-\pi}\right\}\frac{g(t)}{x-t}\,e^{i\omega(x-t)}dt = \sum_{k=1}^{n}\int_{-\pi}^{\pi}g(t)e^{i\omega(x-t)}\left\{\frac{e^{2k\pi i\omega}}{2k\pi+(x-t)}-\frac{e^{-2k\pi i\omega}}{2k\pi-(x-t)}\right\}dt.$$

By direct inspection of the right-hand side of this formula we get (a) in the special case $\omega=$ an integer. Now, let ω be non-integral. Summation by parts gives

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{e^{\pm 2k\pi i\omega}}{2k\pi\pm(x-t)} &= \sum_{k=1}^{\infty}\left\{\sum_{\nu=1}^{k}e^{\pm 2\nu\pi i\omega}\right\}\left\{\frac{1}{2k\pi\pm(x-t)}-\frac{1}{2(k+1)\pi\pm(x-t)}\right\}\\ &= \frac{2\pi e^{\pm 2\pi i\omega}}{1-e^{\pm 2\pi i\omega}}\sum_{k=1}^{\infty}\frac{1-e^{\pm 2k\pi i\omega}}{\{2k\pi\pm(x-t)\}\{2(k+1)\pi\pm(x-t)\}}\end{aligned}$$

so that

$$\left|\sum_{k=1}^{\infty}\frac{e^{\pm 2k\pi i\omega}}{2k\pi\pm(x-t)}\right|<\frac{1}{|1-e^{\pm 2\pi i\omega}|}\left(\frac{2}{\delta}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\right),$$

and from this result (a) and (b) follow.

To obtain (c), let us consider the expression

$$\sum_{k=1}^{n}\left\{\frac{e^{2k\pi i\omega}}{2k\pi+(x-t)}-\frac{e^{-2k\pi i\omega}}{2k\pi-(x-t)}\right\}$$
$$=2i\sum_{k=1}^{n}\frac{\sin 2k\pi\omega}{2k\pi-(x-t)^2/2k\pi}-2(x-t)\sum_{k=1}^{n}\frac{\cos 2k\pi\omega}{4k^2\pi^2-(x-t)^2}$$
$$=2iP_n(\omega,x-t)-2(x-t)Q_n(\omega,x-t).$$

Obviously $\lim_{n\to\infty} Q_n(\omega, x-t)$ exists uniformly for $-\pi+\delta\leqq x\leqq\pi-\delta$ and any range of variation of ω. Furthermore the difference

$$P_n(\omega,x-t)-\sum_{k=1}^{n}\frac{\sin 2k\pi\omega}{2k\pi}$$
$$=(x-t)^2\sum_{k=1}^{n}\frac{\sin 2k\pi\omega}{(2k\pi)^2\{2k\pi-(x-t)^2/2k\pi\}},$$

as $n\to\infty$, converges also uniformly for $-\pi+\delta\leqq x\leqq\pi-\delta$ and any range of variation of ω. Since it is known that the series $\sum_{k=1}^{\infty}\sin 2k\pi\omega/2k\pi$ is boundedly convergent in any interval of ω, (c) is proved.

(iii) *Let* $v\neq 0$. *Then for* $q=1, 2, \cdots; k=1, 2, \cdots,$

$$\frac{d^q}{dv^q}\int_0^{\Omega}\left(1-\frac{\omega}{\Omega}\right)^k e^{i\omega v}d\omega$$
$$=(-1)^q\frac{q!}{v^{q+1}}i+\sum_{\nu=0}^{k-2}\frac{(-1)^q k(k-1)\cdots(k-\nu)(\nu+1+q)!}{i^{\nu}(\nu+1)!\,\Omega^{\nu+1}v^{\nu+q+2}}$$
$$(7)\qquad +\sum_{\nu=0}^{q-1}\frac{(-1)^{\nu}C_{q,\nu}(k+\nu)!\,e^{i\Omega v}}{i^{(k+\nu+1)-q}\Omega^{k+\nu-q}v^{k+\nu+1}}+\frac{(-1)^q(k+q)!}{i^{k+1}\Omega^k v^{k+q+1}}\left(e^{i\Omega v}-1\right)$$

where we put $\sum_{\nu=0}^{-1}\psi(\nu)=0$.[5]

We now need to prove that

$$(8)\qquad \int_0^{\Omega}\left(1-\frac{\omega}{\Omega}\right)^k e^{i\omega v}d\omega=-\frac{1}{iv}-\sum_{\nu=0}^{k-2}\frac{k(k-1)\cdots(k-\nu)}{\Omega^{\nu+1}(iv)^{\nu+2}}+\frac{k!}{\Omega^k}\frac{e^{i\Omega v}-1}{(iv)^{k+1}},$$

because by differentiation of this formula with respect to v we obtain the preceding one.

Now, (8) is certainly true for $k=1$. Let us assume that it is true for

---

[5] $\psi(\nu)$ stands in general for any function of the non-negative integer ν appearing in this paper.

$k=n$. We shall prove that on this hypothesis the formula holds also true for $k=n+1$. Indeed,

$$\begin{aligned}\int_0^\Omega \left(1-\frac{\omega}{\Omega}\right)^{n+1} & e^{i\omega v} d\omega \\ &= \left[\left(1-\frac{\omega}{\Omega}\right)^{n+1}\frac{e^{i\omega v}}{iv}\right]_0^\Omega + \frac{n+1}{iv\Omega}\int_0^\Omega \left(1-\frac{\omega}{\Omega}\right)^{n} e^{i\omega v}d\omega \\ &= -\frac{1}{iv} - \sum_{\nu=0}^{n-1}\frac{(n+1)n\cdots(n+1-\nu)}{\Omega^{\nu+1}(iv)^{\nu+2}} + \frac{(n+1)!}{\Omega^{n+1}}\frac{e^{i\Omega v}-1}{(iv)^{n+2}},\end{aligned}$$

which was to be proved.

In particular, if $q=k$ and A_ν, B_ν, and C are constants, (7) can be written in the form

$$\frac{d^k}{dv^k}\int_0^\Omega \left(1-\frac{\omega}{\Omega}\right)^k e^{i\omega v}d\omega = (-1)^k\frac{k!}{v^{k+1}}\,i + \Phi_\Omega(k, v) \tag{9}$$

where

$$\begin{aligned}\Phi_\Omega(k, v) = \sum_{\nu=0}^{k-2}\frac{A_\nu}{\Omega^{\nu+1}v^{\nu+k+2}} &+ \sum_{\nu=0}^{k-1}\frac{B_\nu}{\Omega^{\nu}v^{\nu+k+1}}\,e^{i\Omega v} \\ &+ \frac{C}{\Omega^k v^{2k+1}}\,(e^{i\Omega v}-1).\end{aligned} \tag{10}$$

These lemmas having been proved, we pass to the proof of the theorem. For the sake of simplicity, let us assume $a=0$ and let us write f_0, J_0, J_0' instead of f_a, J_a, J_a'. Moreover let us put

$$\overset{*}{s}_\omega(x) = \overset{*}{s}_\omega(x, f_0) = \frac{a_0}{2} + \sum_{n\leqq\omega}{}'(a_n\cos nx + b_n \sin nx), \quad \omega \geqq 0,$$

where the dash ′ on the right indicates that if ω is an integer, we must multiply the last term by one-half. Thus, for ω positive and not integral,

$$\overset{*}{s}_\omega(x) = s_\omega(x),$$

and therefore instead of considering the (C, α) means of the difference $S_\omega(x)-s_\omega(x)$, we may consider the (C, α) means of the difference $S_\omega(x)-\overset{*}{s}_\omega(x)$.

Now, it is well known that if $g(t)$ is any periodic function of period 2π, integrable in the interval $(0, 2\pi)$, then[6]

---

[6] G. H. Hardy, *Notes on some points in the integral calculus*, Messenger of Mathematics vol. 52 (1922) pp. 49–53.

$$s_\omega^*(x, g) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\,\frac{\sin \omega(x-t)}{x-t}\,dt = \lim_{T\to\infty}\frac{1}{\pi}\int_{-T}^{T} g(t)\,\frac{\sin \omega(x-t)}{x-t}\,dt.$$

Consequently, the first part of the theorem will be proved if we prove that, as $\Omega \to \infty$,

$$\frac{k}{\pi\Omega^k}\int_0^\Omega (\Omega-\omega)^{k-1}d\omega\,\frac{d^k}{dx^k}\int_{-\infty}^{\infty}\{f(t)-f_0(t)\}\,\frac{\sin\omega(x-t)}{x-t}\,dt \to 0$$

uniformly in J_0'.

For that purpose, let $h(t) = f(t) - f_0(t)$, so that $h(t) = 0$ in $(-\pi, \pi)$, and let us consider the integral

$$J_\Omega(k, x) = \frac{k}{\pi\Omega^k}\int_0^\Omega (\Omega-\omega)^{k-1}d\omega\,\frac{d^k}{dx^k}\int_{|t|\geqq\pi} h(t)\,\frac{\sin\omega(x-t)}{x-t}\,dt.$$

In virtue of assumption (3) and lemmas (i) and (ii), the integral

$$\int_{|t|\geqq\pi} h(t)\left(\frac{d^k}{dx^k}\,\frac{e^{i\omega(x-t)}}{x-t}\right)dt$$

is, for every fixed ω, uniformly convergent in J_0', so that we may write, $x \in J_0'$,

$$J_\Omega(k, x) = \frac{k}{\pi\Omega^k}\int_0^\Omega (\Omega-\omega)^{k-1}d\omega\int_{|t|\geqq\pi} h(t)\left(\frac{d^k}{dx^k}\,\frac{\sin\omega(x-t)}{x-t}\right)dt$$

and also

$$J_\Omega(k, x) = \frac{1}{\pi}\int_{|t|\geqq\pi} h(t)dt\,\frac{k}{\Omega^k}\int_0^\Omega (\Omega-\omega)^{k-1}\left(\frac{d^k}{dx^k}\cdot\frac{\sin\omega(x-t)}{x-t}\right)d\omega,$$

the inversion of integrations being justified by (3) and lemmas (i) and (ii).

Integrating the inner integral by parts, we get

$$\begin{aligned} J_\Omega(k, x) &= \frac{1}{\pi}\int_{|t|\geqq\pi} h(t)dt\int_0^\Omega\left(1-\frac{\omega}{\Omega}\right)^k\left(\frac{d^k}{dx^k}\cos\omega(x-t)\right)d\omega \\ &= \frac{1}{\pi}\int_{|t|\geqq\pi} h(t)dt\,\frac{d^k}{dx^k}\int_0^\Omega\left(1-\frac{\omega}{\Omega}\right)^k\cos\omega(x-t)d\omega \\ &= \frac{1}{\pi}\Re\left\{\int_{|t|\geqq\pi} h(t)dt\,\frac{d^k}{dx^k}\int_0^\Omega\left(1-\frac{\omega}{\Omega}\right)^k e^{i\omega(x-t)}d\omega\right\} \end{aligned}$$

or, according to (9),

$$J_\Omega(k, x) = \frac{1}{\pi}\Re\left\{\int_{|t|\geq\pi} h(t)\Phi_\Omega(k, x-t)dt\right\};$$

Let us consider the integral $\int_{|t|\geq\pi} h(t)\Phi_\Omega(k, x-t)dt$ and let us suppose we have replaced $\Phi_\Omega(k, x-t)$ by its expression (10). If so, we shall obtain integrals belonging to any one of the following types:

$$\frac{1}{\Omega^s}\int_{|t|\geq\pi}\frac{h(t)}{(x-t)^r}dt; \quad \frac{1}{\Omega^s}\int_{|t|\geq\pi}\frac{h(t)}{(x-t)^r}e^{i\Omega(x-t)}dt;$$
$$\int_{|t|\geq\pi}\frac{h(t)}{(x-t)^2}e^{i\Omega(x-t)}dt$$

where s and r are positive integers such that $s\geq 1$, $r\geq 3$.

Clearly every integral belonging to either of the two first types tends to zero uniformly in J_0' as $\Omega\rightarrow\infty$. As for $\int_{|t|\geq\pi}(h(t)/(x-t)^2)e^{i\Omega(x-t)}dt$, we have

$$\int_{|t|\geq\pi}\frac{h(t)}{(x-t)^2}e^{i\Omega(x-t)}dt = \left\{\int_{A\geq|t|\geq\pi} + \int_{|t|\geq A}\right\}\frac{h(t)}{(x-t)^2}e^{i\Omega(x-t)}dt$$
$$= I_1 + I_2.$$

If we choose A large enough, by (3) and Lemma (i), $|I_2| < \epsilon/2$ for any $x\in J_0'$. On the other hand, $1/(x-t)^2$ being for $x\in J_0'$, $\pi\leq|t|\leq A$ a uniformly bounded and uniformly continuous function depending on the parameter x, $I_1\rightarrow 0$ as $\Omega\rightarrow\infty$ uniformly in J_0', so that

$$\int_{|t|\geq\pi}\frac{h(t)}{(x-t)^2}e^{i\Omega(x-t)}dt\rightarrow 0$$

uniformly in J_0' and the first part of the theorem is proved. For the second part, we begin by putting

$$\tilde{s}_\omega^*(x) = \tilde{s}_\omega^*(x, f_0) = \sum_{n\leq\omega}{}'(a_n \sin nx - b_n \cos nx), \qquad \omega\geq 1,$$

where the dash $'$ on the right has the same meaning as before. Now, by analogy with $s_\omega^*(x)$, the expression $\tilde{s}_\omega^*(x)$ can be written in the form

$$\tilde{s}_\omega^*(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} f_0(t)\frac{1-\cos\omega(x-t)}{x-t}dt$$
$$= -\lim\frac{1}{\pi}\int_{-T}^{T} f_0(t)\frac{1-\cos\omega(x-t)}{x-t}dt,$$

$$J_\Omega(k, x) = \frac{1}{\pi}\Re\left\{\int_{|t|\geqq\pi} h(t)\Phi_\Omega(k, x-t)dt\right\}.$$

Let us consider the integral $\int_{|t|\geqq\pi} h(t)\Phi_\Omega(k, x-t)dt$ and let us suppose we have replaced $\Phi_\Omega(k, x-t)$ by its expression (10). If so, we shall obtain integrals belonging to any one of the following types:

$$\frac{1}{\Omega^s}\int_{|t|\geqq\pi}\frac{h(t)}{(x-t)^r}dt;\quad \frac{1}{\Omega^s}\int_{|t|\geqq\pi}\frac{h(t)}{(x-t)^r}e^{i\Omega(x-t)}dt;$$

$$\int_{|t|\geqq\pi}\frac{h(t)}{(x-t)^2}e^{i\Omega(x-t)}dt$$

where s and r are positive integers such that $s\geqq 1$, $r\geqq 3$.

Clearly every integral belonging to either of the two first types tends to zero uniformly in J_0' as $\Omega\to\infty$. As for $\int_{|t|\geqq\pi}(h(t)/(x-t)^2)e^{i\Omega(x-t)}dt$, we have

$$\int_{|t|\geqq\pi}\frac{h(t)}{(x-t)^2}e^{i\Omega(x-t)}dt = \left\{\int_{A\geqq|t|\geqq\pi}+\int_{|t|\geqq A}\right\}\frac{h(t)}{(x-t)^2}e^{i\Omega(x-t)}dt$$
$$= I_1 + I_2.$$

If we choose A large enough, by (3) and Lemma (i), $|I_2|<\epsilon/2$ for any $x\in J_0'$. On the other hand, $1/(x-t)^2$ being for $x\in J_0'$, $\pi\leqq|t|\leqq A$ a uniformly bounded and uniformly continuous function depending on the parameter x, $I_1\to 0$ as $\Omega\to\infty$ uniformly in J_0', so that

$$\int_{|t|\geqq\pi}\frac{h(t)}{(x-t)^2}e^{i\Omega(x-t)}dt\to 0$$

uniformly in J_0' and the first part of the theorem is proved. For the second part, we begin by putting

$$\tilde{s}^*_\omega(x) = \tilde{s}^*_\omega(x, f_0) = \sum_{n\leqq\omega}{}'(a_n\sin nx - b_n\cos nx),\qquad \omega\geqq 1,$$

where the dash $'$ on the right has the same meaning as before. Now, by analogy with $s^*_\omega(x)$, the expression $\tilde{s}^*_\omega(x)$ can be written in the form

$$\tilde{s}^*_\omega(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} f_0(t)\frac{1-\cos\omega(x-t)}{x-t}dt$$
$$= -\lim\frac{1}{\pi}\int_{-T}^{T} f_0(t)\frac{1-\cos\omega(x-t)}{x-t}dt,$$

A NOTE OF CORRECTION

A. A. ALBERT

Dr. Douglas McClay has called my attention to the existence of a two-dimensional commutative absolute-valued real algebra and so to the error of omission of dimension two in Theorem 3 of my paper entitled *Absolute-valued algebraic algebras* (Bull. Amer. Math. Soc. vol. 55 (1949) pp. 763–768). The error is a corollary of the same error in Theorem 3 of my paper *Absolute-valued real algebras* (Ann. of Math. vol. 48 (1947) pp. 495–501) and is due to the omission of the factor of reflection in considering two-rowed orthogonal matrices. The algebra of McClay is actually isomorphic to the commutative isotope of the complex field obtained by defining the products $(1, 1)=1$, $(i, i)=-1$, $(1, i)=-i$.

The University of Chicago

Received by the editors November 5, 1949.

ERRATA, VOLUME 55

Alfred Tarski, *Arithmetical classes and types of mathematical systems.* Abstract 55-1-74.

p. 63, line 12 of the abstract. For "C" read "AC."

Alfred Tarski, *Metamathematical aspects of arithmetical classes and types.* Abstract 55-1-75.

p. 63, lines 2 and 7 of the abstract. For "54-11-74" read "55-1-74."

p. 63, line 6 of the abstract. For "$(\bar{L}\in AC)$" read "(there is a smallest $M\in AC$ with $L\subseteq M$)."

Alfred Tarski, *Arithmetical classes and types of Boolean algebras.* Abstract 55-1-76.

p. 64, line 1 of the abstract. For "54-11-74 and 54-11-75" read "55-1-74 and 55-1-75."

Alfred Tarski, *Arithmetical classes and types of algebraically closed and real-closed fields.* Abstract 55-1-77.

p. 64, line 1 of the abstract. For "54-11-74, 54-11-75" read "55-1-74, 55-1-75."

Andrzej Mostowski and Alfred Tarski, *Arithmetical classes of types of well ordered systems.* Abstract 55-1-78.

p. 65, line 1 of the abstract. For "54-11-74, 54-11-75" read "55-1-74, 55-1-75."

p. 65, line 9 of the abstract. For "has a well ordered basis of the type ω^{ω}" read "has no well ordered basis."

p. 65, line 14 of the abstract. For "$\omega+\omega^{*}\cdot+\tau(\mathfrak{A})$" read "$\omega+\omega^{*}+\tau(\mathfrak{A})$."

Wanda Szmielew, *Arithmetical classes and types of Abelian groups.* Abstract 55-1-79.

p. 65, line 2 of the abstract. For "54-11-74, 54-11-75" read "55-1-74, 55-1-75.

Andrzej Mostowski, *A general theorem concerning the decision problem.* Abstract 55-1-80.

p. 66, line 7 of the abstract. For "54-11-75" read "55-1-75."

Stefan Bergman and M. M. Schiffer, *Kernel functions and conformal mapping.* II. Abstract 55-5-239.

p. 515, line 3 of the abstract. For "$\overline{\phi'(\xi)}$" read "$\phi'(\xi)$."

H. W. Becker, *A new proof and application of Kirchhoff's rules.* Abstract 55-7-437.

p. 718, lines 5 and 6. For "modes" read "nodes."

Alfred Tarski, *A fixpoint theorem for lattices and its applications.* Preliminary report. Abstract 55-11-496.

p. 1052, line 1. For "$\rightarrow$" read "$\geqq$".

INDEX OF VOLUME 55

INDEX OF ABSTRACTS

Subjects:

Algebra and Theory of Numbers, 39, 55, 61, 271, 510, 522, 667, 686, 707, 715, 871, 1042.

Analysis, 41, 55, 66, 279, 514, 524, 669, 689, 708, 716, 871, 1053.

Applied Mathematics, 48, 57, 69, 291, 519, 525, 673, 698, 710, 718, 1068.

Geometry, 50, 58, 70, 297, 526, 674, 699, 712, 719, 1072.

Logic and Foundations, 50, 70, 298, 526, 702, 875, 1074.

Numerical and Graphical Methods, 71.

Statistics and Probability, 50, 71, 300, 520, 676, 702, 713, 1075.

Topology, 52, 58, 301, 521, 527, 676, 703, 713, 875, 1078.

Authors:

Abbott, J. C., 1042; Abramowitz, M., 48, 291, 698; Aczel, J., 689; Albert, A. A., 686; Ames, D. B., 514; Anderson, R. D., 301, 301, 1078; Andree, R. V., 1043; Apostol, T. M., 715; Aronszajn, N., 279.

Bagemihl, F., 279; Barlaz, J., 1053; Barnett, I. A., 1043; Bateman, P. T., 686, 686, 686, 1043; Beckenbach, E. F., 41, 280, 1053; Becker, H. W., 69, 718, 718, 1192; Beesley, E. M., 716; Begle, E. G., 58; Bell, J. H., 1044; Bell, P. O., 712, 712; Bellman, R., 510, 510, 510, 511, 686; Benac, T. J., 1042; Bergman, S., 515, 515, 515, 515, 1192; Bernhart, A., 707; Bernstein, F., 669; Bers, L., 516, 690, 1053; Bing, R. H., 58, 713, 1078; Birkhoff, G., 41, 42, 48, 871; Blakers, A. L., 703; Blankinship, W. A., 703; Blau, J. H., 704; Block, H. D., 280, 1053; Blumberg, H., 298; Blumen, I., 676; Blumenthal, L. M., 271; Boas, R. P., 55; Bochner, S., 690; Boldyreff, A. W., 1044; Boothby, W. M., 690; Bott, R., 291; Bourgin, D. G., 690, 704; Brauer, A. T., 272, 667; Brown, B., 272; Buchi, J. R., 1044; Buck, R. C., 687, 687.

Cairns, S. S., 301; Calabi, E., 280; Calderón, A. P., 1054, 1054; Calkin, J. W., 280; Cameron, R. H., 281; Carlitz, L., 272, 272, 667, 667, 687, 1044; Carruth, P. W., 42; Cashwell, E. D., 708; Castellani, M., 1075; Cesari, L., 716; Chandrasekharan, K., 690; Chang, S. H., 871, 872, 872, 872; Charnes, A., 704; Chatland, H., 55; Chen, Y. W., 292; Chernoff, H., 1075; Cherry, T. M., 516; Chin, L. H., 61; Chowla, S., 273, 273, 273, 273, 274, 511, 511, 686, 686, 686; Chung, K. L., 702, 1076; Civin, P., 66; Clark, R. A., 673; Clarke, F. M., 274, 274, 707; Coburn, N., 1068; Cohen, A. C., 676; Cohen, C., 512, 698; Cohen, E., 274; Cohen, H. J., 713; Cohen, L. W., 52, 56, 281; Cohen, H., 48, 274, 522; Conrad, K. L., 1069; Copeland, A. H., 698; Court, N. A., 297, 1072; Cowling, V. F., 281, 873; Craig, H. V., 56, 674; Curry, H. B. 299, 299, 299.

Darling, D. A., 300, 702, 703; Davenport, H., 55; Day, M. M., 1054, 1054; DeCicco, J., 50, 526, 526, 699, 700, 712; Dekker, D. B., 719; Dietrich, V. E., 55; Dilworth, R. P., 62, 63, 301, 1079; Dolciani, M. P., 39, 687; Dolph, C. L., 713, 1076; Dowker, C. H., 301; Doyle, T. C., 519; Dresden, M., 1069; Dresher, M., 275; Dubisch, R., 715, 1045; Duffin, R. J., 282, 282, 291, 292; Dvoretzky, A., 42, 50, 280, 1055.

Eberlein, W. F., 292, 1055; Eichler, M. M. E., 282; Elliott, H. M., 43, 43, 47, 283, 874; Ellis, D. O., 271; Engel, J. H., 707; Erdös, P., 1076; Estill, M. E., 875; Evans, G. C., 1055; Everett, C. J., 51, 51, 51; Ewing, G. M., 709.

Fan, K., 59, 59; Farnell, A. B., 1059; Farrell, O. J., 691, 691; Federer, H., 688; Fejér, L., 1055; Ficken, F. A., 674; Fine, N. J., 275; Finkbeiner, D. T., 62; Fisher, R. C., 712; Floyd, E. E., 704; Forsythe, G. E., 66, 293; Fort, M. K., 59, 59, 527; Foster, A. L., 39; Fox, R. H., 52, 276, 512, 521, 704; Frame, J. S., 1045; Frank, E., 283; Freilich, G., 691; Freundlich, M. R., 302; Fuchs, W. H. J., 283; Fullerton, R. E., 302.

Gadsden, C. P., 669; Galbraith, A. S., 43; Gale, D., 512, 512, 688, 1045; Garabedian, P. R., 67; Gentry, I. C., 676; Germond, H. H., 1076; Gilbarg, D., 525; Givens, W., 1046; Gleason, A. M., 283, 1079; Goffman, C., 52, 56, 281; Goodman, A. W., 284; Goldberg, M., 297; Goodner, D. B., 1056; Gorciu, V. G., 302; Goss, R. N., 710, 1069; Gottschalk, W. H., 1079; Graves, L. M., 1056; Green, J. W., 284, 302, 720; Grove, V. G., 58, 1073; Guenther, P. E., 673; Gustin, W., 284, 302, 524, 1056; Guy, W. T., 56.

Hall, M., 276, 300; Hamilton, O. H., 714; Harary, F., 55, 276, 276, 277; Harrold, O. G., 302, 677; Heins, A. E., 698; Helsel, R. G., 284; Herstein, I. N., 523, 707; Herzog, F., 1056; Hestenes, M. R., 67; Hewitt, E., 303, 1057, 1057; Higman, G., 667; Hille, E., 1058, 1058; Hirschman, I. I., 516, 873; Hoel, P. G., 1076; Hoffman, A. J., 297; Hopf, E., 1069; Horn, A., 70; Hostinsky, L. A., 1046; Hua, L. K., 1046, 1046, 1047, 1047, 1047; Huff, G. B., 277, 674; Hughes, H. K., 709; Hutcherson, W. R., 1073; Hutchinson, L. C., 277; Hyman, M. A., 284.

Ingram, W. H., 692; Isaacson, E., 48.

Jacobson, F. D., 513; Jacobson, N., 513, 1047; James, R. D., 717; Jehle, H., 293; Jenkins, J. A., 285, 516, 873; Jerison, M., 1079; Jones, B. W., 277; Jónsson, B., 688.

Kac, M., 703; Kahal, R., 1070; Kakutani, S., 1058, 1059; Kaplan, Samuel, 677; Kaplan, Sidney, 519; Kaplansky, I., 278, 513; Karlin, M., 516; Karlin, S., 524, 717; Kasner, E., 50, 526, 526, 699, 700, 700; Katz, L., 1077; Keesee, J. W., 677; Klee, V. L., 44, 44, 44, 669, 670, 670, 670, 670; Kober, H., 692; Koopman, B. O., 520; Krzywoblocki, M. Z., 57, 525, 711; Kuhn, Harold W., 512, 688, 1045; van der Kulk, W., 700.

Lakness, R. M., 67; Lanczos, C., 717; Lane, R. E., 285, 1059; Langenhop, C. E., 1059; Langer, R. E., 285, 294; Leavitt, W. G., 709; Lehmer, D. H., 715, 1047; Lehmer, E., 62; Lehner, J., 40, 286, 1060; Leighton, W., 718, 1060; Lepson, B., 1048; Leutert, W., 44; LeVeque, W. J., 688, 688; Levine, J., 675, 701; Levinson, N., 517, 517, 692, 693, 693, 693; Levit, R. J., 667; Levitzki, J., 513; Lieblein, J., 1077; Lin, C. C., 49, 49; Linial, S., 50; Loomis, L. H., 517; Lorch, E. R., 693; Lyndon, R. C., 513.

McCoy, N. H., 272; McKinsey, J. C. C., 278; McLaughlin, J., 63; McShane, E. J., 49, 303, 873; Macintyre, S. S., 44; Mackey, G. W., 286, 286, 694; MacLane, G. R., 287; Macon, N., 272; Magnus, W., 1048; Malin, H., 69; Mandelbrojt, S., 287; Mann, H. B., 276; Maple, C. G., 294; Martin, M. H., 673, 1070; Martin, W. T., 281; Massey, W. S., 303, 703, 1080; Mautner, F. I., 40, 52, 287; Mendel, C. W., 1043; Mersman, W. A., 718; Metropolis, N., 670; Min, S. H., 517; Mitchell, B. E., 675, 1073; Mitchell, J. M., 524, 709; Mittleman, D., 700, 701; Moise, E. E., 527; Montgomery, D., 60; Moore, C. N., 45, 287; Moore, E. F., 694; Morrey, C. B., 1060; Morrison, W. J. 1077; Mostowski, A., 65, 65, 1192; Motzkin, T. S., 40, 1073; Myers, S. B., 1080.

Nachbin, L., 40, 45, 873; Nehari, Z., 525; Nemenyi, P. F., 1070; Nesbeda, P., 671; Neumann, B. H., 667; Neumann, H., 667; Neustadter, S. F., 68; Nicholson, G. E., 676; Nickerson, H. K., 1061, 1061; Nikodým, O. M., 288, 1080, 1080; Niven, I., 715; Northam, J. I., 713; Norton, D. A., 1048; Novak, I. L., 300, 702.

O'Brien, G., 519; O'Donnell, R. E., 1061; Ostrowski, A. M., 719, 1061; Oxtoby, J. C., 694.

Pall, G., 278, 689; Parker, W. V., 668, 871; Parrish, H. C., 288; Payne, L. E., 1071; Peiser, A. M., 695; Perkins, F. W., 695; Perlis, S., 41, 1045; Peters, G. O., 671; Phillips, R. S., 69, 719; Pinney, E., 69, 719; Piranian, B., 1062; Piranian, G., 1056, 1062; Pitcher, E., 45; Pitman, E. J. G., 676; Pollard, H., 703; Pólya, G., 294, 294; Porcelli, P., 689; Post, E. L., 50; Price, C. M., 523.

Rademacher, H., 695; Rado, T., 1062; Ramanathan, K. G., 273, 523, 668, 1049; Ratner, L. T., 1080; Rauch, H. E., 518; Rauch, L. L., 1062; Reade, M. O., 288, 289, 297; Rechard, O. W., 56; Regan, F., 1063; Reichelderfer, P. V., 1062; Reid, W. T., 57, 1049, 1063; Reissner, E., 49, 295, 295, 519, 673, 674; Reynolds, C. N., 705, 1081; Ribeiro, H., 278; Rickart, C. E., 279, 279, 1047, 1049, 1050; Riggs, L. G., 1063; Ritt, J. F., 695; Ritter, E. K., 1063; Ritter, I. F., 519, Robbins, H. E., 676; Robinson, J., 70; Robinson, L. V., 289, 671; Robinson, R. M., 1050; Rogosinski, W. W., 1064; Room, T. G., 298; Rosenbloom, P. C., 46, 289, 689; Rosenthal, A., 55; Rosser, J. B., 875, 1050; Roth, W. E., 1050, 1050; Rothe, J. C., 874; Rubin, H., 1050, 1064, 1064; Rudin, W., 668, 671, 1064; Rust, C., 1063; Rutledge, W. A., 1046; Ryser, H. J., 511, 511, 871.

Salzer, H. E., 41, 57, 295; Sard, A., 695; Schaeffer, A. C., 282, 1065; Schaerf, H. M., 1065, 1065, 1065, 1066, 1066; Schafer, R. D., 689; Schatten, R., 289, 1066; Scherk, P., 1051; Schiffer, M. M., 515, 515, 515, 515, 1192; Schoenberg, I. J., 695, 696; Schoenfeld, L., 525; Schwartz, H. M., 290, 518, 672; Schweigert, G. E., 304; Schweitzer, A. R., 526, 527, 1074, 1074; Scott, W. T., 1063; Segal, I. E., 46, 46, 57; Seidel, W., 43, 280; Seth, B. R., 295, 711, 1071; Sewell, W. E., 47; Shanks, D., 696; Shapley, L. S., 717; Sherman, J., 1077; Sherman, S., 290, 514; Silverman, E., 518; Simmons, H. A., 523; Sinclair, A., 1066; Sips, R., 47; Smiley, M. F., 524, 1051; Smith, H. L., 1067; Snapper, E., 300; Snyder, W. S., 672; Specht, E. J., 1067; Spencer, D. E., 1071; Springer, C. E., 710; Springer, G. E., 290; Steel, R. G. D., 707; Steinberg, R., 514; Stephens, C. F., 697, 874; Stone, A. H., 52; Strehler, A. F., 708; Strodt, W., 697; Studley, D. M., 711, 1074, 1078; Sugar, A. C., 70, 296; Sumner, D. B., 697; Szasz, O., 47, 280, 290, 1067; Szegö, G., 294, 1055, 1064; Szekeres, G., 689; Szmielew, W., 65, 1075, 1192.

Tarski, A., 61, 63, 63, 64, 64, 65, 1051, 1075, 1192; Taylor, A. E., 68; Thielman, H. P., 291, 710, 1067; Thomas, G. B., 673; Thomas, J. M., 675; Thompson, F. B., 66; Thorne, C. J., 69, 71; Thrall, R. M., 1052; Todd, J., 1071; Tompkins, C., 520; Tornheim, L., 1052; Townsend, B. B., 674; Truesdell, C. A., 296, 699, 1072; Tsien, H. S., 49; Tucker, A. W., 512, 688, 1045; Turán, P., 274.

Ulam, S. M., 51, 51, 51, 670; Ullman, J. L., 291, 1067; Utz, W. R., 298, 1081.

Valentine, F. A., 720; Van Tuyl, A. H., 699, 1070; Vaught, R. L., 716; Vinograde B., 708.

Wade, L. I., 668; Walker, G. L., 41; Wall, H. S., 285; Walsh, J. E., 1078; Walsh, J. L., 43, 47, 874; Wang, H., 875; Ward, J. A., 669; Ward, M., 716; Weinstein, A., 294, 520, 699; White, P. A., 1081; Whitehead, G. W., 521; Whiteman, A.L., 68, 71; Whitney, A. M., 696; Whyburn, G. T., 60, 60, 304, 304, 305, 705; Whyburn, W. M., 672; Widder, D. V., 874; Wilcox, L. R., 1052; Wolfgang, W. G., 701, 702; Woodbury, M. A., 713, 1076.

Yood, B., 673.

Zeichner, A., 296; Zippen, L., 521; Zorn, M. A., 300; Zuckerman, H. S., 1057, 1057; Zygmund, A., 1068.

INDEX OF REPORTS OF MEETINGS AND MISCELLANEOUS ARTICLES

Churchill, R. V., and Youngs, J. W. T. *Third Symposium on Applied Mathematics*, 861.

Davenport, H. See Littlewood, J. E.

Dresden, A. *Report of Committee on Aid to Devastated Libraries*, 1.

Green, J. W. Reports of Meetings of the American Mathematical Society: *November Meeting in Los Angeles*, 61; *April Meeting in Stanford University*, 714; *Summer Meeting in Boulder*, 1035.

Hardy, G. H. See Wiener, N.
Hollcroft, T R. Reports of Meetings of the American Mathematical Society: *October Meeting in New York*, 38; *February Meeting in New York*, 509; *April Meeting in Durham*, 666; *April Meeting in Philadelphia*, 678.
Kline, J. R. *Memoirs of the American Mathematical Society*, 1013.
Littlewood, J. E., Pólya, G., Mordell, L. J., Titchmarsh, E. C., Davenport, H., and Wiener, N. *Two statements concerning the article on G. H. Hardy*, 1082.
Mordell, L. J. See Littlewood, J. E.
Pólya, G. See Littlewood J. E.
Titchmarsh, E. C., See Littlewood, J. E.
Wiener, N. *Godfrey Harold Hardy, 1877–1947*, 72.
—— See Littlewood, L. J.
Youngs, J. W. T. Reports of meetings of the American Mathematical Society: *November Meeting in Chicago*, 54; *Annual Meeting of the Society*, 261, *February Meeting in Chicago*, 522; *April Meeting in Lawrence*, 706.
—— See Churchill, R. V.

INDEX OF BOOK REVIEWS

Allendoerfer, C. B. See Coxeter, H. S. M.
Bass, J. *Les méthodes modernes du calcul des probabilités et leur application au problème de la turbulence.* M. Loève, 864.
Behari, R. *The differential geometry of ruled surfaces.* J. DeCicco, 313.
Bochner, S. See Hille, E.
Cambi, E. *Eleven and fifteen-place tables of Bessel functions of the first kind, to all significant orders.* E. R. Lorch, 78.
Coxeter, H. S. M. *Regular polytopes.* C. B. Allendoerfer, 721.
—— See Hodge, W. V. D.
DeCicco, J. See Behari, R.
Dieudonné, J. *Sur les groupes classiques.* E. R. Kolchin, 317.
Federer, H. See Hahn, H.
Fort, T. *Finite differences and difference equations in the real domain.* W. Strodt, 312.
Hahn, H., and Rosenthal, A. *Set functions.* H. Federer, 316.
Hlavaty, V. See Vranceanu, G.
Hille, E. *Functional analysis and semi-groups.* S. Bochner, 528.
Hodge, W. V. D., and Pedoe D. *Methods of algebraic geometry.* H. S. M. Coxeter 315.
Householder, A. S. See Rashevsky, N.
Kolchin, E. R. See Dieudonné, J.
Landahl, H. D. See Luneburg, R. K.
Loève, M. See Bass, J.
Lorch, E. R. See Cambi, E., Parke, N. G., Tables.
Luneburg, R. K. *Mathematical analysis of binocular vision.* H. D. Landahl, 866.
McCoy, N. H. *Rings and ideals.* C. C. MacDuffee, 864.
MacDuffee, C. C. See McCoy, N. H.
Parke, N. G. *Guide to the literature of mathematics and physics including related works on engineering science.* E. R. Lorch, 78.
Pedoe, D. See Hodge, W. V. D.
Rashevsky, N. *Mathematical theory of human relations. An approach to a mathematical biology of social phenomena.* A. S. Householder, 722.
Robinson, G. de B. See Rutherford, D. E.
Rosenthal, A. See Hahn, H.
Rutherford, D. E. *Substitutional analysis.* G. de B. Robinson, 1085.

Strodt, W. See Fort, T.
Tables of spherical Bessel functions. Vol. 2, E. R. Lorch, 78.
Tables of the Bessel functions of the first kind of orders four, five, six, seven, eight, and nine. E. R. Lorch, 79.
Tables of Bessel functions of fractional order. Vol. 1. E. R. Lorch, 79.
Thron, W. J. See Wall, H. S.
Vranceanu, G. *Leçons de géométrie différentielle*, V. Hlavaty, 867.
Wall, H. S. *Analytic theory of continued fractions.* W. J. Thron, 1083.

INDEX OF ADDRESSES[1]

Bergman, S., 510.
Besicovitch, A. S., 263.
Busemann, H., 61.
Cesari, L., 263.
Eilenberg, S. *Topological methods in abstract algebra. Cohomology theory of groups*, 3.
Halmos, P. R., 54.
—— *Measurable transformations*, 1015.
Hedlund, G. A., 1036.
Hestenes, M. R., 714.
Hille, E., 263.
Householder, A. S., 706.
James, R. D. *Recent progress in the Goldbach problem*, 246.
Jessen, B., 1036.
Jones, F. B., 1036.
Kac, M., 263.
—— *Probability methods in some problems of analysis and number theory*, 641.
Mackey, G. W., 679.
MacLane, S., 54.
Mickle, E. J., 706.
Myers, S. B., 522.
Reidemeister, K. W., 679.
Reissner, E. *Boundary value problems in aerodynamics of lifting surfaces in non-uniform motion*, 825.
Roberts, J. H., 666.
Salem, R., 39.
—— *Convexity theorems*, 851.
Thrall, R. M., 706.
Tukey, J. W., 706.
Weyl, H., 262.
Whitehead, J. H. C. *Combinatorial homotopy.* I, 213.
—— *Combinatorial homotopy.* II, 453.
Woodbury, M. A., 706.

INDEX OF RESEARCH PAPERS

Abdelhay, J. *On a theorem of representation*, 408.
Albert, A. A. *Absolute-valued algebraic algebras*, 763, 1191.
Ankeny, N. C. *A note on the minimum of a binary form*, 615.
Baer, R., and Williams, C. *Splitting criteria and extension types*, 729.

---

[1] Where title of address is not given, the reference is to the Report of the Meeting at which the address was given.

Bateman, P. T. *Note on the coefficients of the cyclotomic polynomial*, 1180.

Beckenbach, E. F., Gustin, W., and Shniad, H. *On the mean modulus of an analytic function*, 184.

Begle, E. G. *A note on S-spaces*, 577.

Bergman, S. *An initial value problem for a class of equations of mixed type*, 165.

Bing, R. H. *A convex metric for a locally connected continuum*, 812.

—— *Partitioning a set*, 1101.

Borel, A. *Some remarks about Lie groups transitive on spheres and tori*, 580.

Brauer, R. *On a theorem of H. Cartan*, 619.

Brunk, H. D. *A consistency theorem*, 204.

Bruynes, H., and Raisbeck, G. *A method of analytic continuation suggested by heuristic principles*, 193.

Cameron, R. H., and Hatfield, C. *On the summability of certain orthogonal developments of nonlinear functionals*, 130.

Campbell, R. C. *A simple solution of the diophantine equation* $x^3+y^3=z^2+t^3$, 442.

Carmichael, R. D. *Errata for On Euler's* ϕ *function*, 212.

Chatland, H. *On the Euclidean algorithm in quaratic number fields*, 948.

Chatland, H., and Mann, H. B. *Integral extensions of a ring*, 592.

Cheng, T. T. *The normal approximation to the Poisson distribution and a proof of a conjecture of Ramanujan*, 396.

Civin, P. *Approximation in* Lip (α, p), 794.

Cohn, R. M. *A theorem on difference polynomials*, 595.

—— *Inversive difference fields*, 597.

Dekker, D. B. *Hypergeodesic curvature and torsion*, 1151.

Dietrich, V. E., and Rosenthal, A. *Transcendence of factorial series with periodic coefficients*, 954.

Diliberto, S. P. *Special properties of measure preserving transformations*, 554.

Dilworth, R. P., and Ward, M. *Note on a paper by C. E. Rickart*, 1141.

Dowker, Y. N. *A note on the ergodic theorems*, 379.

Duffin, R. J. *Nonlinear networks*. III, 119.

Durfee, William H. See Jones, B. W.

Eckmann, B. *Coverings and Betti numbers*, 95.

Eckmann, B., Samelson, H., and Whitehead, G. W. *On fibering spheres by toruses*, 433.

Eilenberg, S. See Index of Addresses.

Ellis, D. *Superposibility properties of naturally metrized groups*, 639.

Erdös, P., Feller, W., and Pollard, H. *A property of power series with positive coefficients*, 201.

Feller, W. See Erdös, P.

Floyd, E. E. *A nonhomogeneous minimal set*, 957.

Fort, M. K. *A note on equicontinuity*, 1098.

Frank, E. *Orthogonality properties of C-fractions*, 384.

Frucht, R. *On the groups of repeated graphs*, 418.

Fullerton, R. E. *Integral distances in Banach spaces*, 901

Garabedian, P. R. *A problem of Robinson*, 917.

Gleason, A. M. *Square roots in locally Euclidean groups*, 446.

—— *A note on locally compact groups*, 744.

González, M. O. *On the solution of ordinary differential equations of the first order invariant under contact transformations*, 355.

Goodman, A. W. *Note on regions omitted by univalent functions*, 363.

Gustin. W. See Beckenbach, E. F.

Halmos, P. R. See Index of Addresses.

Hatfield, C. See Cameron, R. H.

Herrera, F. E. *On the equiconvergence of Fourier series and Fourier integrals*, 1182.

Hewitt, E. *A class of topological spaces*, 421.

Hille, E. *Remarks on a paper by Zeev Nehari*, 552.

Hillman, A. *On the reality of zeros of Bessel functions*, 198.

Horn, A. *Some generalizations of Helly's theorem on convex sets*, 923.

Hsiung, C. C. *Invariants of intersection of certain pairs of space curves*, 623.

Hsü, H. Y. *On Sonine's integral formula and its generalization*, 370.

Jacobson, A. W. *A generalized convolution for finite Fourier transformations*, 804.

James, R. D. See Index of Addresses.

Jones, B. W. *A theorem on integral symmetric matrices*, 620.

Jones, B. W., and Durfee, William H. *A theorem on quadratic forms over the ring of 2-adic integers*, 758.

Jones, F. B. *A note on homogeneous plane continua*, 113.

Kac, M. See Index of Addresses.

Kelly, P. J. *On Minkowski bodies of constant width*, 1147.

Kosambi, D. D. *The differential invariants of a two-index tensor*, 90.

Kuo, H. T. *A recurrence formula for* $\zeta(2n)$, 573.

Leavitt, W., and Whaples, G. *On matrices with elements in a principal ideal ring*, 117.

Leighton, W., *A substitute for the Picone formula*, 325.

Loomis, L. H., and Whitney, H. *An inequality related to the isoperimetric inequality*, 961.

Lu, C. K. *Classification of 2-manifolds with singular points*, 1093.

Mann, H. B. See Chatland, H.

Mark, A. M. *Some probability limit theorems*, 885.

Mickle, E. J. *On the extension of a transformation*, 160.

Mirsky, L. *On the frequency of pairs of square-free numbers with a given difference*, 936.

Moise, E. E. *A theorem on monotone interior transformations*, 810.

—— *Grille decomposition and convexification theorems for compact metric locally connected continua*, 1111.

Morse, M. *L-S-homotopy classes on the topological image of a projective plane*, 981.

Mostow, G. D. *A new proof of E. Cartan's theorem on the topology of semi-simple groups*, 969.

Motzkin, Th. *Approximation by curves of a unisolvent family*, 789.

—— *The Euclidean algorithm*, 1142.

Muhly, H. T. *The irregularity of an algebraic surface and a theorem on regular surfaces*, 940.

Myers, S. B. *Spaces of continuous functions*, 402.

Nachbin, L. *On strictly minimal topological division rings*, 1128.

Nagy, Gy. v. Sz. *Über die Lage der Nullstellen eines Abstandspolynoms und seiner derivierten*, 329.

Nehari, Z. *The Schwarzian derivative and schlicht functions*, 545.

Ostrom, T. G. *The solution of linear integral equations by means of Wiener integrals*, 343.

Parker, W. V. *On the characteristic equations of certain matrices*, 115.

Peixote, M. M. *Generalized convex functions and second order differential inequalities*, 563.

Pollard, H. See Erdős, P.

Radó, T. *On essentially absolutely continuous plane transfomations*, 629.

Raisbeck, G. See Bruynes, H.

Ramaswami, V. *On the number of positive integers less than* x *and free of prime divisors greater than* x^c, 1122.

Ree, R. *On a problem of Max A. Zorn*, 575.
Reissner, E. See Index of Addresses.
Rosenbaum, J. See Rosenbaum, R. A.
Rosenbaum, R. A., and Rosenbaum, J. *Some consequences of a well known theorem on conics*, 933.
Rosenthal, A. See Dietrich, V. E.
Rosser, J. B. *Real roots of Dirichlet L-series*, 906.
Salem, R. See Index of Addresses.
Salzer, H. E., and Zucker, R. *Table of the zeros and weight factors of the first fifteen Laguerre polynomials*, 1004.
Samelson, H. See Eckmann, B.
Schafer, R. D. *The Wedderburn principal theorem for alternative algebras*, 604.
—— *Inner derivations of non-associative algebras*, 769.
Scott, W. R. *Some elementary topological properties of essential maximal model continua*, 963.
Sewell, W. E. See Walsh, J. L.
Shapiro, H. N. *Note on a theorem of Dickson*, 450.
Sheffer, I. M. *On the theory of sum-equations*, 777.
Shniad, H. See Beckenbach, E. F.
Sholander, M. *On the existence of the inverse operation in alternation groupoids*, 746.
Spanier, E. H. *The Mayer homology theory*, 102.
Stark, R. H. *On the representation of a function as a Hellinger integral*, 155.
Stewart, B. M. *A note on least common left multiples*, 587.
Sumner, D. B. *An inversion formula for the generalized Stieltjes transform*, 174.
Taub, A. H. *A characterization of conformally flat spaces*, 85.
Thompson, F. B. *A note on the unique factorization of abstract algebras*, 1137.
Turán, P. *On Descartes-Harriot's rule*, 797.
Wall, H. S. *Convergence of continued fractions in parabolic domains*, 391.
Wallace, A. D. *Cyclic invariance under multi-valued maps*, 820.
Walsh, J. L., and Sewell, W. E. *On interpolation to an analytic function in equidistant points: Problem β*, 1177.
Ward, M. See Dilworth, R. P.
Weil, A. *Numbers of solutions of equations in finite fields*, 497.
Whaples, G. See Leavitt, W.
Whitehead, G. W. See Eckmann, B.
Whitehead, J. H. C. See Index of Addresses.
Whitney, H. See Loomis, L. H.
Wilansky, A. *A necessary and sufficient condition that a summability method be stronger than convergence*, 914.
Wilkins, J. E. *An integration scheme of Maréchal*, 191.
—— *A bound for the mean value of a function*, 801.
—— *Some remarks on ruled surfaces*, 1169.
Williams, C. See Baer, R.
Wing, G. M. *Summability with a governor of integral order*, 146.
Wolfowitz, J. *Remarks on the notion of recurrence*, 394.
Wylie, C. R. *The uniqueness of a certain line involution*, 633.
Yang, C. T. *A theorem in finite projective geometry*, 930.
Young, G. S. *On continuous curves irreducible about compact sets*, 439.
Youngs, J. W. T. *Remarks on cyclic additivity*, 427.
Zucker, R. See Salzer, H. E.